McGill University
Department of Electrical and Computer Engineering

Lecture 33

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Ideal LPF, HPF and BPF

Infinite Impulse Response (IIR) Filters

Finite Impulse Response (FIR) Filters

Moving-Average Filters

Ideal LPF

The frequency response of the ideal low-pass DT filter with cutoff frequency ω_c is described by:

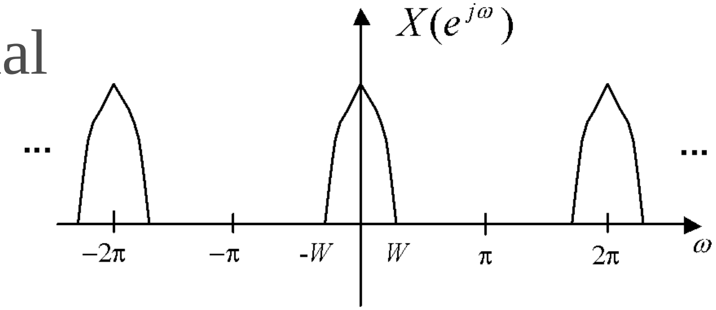
$$H(e^{j\omega}) = \begin{cases} 1, & |\omega - k2\pi| \le \omega_c, k = 0, \pm 1, \pm 2, \ldots. \\ 0, & otherwise \end{cases}$$

The impulse response of the ideal LPF is:

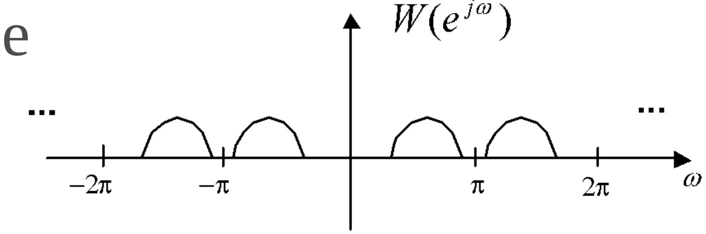
$$h_{lp}[n] = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi}\mathrm{sinc}(\frac{\omega_c n}{\pi}), \quad n = \ldots, -2, -1, 0, 1, 2, \ldots.$$

The impulse response is anti-causal, real and even. The anti-causality makes a real-time implementation of this filter impossible!

