McGill University
Department of Electrical and Computer Engineering

Lecture 32

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Hui Qun Deng, PhD

- Realization (Block Diagram) of Difference Systems
- Unilateral z-Transform
- Inverse Unilateral z-Transform
- Solutions to Difference Equations Initially Not at Rest

Transfer Function Characterization of LTI Difference Systems

Consider the *N*th-order difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

Using the time-shifting property of the *z*-transform, we obtain

$$\sum_{k=0}^{N} a_k z^{-k} Y(z) = \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$(a_0 + a_1 z^{-1} + \cdots + a_N z^{-N})Y(z) = (b_0 + b_1 z^{-1} + \cdots + b_M z^{-M})X(z)\,.$$

The transfer function is then given by the *z*-transform of the output divided by the *z*-transform of the input:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{(b_0+b_1z^{-1}+\cdots+b_Mz^{-M})}{(a_0+a_1z^{-1}+\cdots+a_Nz^{-N})}.$$

Hence the transfer function of an LTI difference system is always rational.

The ROC of $H(z)$ must be consistent with the ROCs of $Y(z)$ and $X(z)$. Namely, it must satisfy $R_Y \supset R_X \cap R_H$

Example

Consider a DLTI system defined by the difference equation:

$$y[n] + \frac{1}{3} y[n-1] = 2x[n-1] .$$

Taking the z-transform, we get:

$$Y(z) + \frac{1}{3} z^{-1} Y(z) = 2z^{-1} X(z) ,$$

which yields the transfer function

$$H(z) = \frac{2z^{-1}}{1+\frac{1}{3}z^{-1}},$$

This provides the algebraic expression for $H(z)$, but not the ROC.

As a matter of fact, there are two impulse responses that are consistent with the difference equation.

A right-sided impulse response corresponds to the ROC $|z| > \frac{1}{3}$.

Using the time-shifting property, we get:

$$h[n] = 2\left(-\frac{1}{3}\right)^{n-1} u[n-1]$$

In this case the system is causal and stable.

A left-sided impulse response corresponds to the ROC $|z| < \frac{1}{3}$.

Using the time-shifting property again, we get:

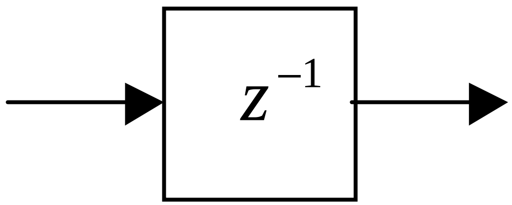
$$h[n] = -2\left(-\frac{1}{3}\right)^{n-1} u[-n].$$

This case leads to an unstable, anticausal system.

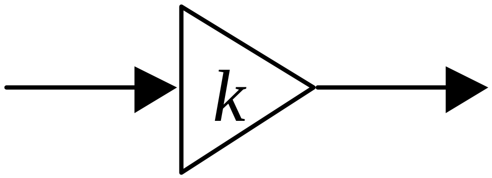
Block Diagrams (Realization) of $H(z)$ for Difference Systems

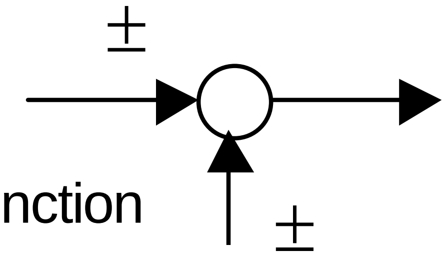
The transfer function of a DLTI difference system can be realized using a combination of three basic elements:

the unit delay,



the gain,





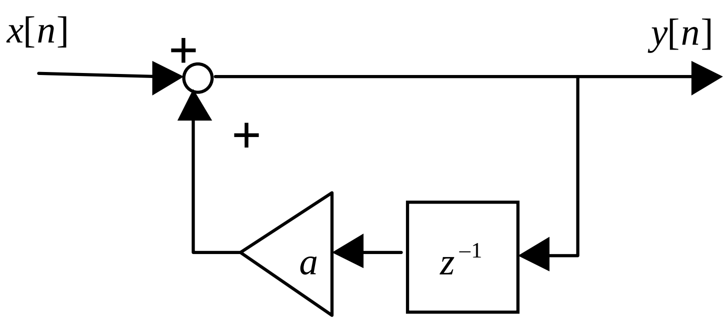
the summing junction

Simple First-Order Transfer Function

Consider the transfer function $H(z)=\dfrac{1}{1-az^{-1}}$, which corresponds to the first-order difference equation

$$y[n]-ay[n-1]=x[n]$$

$$y[n]=ay[n-1]+x[n]$$



Simple Second-Order Transfer Function

Consider the transfer function $H(z) = \dfrac{1}{1 + a_1 z^{-1} + a_2 z^{-2}}$.

