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Lecture 29

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FT of DT periodic signals

- Exponential DT periodic signals
- General DT periodic signals
- DT Impulse train

FT of DT exponential periodic signals

DT periodic signals are not absolutely summable. But their Fourier transforms exist and can be defined using impulses in the frequency domain.

Consider a DT periodic complex exponential signal:

$$x[n] = e^{j\omega_0 n}$$

Its Fourier transform is

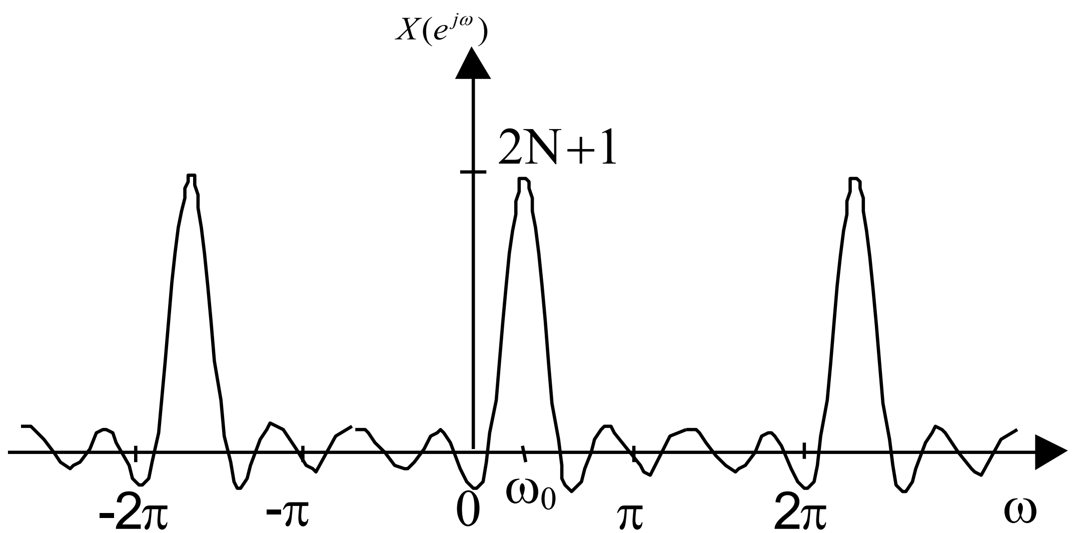
$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} e^{j\omega_0 n} e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} e^{-j(\omega-\omega_0)n} .$$

This last sum does not converge to a regular function. But, this sum is an impulse train as shown below.

First, consider the finite sum of exponential series

$$X_N(e^{j\omega}) = \sum_{n=-N}^{+N} e^{-j(\omega-\omega_0)n} = e^{j(\omega-\omega_0)N} \sum_{m=0}^{2N} e^{-j(\omega-\omega_0)m}$$

$$= \frac{\sin[(N+1/2)(\omega-\omega_0)]}{\sin[(\omega-\omega_0)/2]}$$

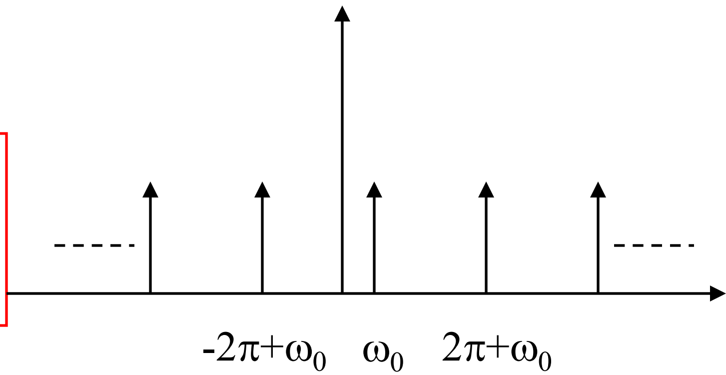


Second, take the limit of the finite sum as $N \rightarrow \infty$,

$$\lim_{N\rightarrow\infty} X_N(e^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l).$$

Thus,

$$e^{j\omega_0 n} \overset{FT}{\leftrightarrow} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l)$$



Varify the inverse FT of the above $X(e^{j\omega})$ is $e^{j\omega_0 n}$:

$$x[n] = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega-\omega_0-2\pi l)e^{j\omega n}d\omega = e^{j\omega_0 n}$$

Note that there is only one impulse per interval of width 2π.

FT of general DT periodic signal

Consider a periodic signal $x[n]$ with period N and with the Fourier series representation

$$x[n] = \sum_{k=<N>} a_k e^{jk\frac{2\pi}{N}n}$$

The above signal is a linear combination of $e^{jk\omega_0 n}$.

Its Fourier transform can be derived from the linear combination of the FT of $e^{jk\omega_0 n}$:

$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - \frac{2\pi k}{N})$$

Note: a_k is periodic of N.