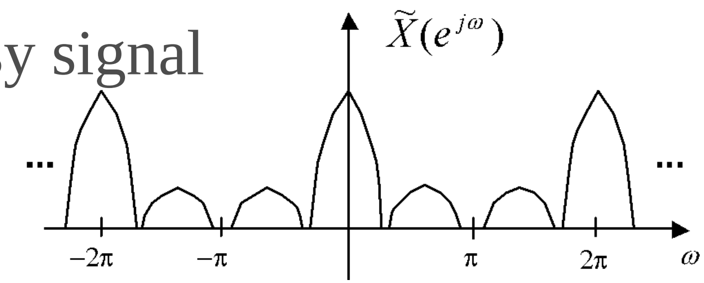
signal



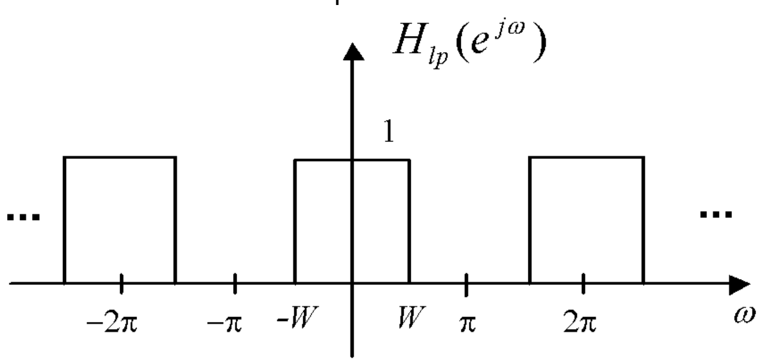
Noise

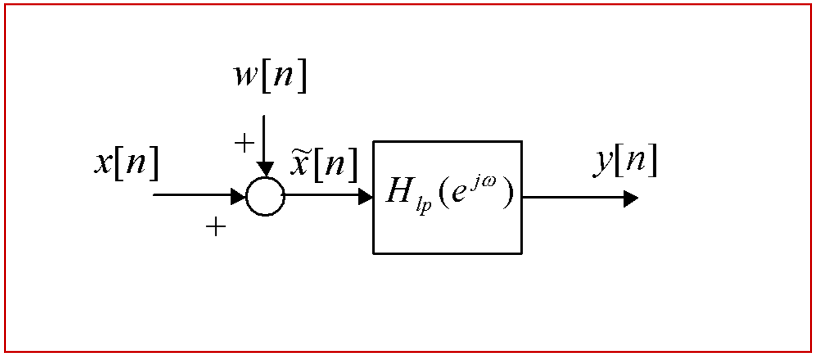


Noisy signal

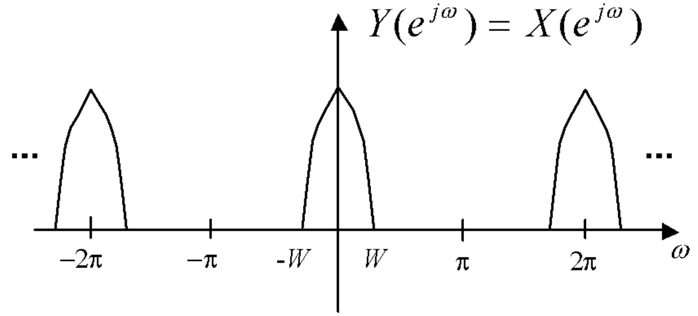


LPF





LP filtered signal

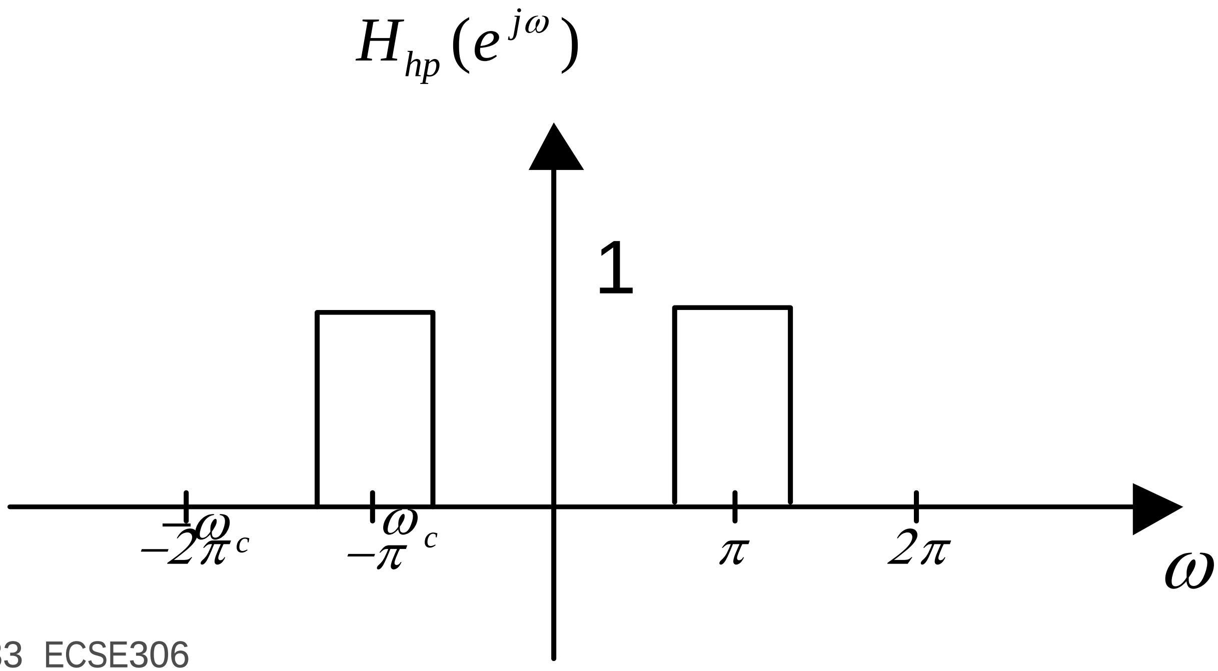


Note: when the spectra of the signal and the noise overlap, LPF can’t eliminate the noise from the noisy signal completely.

