ECSE 306 - Fall 2008

Fundamentals of Signals and Systems

McGill University
Department of Electrical and Computer Engineering

Lecture 30

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Hui Qun Deng, PhD

z-transform

ROC of the z-transform

Properties of z-transform

Two-sided z-transform

The response of a DLTI system to a ***complex exponential*** input z^n is the same complex exponential with only a change in (complex) amplitude: $z^n \to H(z)z^n$. The complex amplitude factor is in general a function of the complex variable z.

$$\begin{aligned} y[n] &= \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = \sum_{k=-\infty}^{+\infty} h[k]z^{n-k} \\ &= z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k} \\ &= H(z)z^n \end{aligned}$$

Recall Lecture 10 that z^n is an eigenfunction of DT LTI system.

The system's response has the form $y[n] = H(z)z^n$, where

$$H(z) = \sum_{n=-\infty}^{+\infty} h[n]z^{-n} ,$$

The function $H(z)$ is the *z-transform* of the impulse response of the system. The z-transform is also defined for a general DT signal $x[n]$:

$$X(z) := \sum_{n=-\infty}^{+\infty} x[n]z^{-n} .$$

The region of convergence of the z-transform

Writing $z = re^{j\omega}$, we analyze the region of z where the Z transform converge.

$$X(z)\Big|_{z=re^{j\omega}} = \sum_{n=-\infty}^{\infty} x[n]r^{-n}e^{-j\omega n} = \mathcal{F}\{x[n]r^{-n}\}$$

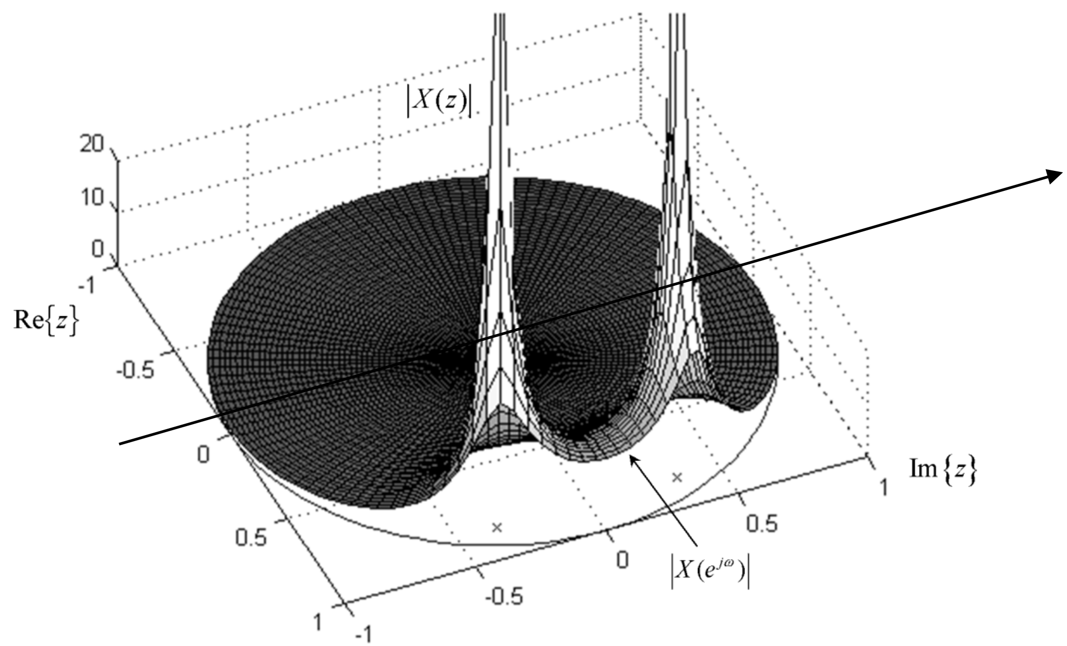
The ROC is the region of the *z*-plane ($z = re^{j\omega}$) where the signal $x[n]r^{-n}$ has a DTFT, i.e., $x[n]r^{-n}$ is absolutely summable, i.e., $\sum_{k=-\infty}^{+\infty} |x[k]| r^{-k} < \infty$.

Relationship between Z transform and Fourier transform

Note that the DTFT is a special case of the z-transform:

$$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}} = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}.$$

The DTFT is simply $X(z)$ evaluated on the unit circle in the z-plane.



Example of z-transform

Consider the signal $x[n] = a^n u[n]$. Then,

$$X(z) = \sum_{n=0}^{+\infty} a^n z^{-n} = \sum_{n=0}^{+\infty} (az^{-1})^n$$

We need to specify the region of convergence (ROC) where the above sum is finite.

