McGill University
Department of Electrical and Computer Engineering

Lecture 31

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Hui Qun Deng, PhD

Inverse z-transform

Application of z-transform to difference systems

Inverse z-transform using partial fraction expansion

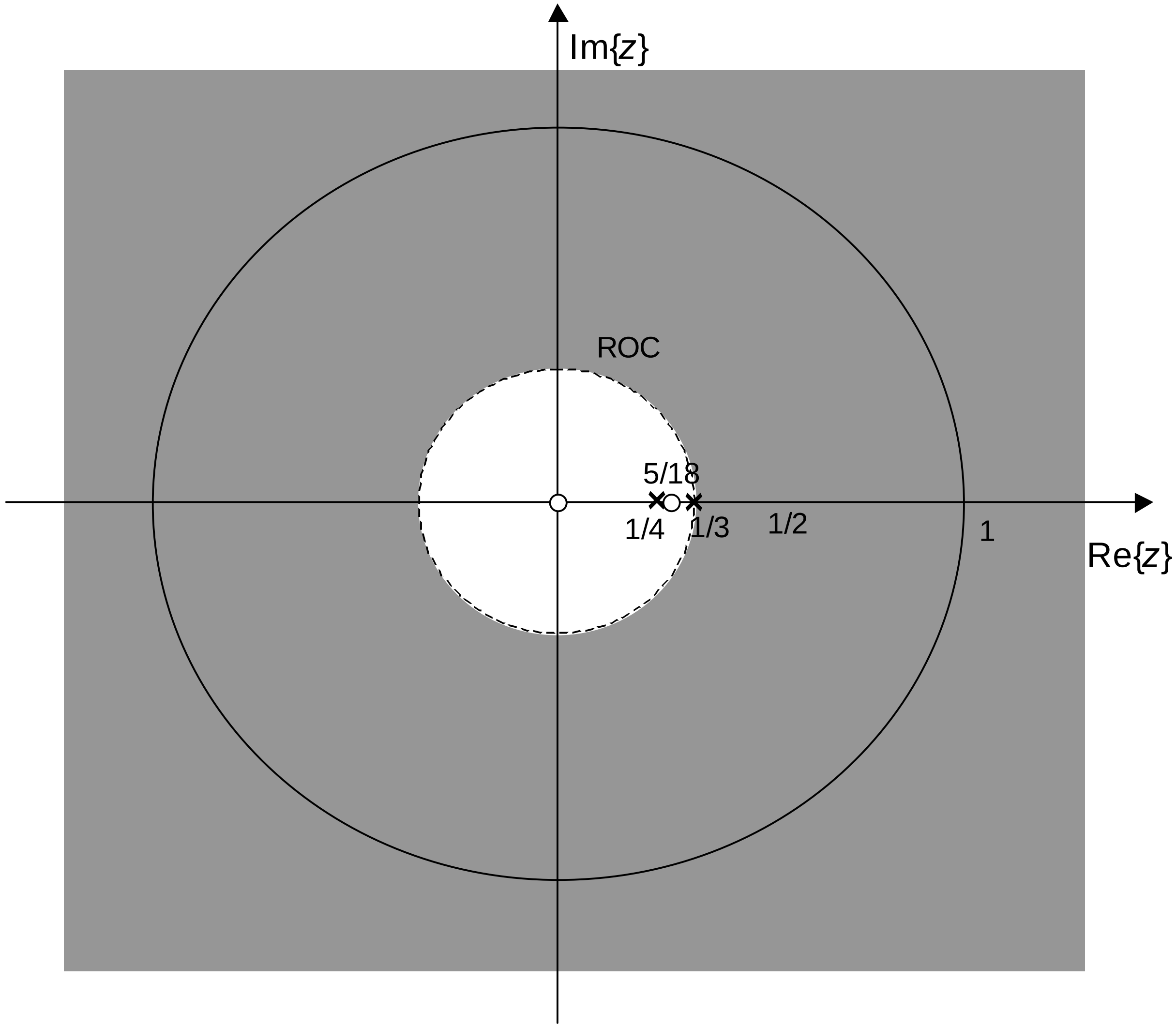
An easier way to get the inverse *z*-transform is to expand *x*[*z*] in partial fractions and use the table of *z*-transform pairs.