Ideal High-Pass Filter

An ideal high-pass filter with cutoff frequency ω_c is given by:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega - (2k+1)\pi| \le \pi - \omega_c, k = 0, \pm 1, \pm 2, \ldots \\ 0, & otherwise \end{cases}$$



From the frequency shifting property of the DTFT, we find that the impulse response of the ideal high-pass filter is:

$$h_{hp}[n] = \mathcal{F}^{-1}\left\{H_{lp}(e^{j(\omega-\pi)})\right\}$$

$$= e^{j\pi n} h_{lp}[n]$$

$$= (-1)^n \frac{\sin(\pi - \omega_c)n}{\pi n}$$

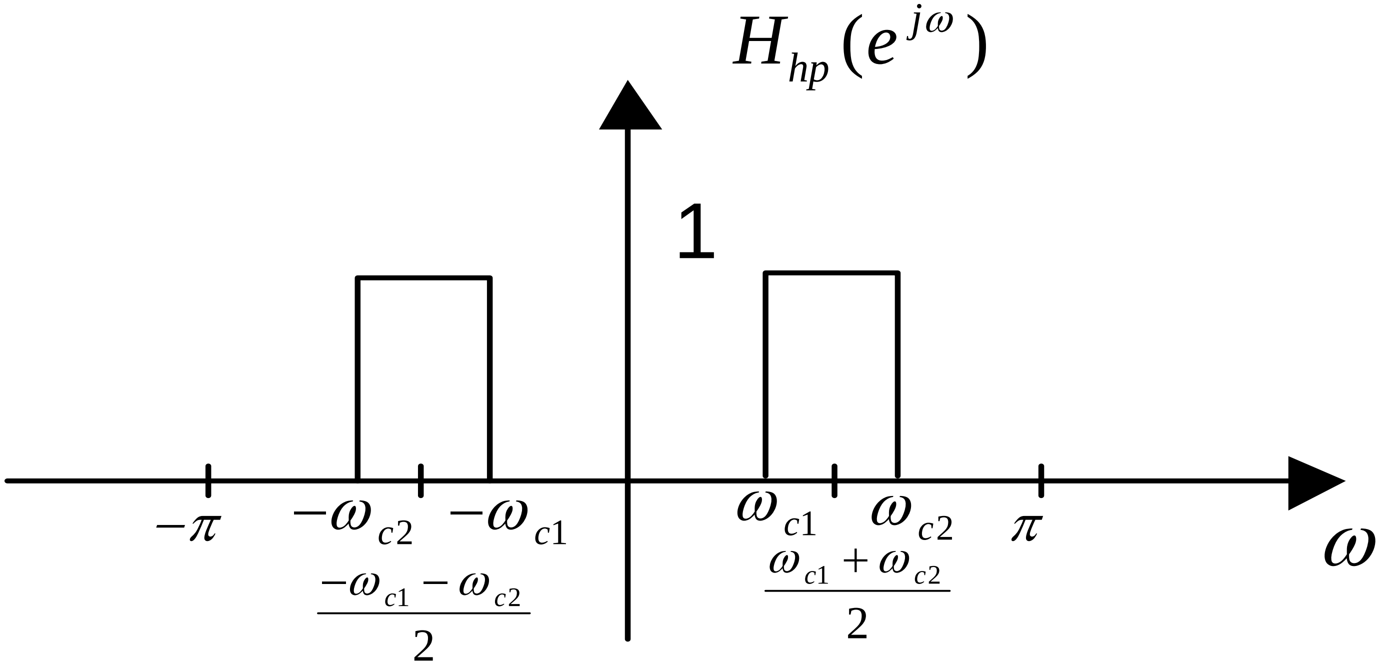
$$= (-1)^n \frac{\sin \omega_c n}{\pi n} = (-1)^n \frac{\omega_c}{\pi} \operatorname{sinc}\left(\frac{\omega_c n}{\pi}\right)$$

n=…, -2, -1,0,1, 2, …

This high-pass filter is not causal, and impossible to filter signals in real time.

Ideal Band-Pass Filter

An ideal band-pass filter with passband between ω_{c1}, ω_{c2} :



IIR (Infinite Impulse Response) Filters

IIR filters have impulse responses extending to $n \to +\infty$.

This includes the class of DLTI *recursive* filters, that is, filters represented by difference equations including delayed versions of the output $y[n]$.