The transfer function can be realized as a sum of two first-order transfer functions (partial fraction expansion): the *parallel form*, which is a parallel interconnection of the two first-order transfer functions.

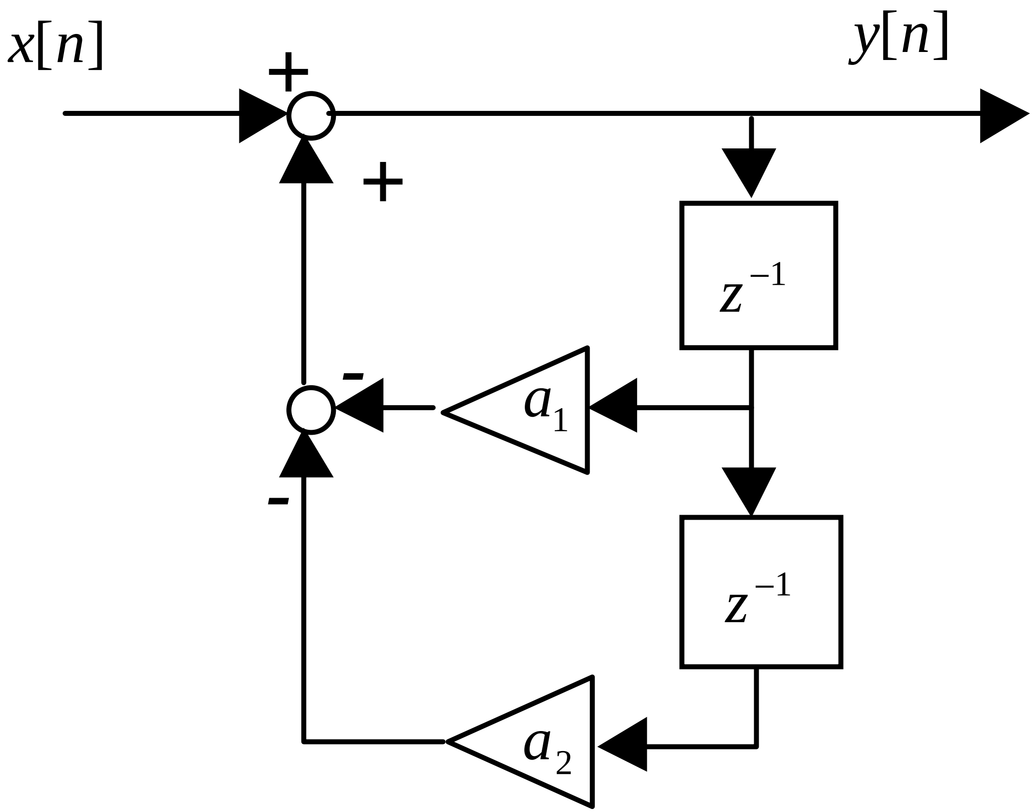
Another way is to break up the transfer function as a cascade (multiplication) of two first-order transfer functions.

Yet another way to realize the second-order transfer function is the so-called *direct form* or *controllable canonical form*. To develop this form, consider the system equation

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) + X(z).$$

$$Y(z) = -a_1 z^{-1} Y(z) - a_2 z^{-2} Y(z) + X(z) .$$

This equation can be realized as follows with two unit delays:

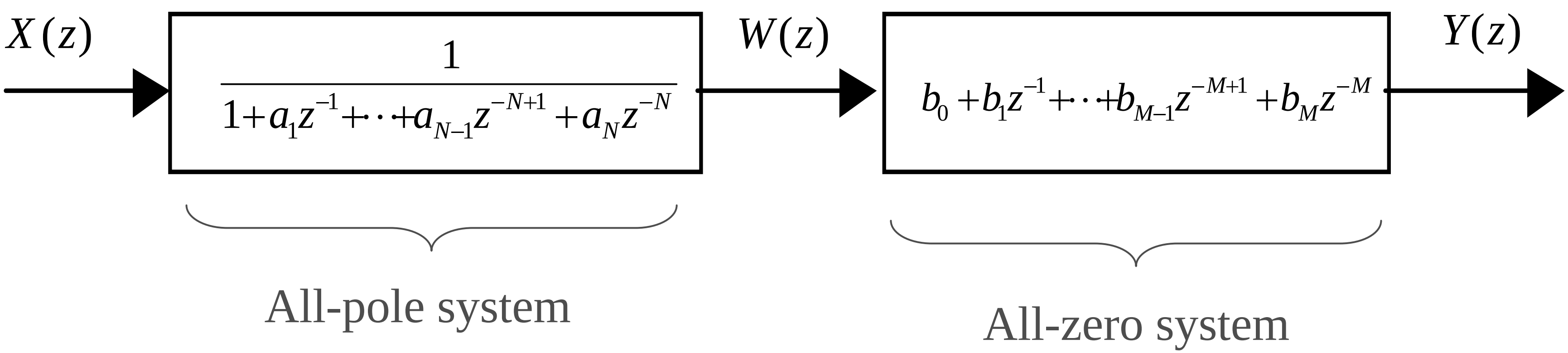


Direct Form (Controllable Canonical Form)

A direct form can be obtained by breaking up a general transfer function into two subsystems as follows:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{(b_0+b_1z^{-1}+\cdots+b_Mz^{-M})}{(a_0+a_1z^{-1}+\cdots+a_Nz^{-N})}.$$

Assume without loss of generality that $a_0=1$.