Example

Find the signal with *z*-transform

$$X(z) = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})(1 - \frac{1}{3} z^{-1})}, \quad |z| > \frac{1}{3}$$

The pole-zero plot with ROC is shown below.



The partial fraction expansion of $X(z)$ has the form

$$X(z) = \underbrace{\frac{A}{1-\frac{1}{4}z^{-1}}}_{|z|>\frac{1}{4}} + \underbrace{\frac{B}{1-\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}},$$

where the ROCs of the individual terms are selected for consistency with the ROC $|z| > \frac{1}{3}$

coefficients *A* and *B* are calculated as follows.

$$A = (1 - \tfrac{1}{4} z^{-1}) X(z) \Big|_{z=\frac{1}{4}} = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{3} z^{-1})} \Bigg|_{z=\frac{1}{4}} = 1$$

$$B = (1 - \tfrac{1}{3} z^{-1}) X(z) \Big|_{z=\frac{1}{3}} = \frac{3 - \frac{5}{6} z^{-1}}{(1 - \frac{1}{4} z^{-1})} \Bigg|_{z=\frac{1}{3}} = 2$$

$$X(z) = \underbrace{\frac{1}{1-\frac{1}{4}z^{-1}}}_{|z|>\frac{1}{4}} + \underbrace{\frac{2}{1-\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}}$$

$$\overset{z}{\leftrightarrow}$$

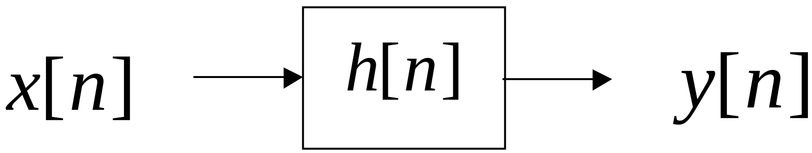
$$x[n] = \left(\frac{1}{4}\right)^n u[n] + 2\left(\frac{1}{3}\right)^n u[n]$$

If the ROC is specified as $\frac{1}{4} < |z| < \frac{1}{3}$, then the inverse *z*-transform is given by

$$X(z) = \underbrace{\frac{1}{1 - \frac{1}{4} z^{-1}}}_{|z| > \frac{1}{4}} + \underbrace{\frac{2}{1 - \frac{1}{3} z^{-1}}}_{|z| < \frac{1}{3}} \quad \overset{z}{\leftrightarrow}$$

$$x[n] = \left(\frac{1}{4}\right)^n u[n] - 2\left(\frac{1}{3}\right)^n u[-n-1]$$

Analysis and Characterization of DLTI Systems Using the *z*-Transform



The convolution property of the *z*-transform allows us to write

$$Y(z) = H(z)X(z), \;\; ROC \supset R_x \cap R_h$$

Response is the inverse *z*-transform of $Y(z)$.

$$y[n] = \mathcal{Z}^{-1}\{Y(z)\}$$

The *z*-transform $H(z)$ of the impulse response $h[n]$ is called the *transfer function* (or *system function*) of the DLTI system.

The transfer function together with its ROC uniquely defines the DLTI system.

Properties of DLTI systems are associated with the characteristics of their transfer functions (poles, zeros, ROC.)

Causality

Recall that $h[n]=0$ for $n<0$ for a causal system, and thus the impulse response is right-sided.

We have seen that the ROC of a right-sided signal is the exterior of a disk. If $z=\infty$ is also included in the ROC, then the signal is also causal.

If $h[n]$ has non-zero values for $n<0$, then $z=\infty$ can not be included in the ROC.

Therefore,

A DLTI system is causal if and only if the ROC of its transfer function $H(z)$ *is the exterior of a circle including infinity.*

If the transfer function $H(z)$ is rational, then we can interpret this result as follows.