In this case, ROC is the range of z for which $|az^{-1}| < 1$, or equivalently $|z| > |a|$. Then

$$X(z) = \sum_{n=0}^{+\infty} (az^{-1})^n = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}, \quad |z| > |a|.$$

The z-transform of unit step signal

The unit step signal $x[n] = u[n]$ has the *z*-transform

$$X(z) = \frac{1}{1 - z^{-1}} = \frac{z}{z - 1}, \quad |z| > 1 .$$

Example

Determine the Z transform of the signal

$$x[n] = \left(\frac{1}{3}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[-n-1]$$

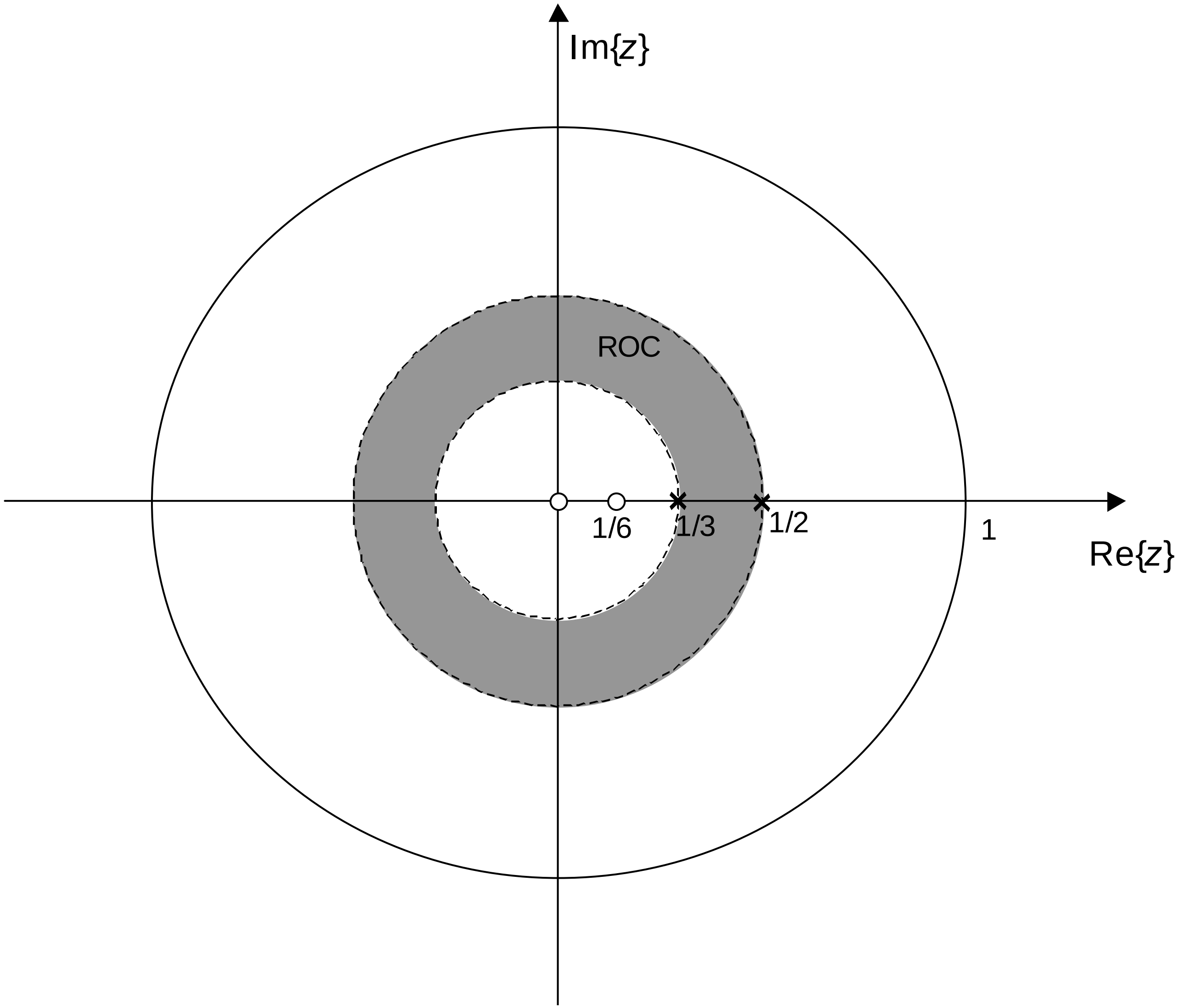
Solution:

$$X(z) = \sum_{n=-\infty}^{+\infty} \left[\left(\frac{1}{3}\right)^n u[n] + 2\left(\frac{1}{2}\right)^n u[-n-1]\right] z^{-n}$$

$$= \sum_{n=0}^{+\infty} \left(\frac{1}{3}\right)^n z^{-n} + 2\sum_{n=-\infty}^{-1} \left(\frac{1}{2}\right)^n z^{-n} = \underbrace{\frac{1}{1-\frac{1}{3}z^{-1}}}_{|z|>\frac{1}{3}} + \underbrace{\frac{4z}{1-2z}}_{|z|<\frac{1}{2}}$$

$$= \frac{2z(z-\frac{1}{6})}{(z-\frac{1}{3})(1-2z)}, \quad \boxed{\frac{1}{3} < |z| < \frac{1}{2}}$$