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_N y[n-N]$$
$$= b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M].$$

Recursive IIR filters have transfer functions with at least one pole different from 0:

$$H(z) = \frac{(b_0 + b_1 z^{-1} + \cdots + b_M z^{-M})}{(a_0 + a_1 z^{-1} + \cdots + a_N z^{-N})}$$

$$= A \frac{\prod_{k=1}^{M} (1 - z_k z^{-1})}{\prod_{k=1}^{N} (1 - p_k z^{-1})} \quad .$$

$$= A z^{N-M} \frac{\prod_{k=1}^{M} (z - z_k)}{\prod_{k=1}^{N} (z - p_k)}$$

Example: First-order IIR filter

$$H(z) = \frac{1 + \frac{1}{3} z^{-1}}{1 - \frac{1}{2} z^{-1}}, \quad |z| > \frac{1}{2},$$

whose associated recursive equation is

$$y[n] = \frac{1}{2} y[n-1] + x[n] + \frac{1}{3} x[n-1].$$

Benefits of IIR filters include:

- Low-order filters can give relatively sharp transition bands,
- Low memory requirements when implemented as a recursive equation.

Disadvantages include:

- Linear phase difficult to obtain

FIR (Finite Impulse Response) Filters

FIR filters have, as the name implies, impulse responses of finite duration.

We will restrict most of our discussion to *causal* FIR filters

$$a_0 y[n] = b_0 x[n] + b_1 x[n-1] + \cdots + b_M x[n-M].$$

Even though this difference equation is 0th-order according to our original definition, as an FIR filter the system is said to be Mth-order.

Moving-average filters are of the FIR type.

The impulse response of a causal FIR filter (with $a_0 = 1$ without loss of generality) is simply

$$h[n] = b_0\delta[n] + b_1\delta[n-1] + \cdots + b_M\delta[n-M].$$

That is,

$$h[n] = \begin{cases} b_n, & n = 0,\dots,M \\ 0, & otherwise \end{cases}$$

The transfer function of a causal FIR filter is given by:

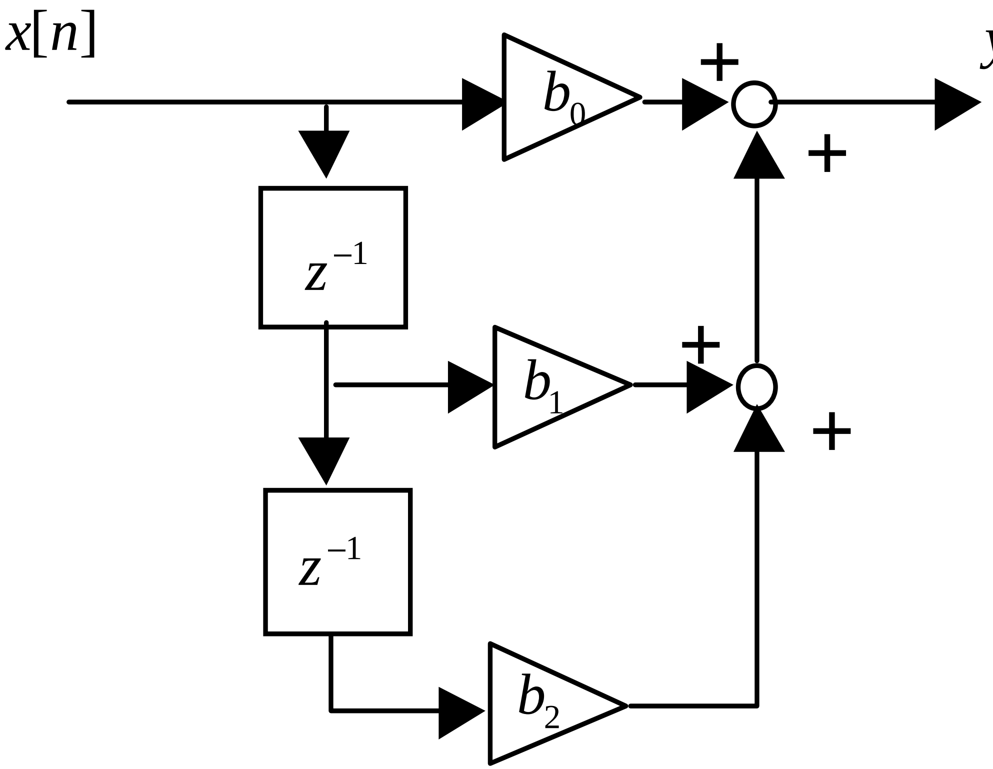
$$H(z) = b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}$$

$$= \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{z^M}$$

$$= A \frac{\prod_{k=1}^{M} (z - z_k)}{z^M}$$

Note that all the poles are at $z = 0$. Thus, only the zeros' locations in the *z*-plane will determine the filter's frequency response.