A DLTI system with a rational transfer function is causal iff

(a) the ROC is the exterior of a circle of radius equal to the magnitude of the outermost pole, and

(b) with $H(z)$ expressed as a ratio of polynomials in z, the order of the numerator is less than or equal to the order of the denominator.

We know this from long division.

Example:

Consider a system with rational transfer function

$$H(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}}, \quad |z| > 2 .$$

Since the ROC is the exterior of a circle outside of the outermost pole, the impulse response is right-sided. The second condition can be checked directly be letting $z \to \infty$:

$$\lim_{z\to\infty} H(z) = 2 ,$$

a finite value from which we conclude that the system is causal.

As a single rational function of *z:*

$$H(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z-2}, \quad |z| > 2$$

$$= \frac{2z^2 - \frac{5}{2}z}{(z-\frac{1}{2})(z-2)}, \quad |z| > 2$$

degree of numerator is no greater than the degree of the denominator.

Stability

Stability

Stability of a DLTI system is equivalent to its impulse response being absolutely summable: $\sum_{n=-\infty}^{+\infty} |h[n]| < +\infty$ in which case its Fourier transform converges.

This also implies that the ROC contains the unit circle.

A DLTI system is stable if and only if the ROC of its transfer function contains the unit circle

For a DLTI system with a *rational* and *causal* transfer function:

A causal DLTI system with rational transfer function $H(z)$ *is stable if and only if*

all of its poles lie inside the unit circle.

Remarks

- A DLTI system can be stable without being causal
- A DLTI system can be causal without being stable

Example

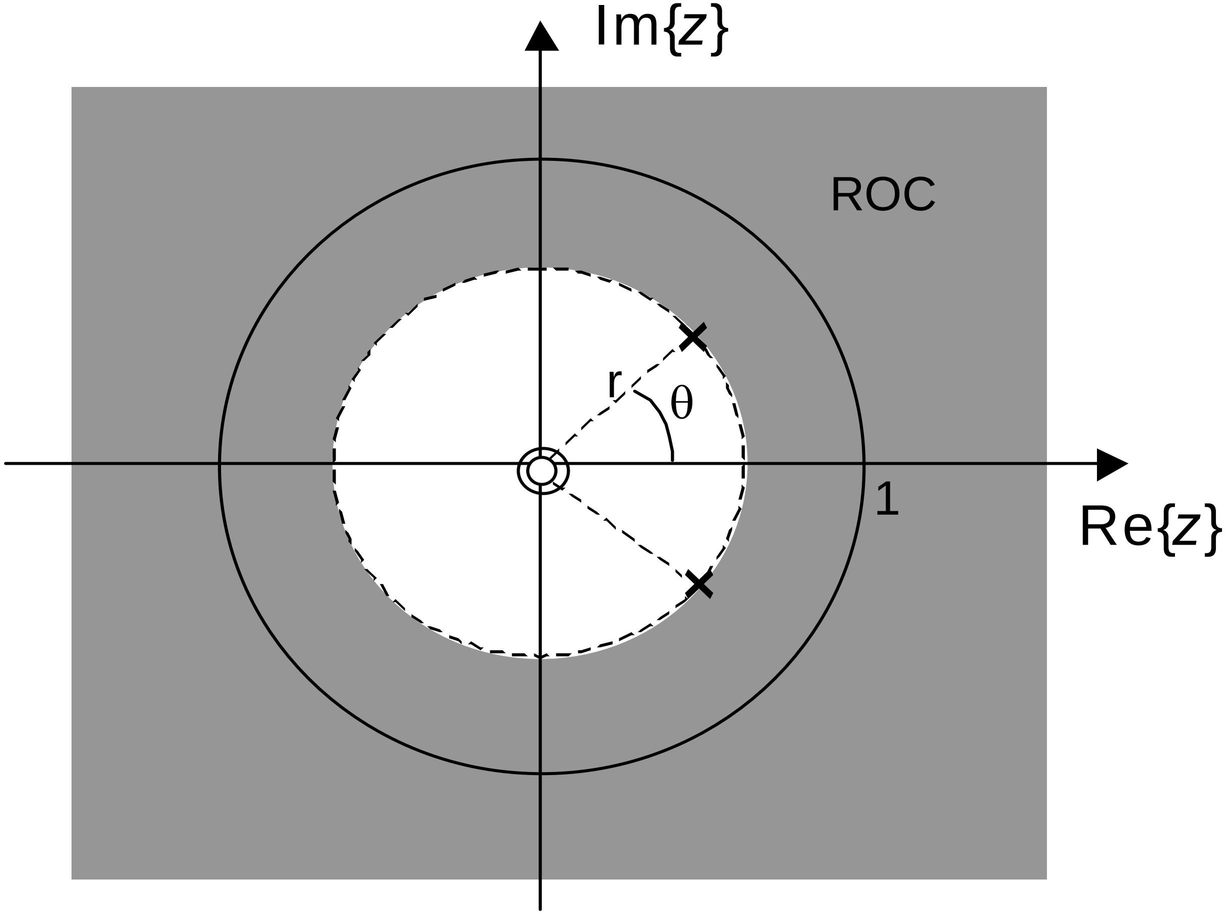
The transfer function for a second-order system with complex poles at $p_1 = re^{j\theta}, \ p_2 = re^{-j\theta}$ is given by