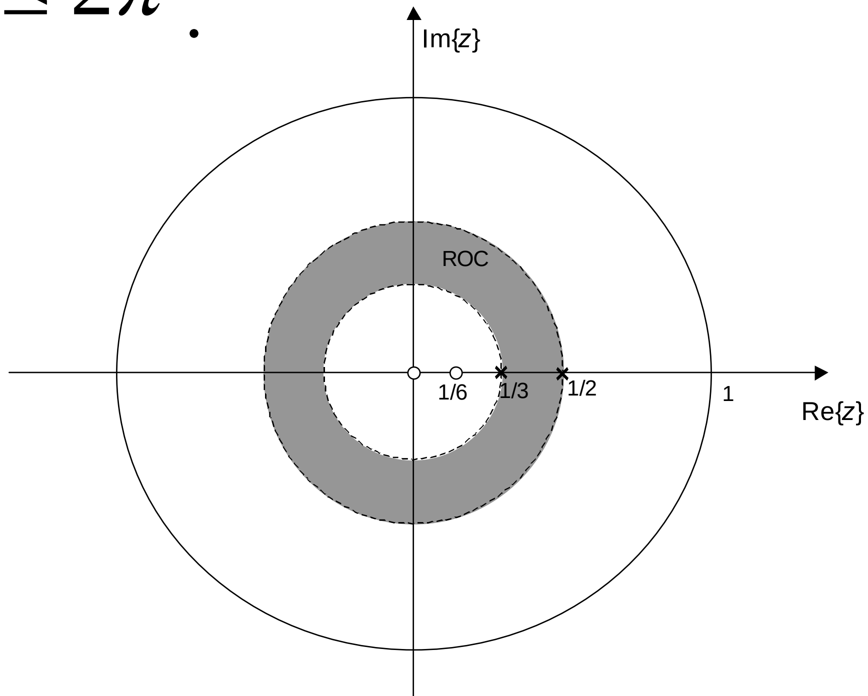
The ROC of $X(z)$ can be displayed on a pole-zero plot as follows:



Properties of ROC

Property 1: *The ROC of x(z) consists of a ring in the z-plane centered around the origin.*

Convergence is dependent only on r, not on ω. Hence, if $X(z)$ exists at the point $z_0 = r_0 e^{j\omega_0}$, then it also converges on the circle $z = r_0 e^{j\omega}, 0 \le \omega \le 2\pi$.



Property 2: *The ROC of* $X(z)$ *does not contain any poles.*

This one is obvious.

Property 3: *If* $x[n]$ *is of finite duration, then the ROC is the entire z-plane, except possibly* $z = 0$ *and/or* $z = \infty$ *.*

In this case, the finite sum of the *z*-transform converges for (almost) all *z*. Two exceptions are $z = 0$ and $z = \infty$ in

$$X(z) = \sum_{n=-N_1}^{N_2} x[n] z^{-n}$$

Property 4

If x[n] is right-sided, and if the circle $|z| = r_0$ *is in the ROC, then all finite values of z for which* $|z| > r_0$ *will also be in the ROC.*

This is because if the signal $x[n]r_0^{-n}$ is absolutely summable, then, for $r_1 > r_0$, we have $|x[n]|r_1^{-n} < |x[n]|r_0^{-n}$ for $n \geq 0$, and $\sum_{n=-N_1}^{-1} |x[n]|r_1^{-n} < \infty$ in case the right-sided signal begins at negative time $-N_1$.

Properties of the Two-Sided *z*-Transform

We use the notation $x[n] \overset{z}{\leftrightarrow} X(z)$ to represent a *z*-transform pair.

Linearity

The operation of calculating the *z*-transform of a signal is linear.