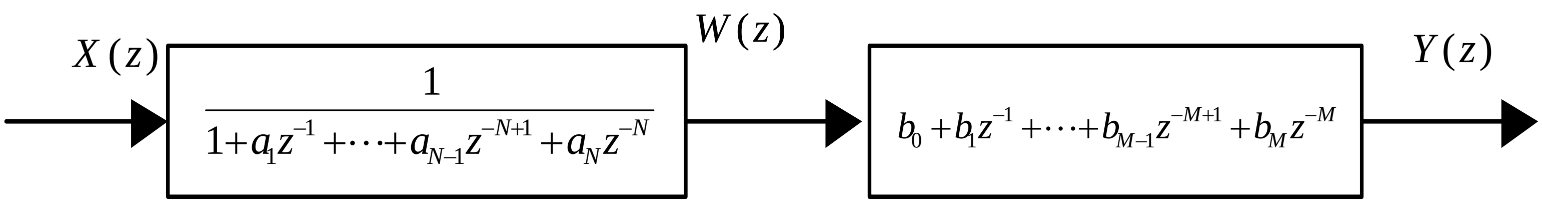


The input-output system equation of the first subsystem is

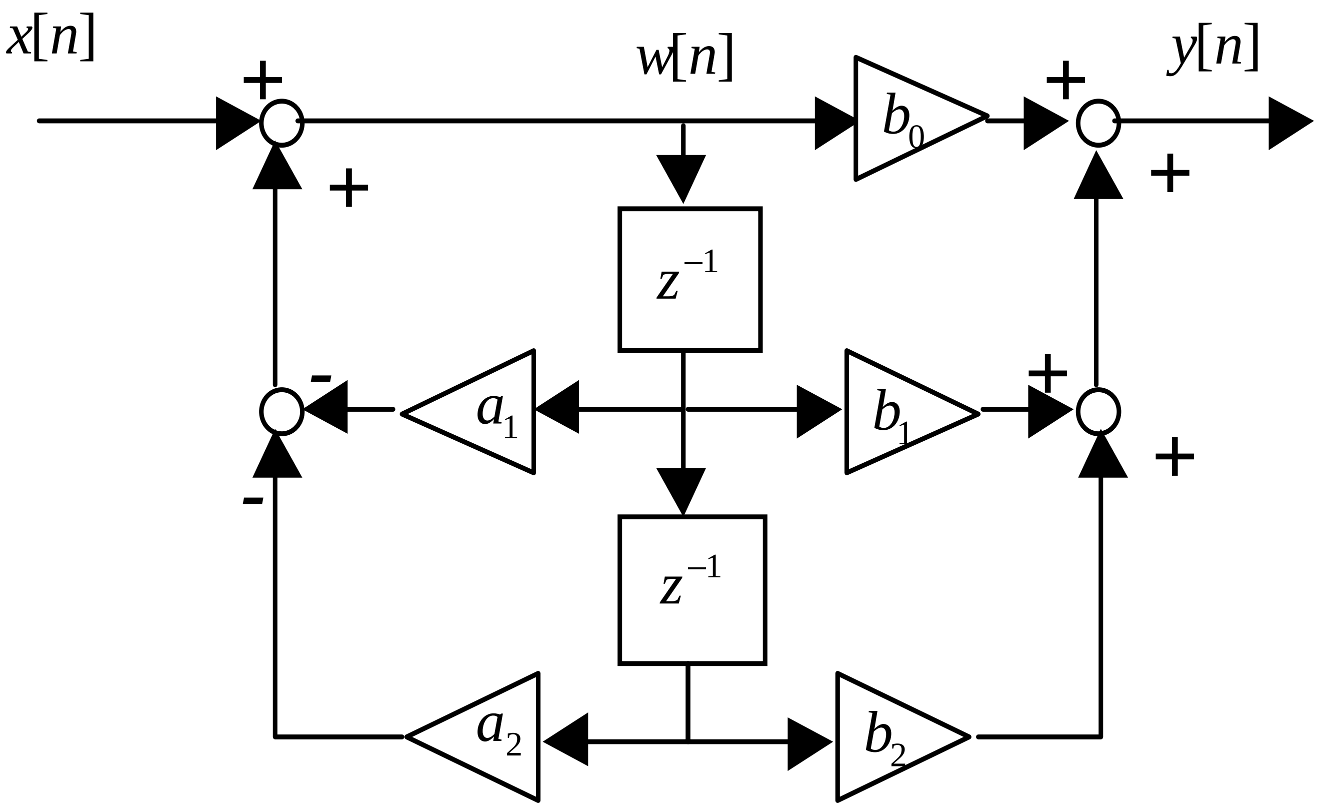
$$W(z) = -a_1 z^{-1} W(z) - \cdots - a_{N-1} z^{-N+1} W(z) - a_N z^{-N} W(z) + X(z)$$

and for the second subsystem we have

$$Y(z) = b_0 W(z) + b_1 z^{-1} W(z) + \cdots + b_{M-1} z^{-M+1} W(z) + b_M z^{-M} W(z) .$$



The direct form realization is then (for a second-order system):



The Unilateral *z*-Transform

The unilateral *z*-transform is defined for the causal part of discrete-time signals.

$$\mathcal{X}(z) := \sum_{n=0}^{+\infty} x[n] z^{-n} ,$$

The signal/transform pair is denoted as

$$x[n] \overset{\mathcal{UZ}}{\longleftrightarrow} \mathcal{X}(z) = \mathcal{UZ}\{x[n]\}$$

The series only has negative powers of *z* since the summation runs over nonnegative times.

One implication is that

$$\mathcal{UZ}\{x[n]\} = \mathcal{UZ}\{x[n]u[n]\}$$

Another implication is that the ROC of a unilateral *z*-transform *is always the exterior of a circle*.

Example

Consider the signal

$$x[n] = a^{n+1}u[n+1].$$

The bilateral *z*-transform of $x[n]$ is obtained by using the time-shifting property:

$$X(z) = \frac{z}{1 - az^{-1}} = \frac{z^2}{z - a}, \quad |z| > a$$

The unilateral z-transform of $x[n]$ is computed as:

$$\mathcal{X}(z) = \sum_{n=0}^{+\infty} a^{n+1} z^{-n}$$

$$= \frac{a}{1 - az^{-1}}, \quad |z| > a$$

The bilateral and unilateral z-transforms are different for non-causal signals.

Inverse Unilateral *z*-Transform

The inverse unilateral *z*-transform can be obtained by

- performing a partial fraction expansion,
- selecting all the ROCs of the individual first-order fractions to be exteriors of disks.

Long division can be used as well. The series must be in negative powers of z.