$$H(z) = \frac{1}{1 - 2r\cos\theta z^{-1} + r^2 z^{-2}}$$

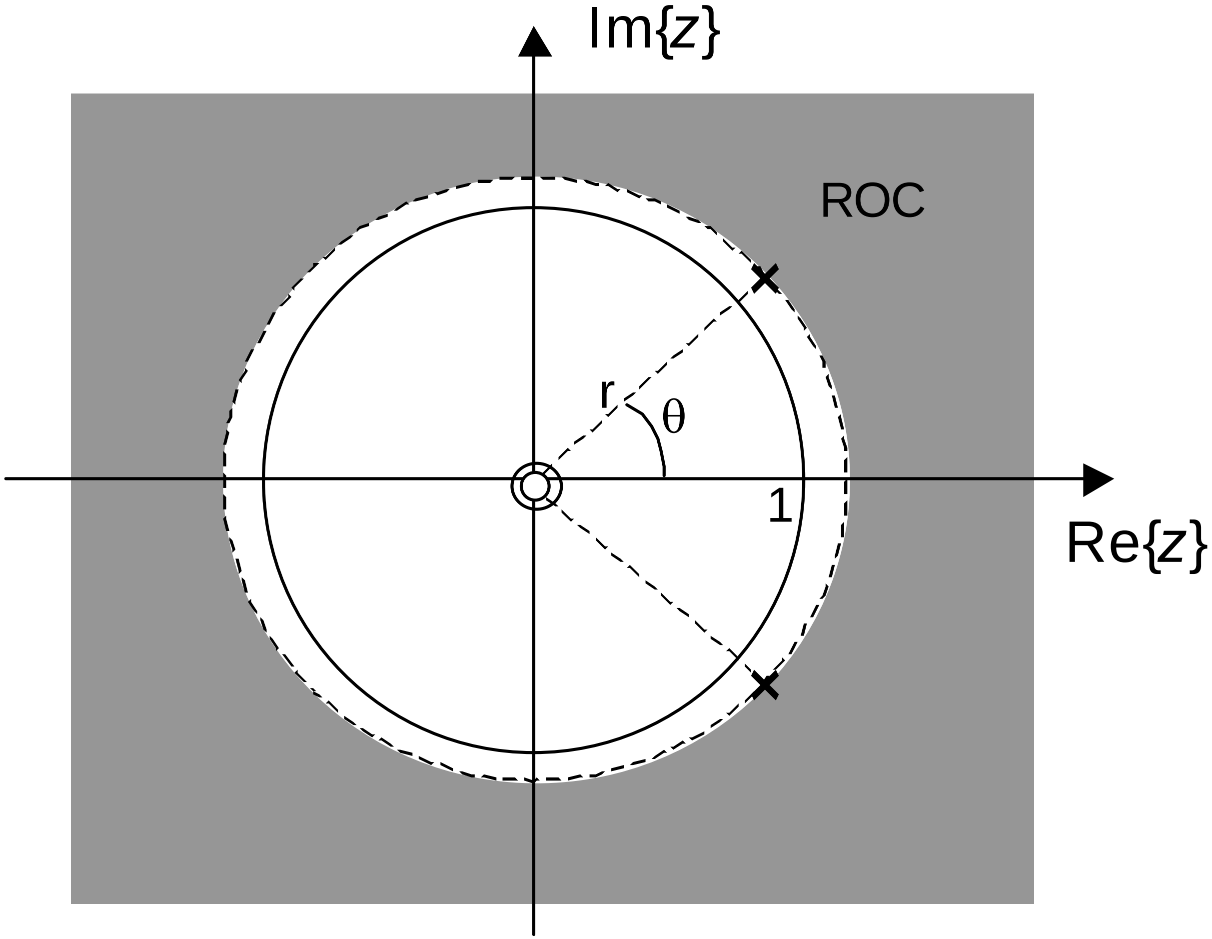
Assuming that the system is causal, ROC is $|z| > r$.