For $x[n] \overset{z}{\leftrightarrow} X(z),\ ROC = R_x$, $y[n] \overset{z}{\leftrightarrow} Y(z),\ ROC = R_y$, let $z[n] = Ax[n] + By[n]$, then

$$z[n] \overset{z}{\leftrightarrow} AX(z) + BY(z),\ \ ROC \supset R_x \cap R_y .$$

Time Shifting

Time shifting leads to a multiplication by a complex exponential.

$$x[n-n_0] \overset{z}{\leftrightarrow} z^{-n_0} X(z),$$

$$ROC = R_x,$$

.

except possible addition/deletion of 0 or ∞

Example:

$$2^n u[n] \leftrightarrow \frac{1}{1-2z^{-1}}, \{z \in Complex, |z| > 2\}$$

$$2^{n+2} u[n+2] \leftrightarrow \frac{z^2}{1-2z^{-1}}, \{z \in Complex, |z| > 2\} not \{\infty\}$$

Scaling in the z-Domain

$$z_0^{\ n} x[n] \overset{z}{\longleftrightarrow} X\left(\frac{z}{z_0}\right), \quad ROC = |z_0| R_x ,$$

where the ROC is the scaled version of R_x .

if $X(z)$ has a pole or zero at $z = a$, then $X(z/z_0)$ has a pole or zero at $z = z_0 a$.

Recall the frequency shifting property of Fourier transform (Lecture 28).

Time Reversal

$$x[-n] \overset{z}{\leftrightarrow} X(z^{-1}), \quad ROC = 1/R_x.$$

That is, if $z \in R_x$, then $\frac{1}{z} \in ROC$.

Time Expansion (upsampling)

The upsampled signal

$$x_{(m)}[n] = \begin{cases} x[n/m], & n/m = \text{integer} \\ 0, & otherwise \end{cases}$$

has a *z*-transform given by:

$$x_{(m)}[n] \overset{z}{\leftrightarrow} X(z^m), \;\; ROC = R_x^{1/m}.$$

Differentiation in the z-Domain

Differentiation of the *z*-transform with respect to z yields

$$nx[n] \overset{z}{\longleftrightarrow} -z\frac{dX(z)}{dz}, \quad ROC = R_x \, .$$

Example:

$$u[n] \overset{z}{\longleftrightarrow} \frac{z}{z-1}, \quad |z| > 1$$

$$nu[n] \overset{z}{\longleftrightarrow} -z\left[\frac{(z-1)-z}{(z-1)^2}\right] = \frac{z}{(z-1)^2}, \quad |z| > 1$$

Convolution of two signals

The convolution of $x[n]$ and $y[n]$ has a resulting z-transform given by

$$x[n] * y[n] = \sum_{m=-\infty}^{\infty} x[m] y[n-m] \overset{z}{\leftrightarrow} X(z)Y(z), \quad ROC \supset R_x \cap R_y \; .$$

Remark:

The ROC can be larger than $R_x \cap R_y$ if pole-zero cancellations occur when forming the product $X(z)Y(z)$.

First Difference

The first difference of a signal has the following *z*-transform:

$$x[n]-x[n-1] \stackrel{z}{\longleftrightarrow} (1-z^{-1})X(z), \quad ROC = R_x \,,$$

with the possible deletion of $z=0$ from the ROC, and/or addition of $z=1$.

Running Sum (accumulation)

The running sum of a signal is the inverse of the first difference.

$$\sum_{m=-\infty}^{n} x[m] \overset{z}{\leftrightarrow} \frac{1}{(1-z^{-1})} X(z), \quad ROC \supset R_x \cap \{z \in \mathbb{C} : |z| > 1\}$$

Conjugation

$$x^*[n] \overset{z}{\leftrightarrow} X^*(z^*), \quad ROC = R_x$$

Remark:

For $x[n]$ real, we have: $X(z) = X^*(z^*)$. Thus if $X(z)$ has a pole (or zero) at $z = a$, then it must also have a pole (or zero) at $z = a^*$.

That is, all complex poles and zeros come in conjugate pairs in the *z*-transform of a real signal.

Initial-Value Theorem

If $x[n]$ is a causal signal, i.e., $x[n] = 0,\ n < 0$, we have

$$x[0] = \lim_{z \to \infty} X(z) .$$

This property follows from the power series representation of $X(z)$:

$$\lim_{z \to \infty} X(z) = \lim_{z \to \infty} [x[0] + x[1]z^{-1} + x[2]z^{-2} + \ldots] = x[0]$$

$$x[0] = \lim_{z \to \infty} X(z) .$$

Consequence:

With $X(z)$ expressed as a ratio of polynomials, the order of its numerator cannot be greater than the order of its denominator (for $x[n]$ causal with $x[0]$ finite.)

Final-Value Theorem

If $x[n]$ is a causal signal, we have

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(1 - z^{-1})X(z)$$

This formula gives us the residue at the pole $z = 1$
(which corresponds to DC).
If this residue is nonzero, then $X(z)$ has a nonzero final value.