Example

The unilateral z-transform

$$\mathcal{X}(z) = \frac{1}{1 - az^{-1}}, \quad |z| > a$$

can be expanded in the power series

$$\begin{array}{r|l} & 1 + az^{-1} + a^2 z^{-2} + \dots \\ \hline 1 - az^{-1} & 1 \\ & \underline{1 - az^{-1}} \\ & \quad az^{-1} \\ & \quad \underline{az^{-1} - a^2 z^{-2}} \\ & \qquad a^2 z^{-2} \end{array}$$

Note that the resulting power series converges because the ROC implies $\left|az^{-1}\right| < 1$.

Here, we can see that the signal is

$$x[n] = a^n u[n].$$

Properties of the Unilateral z-Transform that Differ from Properties of the Bilateral z-Transform

Consider the pair $x[n] \overset{uz}{\leftrightarrow} \mathcal{X}(z)$.

Time Delay

$$x[n-1] \overset{uz}{\leftrightarrow} z^{-1}\mathcal{X}(z) + x[-1]$$

Time advance

Time Advance

$$x[n+1] \overset{uz}{\leftrightarrow} z\mathcal{X}(z) - zx[0]$$

Convolution

For *causal* signals $x_1[n] \overset{uz}{\leftrightarrow} \mathcal{X}_1(z)$ and $x_2[n] \overset{uz}{\leftrightarrow} \mathcal{X}_2(z)$,
we have the familiar result:

$$x_1[n] * x_2[n] \overset{uz}{\leftrightarrow} \mathcal{X}_1(z)\mathcal{X}_2(z)$$

Note that the resulting signal will also be causal since

$$
\begin{aligned}
y[n] &= x_1[n] * x_2[n] \\
&= \sum_{m=-\infty}^{+\infty} x_1[m] x_2[n-m] \\
&= \sum_{m=-\infty}^{+\infty} x_1[m] u[m] x_2[n-m] u[n-m] \\
&= \sum_{m=0}^{n} x_1[m] x_2[n-m]
\end{aligned}
$$

and the last summation is 0 for $n < 0$.

Solution to Difference Equations Initially NOT at Rest

Recall Ch3, we solved difference equations initially at rest. Now, using unilateral z-transform, you can solve difference equations initially NOT at rest, i.e., $y[-1]$, $y[-2]$, $y[-3]$, …, are not zero.

The time delay property can be used recursively to show that

$$x[n-m] \overset{uz}{\longleftrightarrow} z^{-m}\mathcal{X}(z) + z^{-m+1}x[-1] + \cdots + z^{-1}x[-m+1] + x[-m]$$

Thus, we can solve a difference system with initial conditions by using the unilateral z-transform.

Example

Consider the causal difference equation:

$$y[n]-0.8y[n-1]=2x[n],$$

where the input signal is $x[n]=(0.5)^n u[n]$, and the initial condition is $y[-1]=y_{-1}$.

Unilateral *z*-transform:

$$\mathcal{Y}(z)-0.8z^{-1}\mathcal{Y}(z)-0.8y[-1]=2\mathcal{X}(z)$$

$$(1-0.8z^{-1})\mathcal{Y}(z)-0.8y_{-1}=\frac{2}{1-0.5z^{-1}}$$

which yields

$$\mathcal{Y}(z) = \frac{0.8y_{-1}}{(1-0.8z^{-1})} + \frac{2}{(1-0.8z^{-1})(1-0.5z^{-1})}, \quad |z| > 0.8$$

The first term on the right-hand side is the *zero-input response*.

The second term on the right-hand side is the *zero-state response.*

The zero-state response

Expand the zero-state response in partial fractions:

$$\frac{2}{(1-0.8z^{-1})(1-0.5z^{-1})} = \frac{1.23}{1-0.8z^{-1}} + \frac{0.77}{1-0.5z^{-1}}.$$

Finally, the unilateral z-transform of the system is given by:

$$\mathcal{Y}(z) = \underbrace{\frac{0.8y_{-1}+1.23}{1-0.8z^{-1}}}_{|z|>0.8} + \underbrace{\frac{0.77}{1-0.5z^{-1}}}_{|z|>0.5},$$

and its corresponding time-domain signal is:

$$y[n] = (0.8y_{-1}+1.23)(0.8)^n u[n] + 0.77(0.5)^n u[n]$$