For $r \geq 1$, the system is unstable since the poles are outside of or on the unit circle.

For $r < 1$ the system is stable.



Stable $r < 1$



Unstable $r \geq 1$

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Note that the case $r = 1$ is when the two complex conjugate poles are on the unit circle.

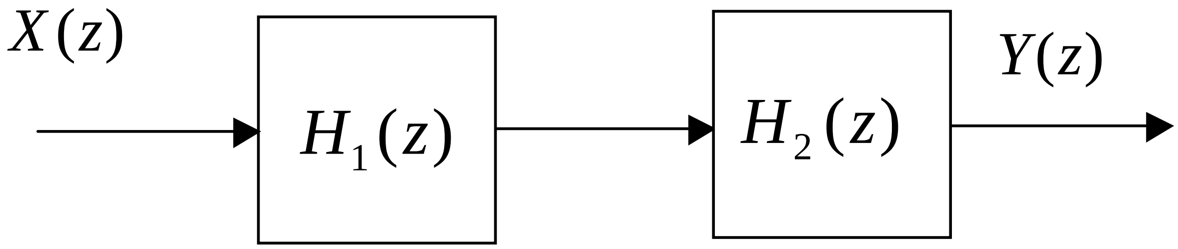
The corresponding impulse response is a sinusoidal signal which is not absolutely summable,

and therefore the system is unstable.

Transfer Function Algebra and Block Diagram Representations

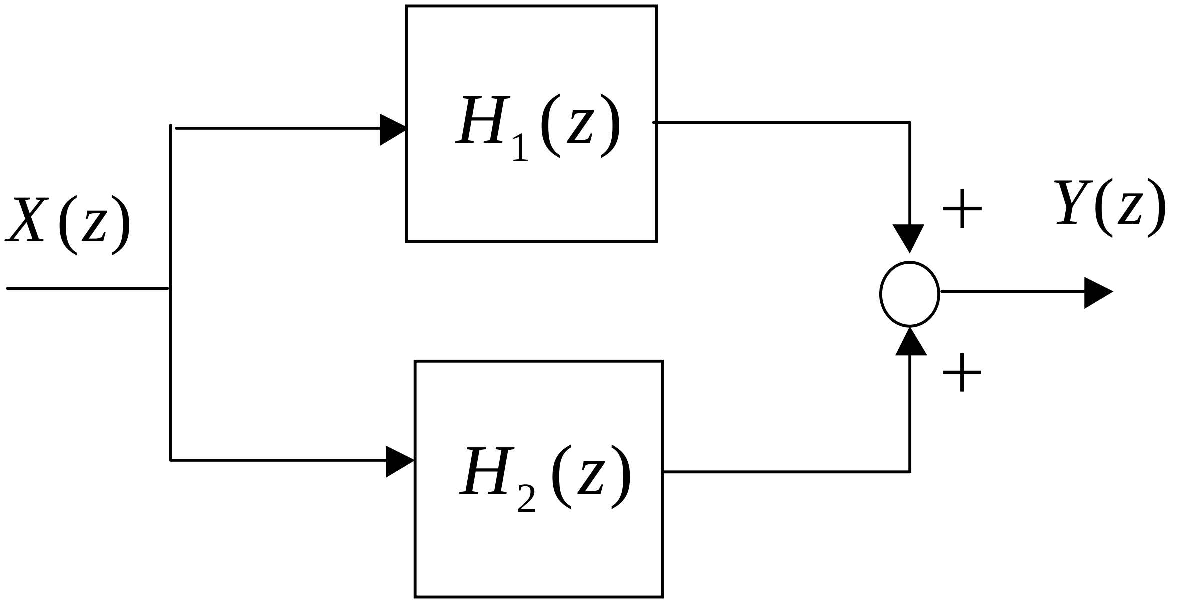
We can interconnect transfer functions in the *z*-domain to form new DLTI systems.