Realization of second-order causal FIR filter



Moving-Average Filters

A special type of FIR filter is the causal *moving-average filter* whose coefficients are all equal to a constant (chosen so that the DC gain is 1), i.e. the impulse response is a rectangular pulse:

$$h[n] = \begin{cases} \dfrac{1}{M+1}, & n = 0, \ldots, M \\ 0, & otherwise \end{cases}$$

This type of filter is often used to smooth economic data to find the underlying trend of a variable. The transfer function is:

$$H(z) = \frac{1}{M+1}[1 + z^{-1} + \ldots + z^{-M}] = \frac{1}{M+1}\frac{z^{M} + z^{M-1} + \ldots + 1}{z^{M}}$$

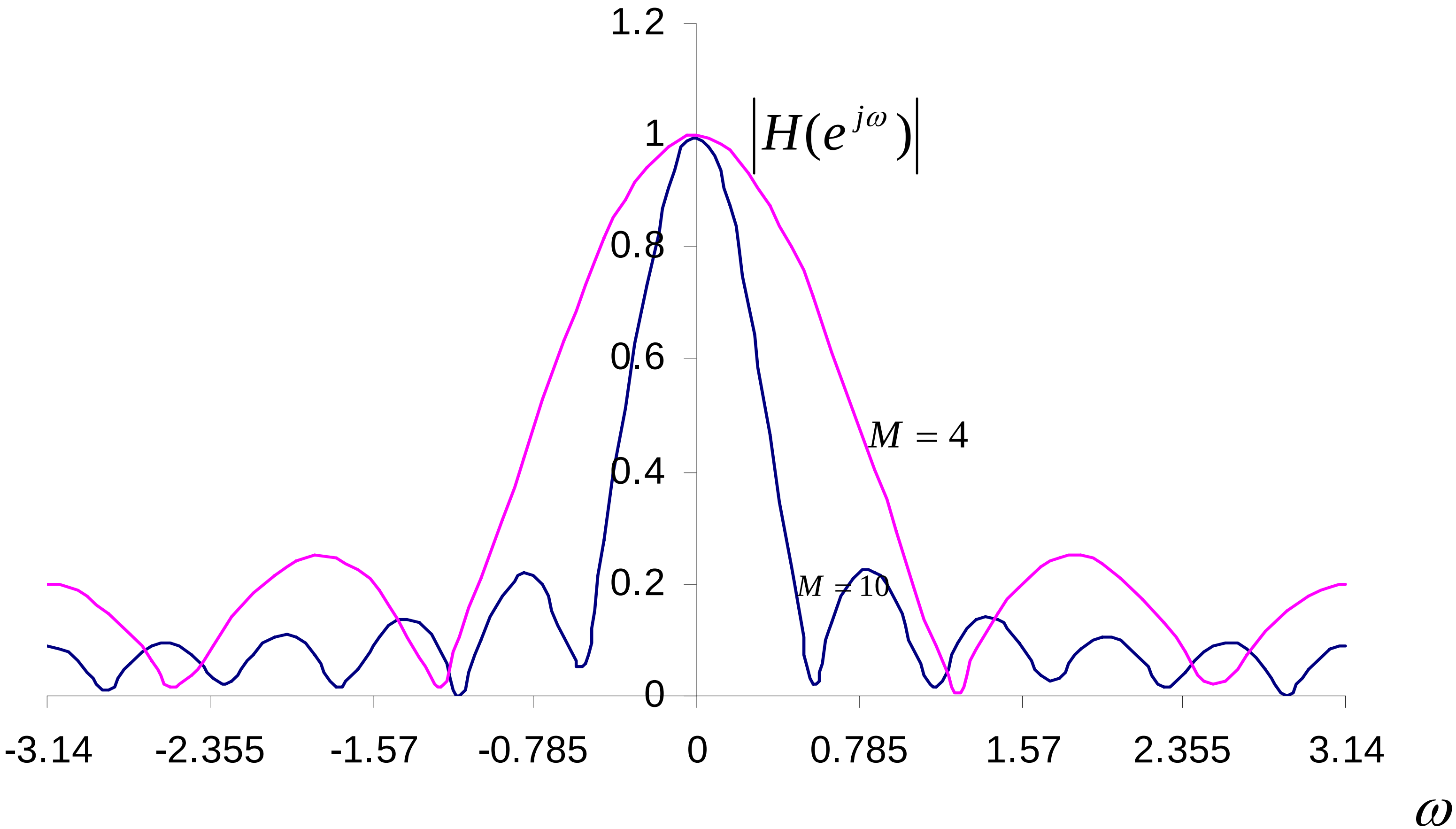
The spectrum of the moving-average filter