Cascade Interconnection



$$Y(z) = H_2(z)H_1(z)X(z)$$

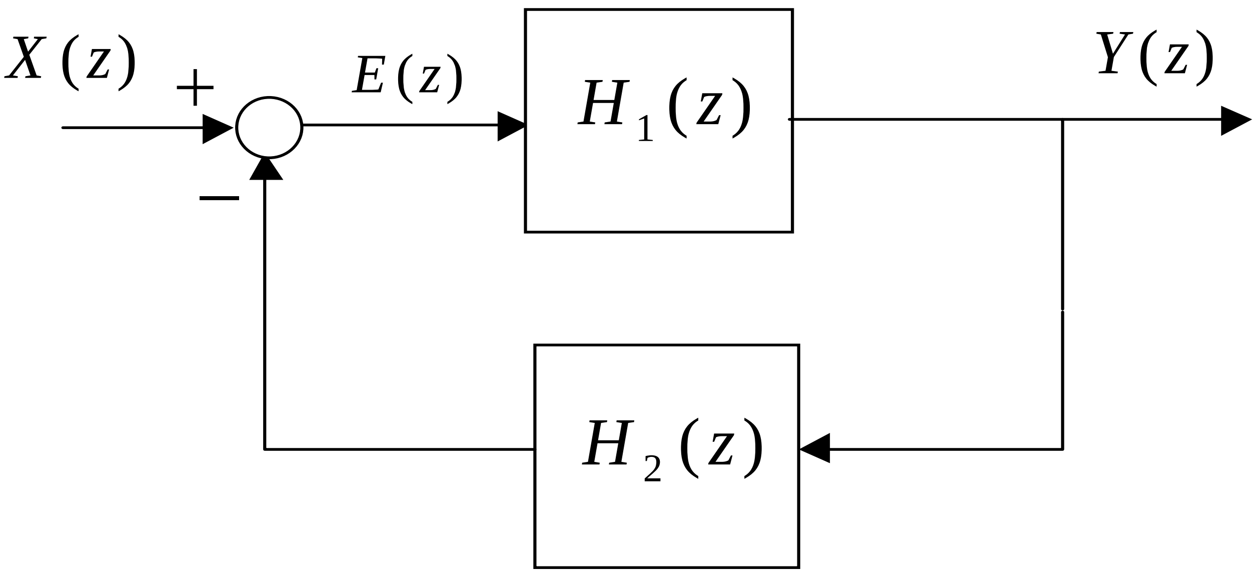
Parallel Interconnection



$$Y(z) = [H_2(z) + H_1(z)]X(z)$$

$$H(z) = H_2(z) + H_1(z)$$

Feedback Interconnection



$$E(z) = X(z) - H_2(z)Y(z)$$
$$Y(z) = H_1(z)E(z)$$

$$E(z) = X(z) - H_2(z)H_1(z)E(z)$$

$$E(z) = \underbrace{\frac{1}{1 + H_1(z)H_2(z)}}_{H(z)} X(z)$$

$$Y(z) = \underbrace{\frac{H_1(z)}{1 + H_1(z)H_2(z)}}_{H(z)} X(z)$$