We can use the finite geometric sum formula to obtain an expression for $H(e^{j\omega})$:

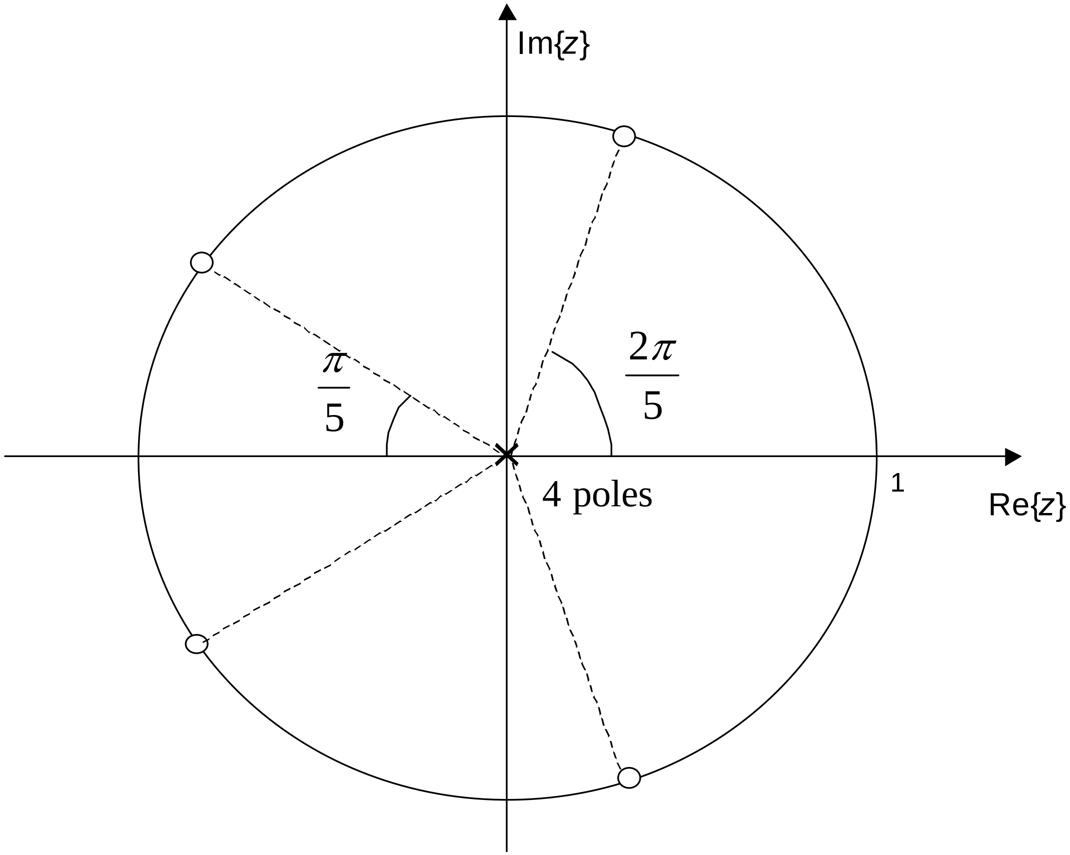
$$H(e^{j\omega}) = \frac{1}{M+1}\sum_{k=0}^{M} e^{-jk\omega} = \frac{1}{M+1}\frac{1-e^{-j\omega(M+1)}}{1-e^{-j\omega}}$$

$$= \frac{e^{-j\omega\frac{M}{2}}}{M+1}\frac{\sin(\omega\frac{M+1}{2})}{\sin(\frac{\omega}{2})}$$

The magnitude of this frequency response displays *M* zeros, the first one being at $\omega = \dfrac{2\pi}{M+1}$. Thus, the bandwidth of this moving-average low-pass filter depends on its length and cannot be specified otherwise.



Also, since there are M zeros in $H(e^{j\omega})$, these zeros must be zeros of $H(z)$ on the unit circle. (case $M = 4$ here)



That is, the zeros are equally spaced on the unit circle as though there were $M+1$ zeros, with the one at z=1 removed.