The FT of a DT periodic signal is derived from the DTFT formula and the DTFT of $e^{j\omega_0 n}$:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n} e^{-j\omega n}$$

$$= \sum_{k=\langle N \rangle} a_k \boxed{\sum_{n=-\infty}^{\infty} e^{-j(\omega - k\frac{2\pi}{N})n}}$$

The FT of $e^{jk\omega_0 n}$

$$= \sum_{k=\langle N \rangle} 2\pi a_k \sum_{l=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{N} - 2\pi l)$$

$$= \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\frac{2\pi}{N})$$

Thus, we can write the DTFT of a periodic signal directly from the knowledge of its Fourier series coefficients

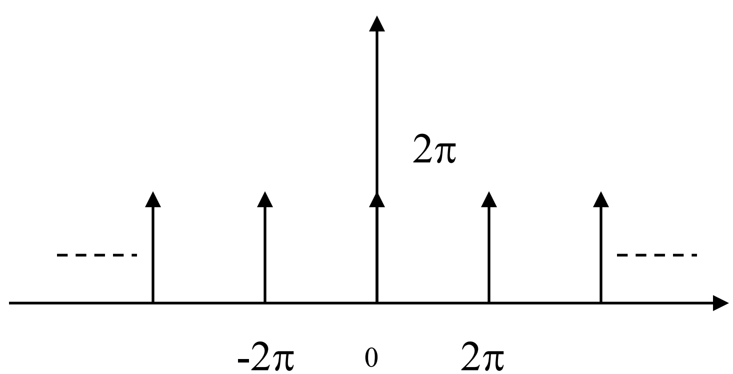
$$X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\frac{2\pi}{N})$$

FT of a constant DT signal

Consider a constant signal $x[n]=1$. It can be written as $x[n]=e^{j0n}$,

Its DTFT in the interval $[-\pi,\pi]$ is an impulse located at $\omega=0$.

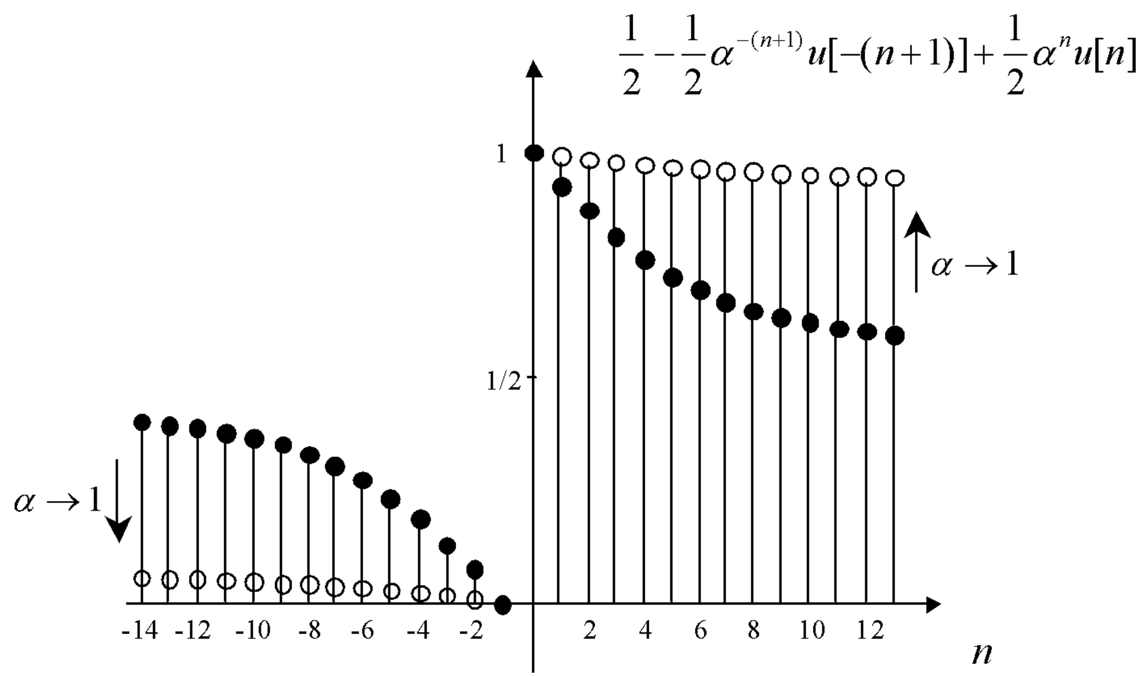
$$x[n]=1 \overset{\mathcal{F}}{\leftrightarrow} \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega-2\pi l)$$



Fourier transform of the unit step signal

$u[n]$ can be written as the limit of two exponential signals pieced together at the origin plus the constant $1/2$:

$$u[n] = \frac{1}{2} + \lim_{\varepsilon \to 1} \left(-\frac{1}{2}\alpha^{-(n+1)} u[-(n+1)] + \frac{1}{2}\alpha^{n} u[n] \right)$$



DTFT of $\frac{1}{2}\alpha^n u[n]$ is $\frac{1}{2}\frac{1}{1-\alpha e^{-j\omega}}$,

and by the time reversal and time shifting properties, we have $-\frac{1}{2}\alpha^{-(n+1)}u[-(n+1)] \overset{\mathcal{F}}{\leftrightarrow} -\frac{1}{2}\frac{e^{j\omega}}{1-\alpha e^{j\omega}}$.

Then,

$$u[n] \overset{FT}{\leftrightarrow} \sum_{l=-\infty}^{\infty}\pi\delta(\omega-2\pi l) + \lim_{\alpha\to 1}\left(-\frac{1}{2}\frac{e^{j\omega}}{1-\alpha e^{j\omega}} + \frac{1}{2}\frac{1}{1-\alpha e^{-j\omega}}\right)$$

$$= \sum_{l=-\infty}^{\infty}\pi\delta(\omega-2\pi l) + \frac{1}{1-e^{-j\omega}}$$

Example

Let us find the DTFT of $x[n] = 1 + \sin(\frac{2\pi}{3}n) - \cos(\frac{2\pi}{7}n)$.

Periodic?

The sine term repeats every 3 time steps, whereas the cosine term repeats every 7 time steps. Thus, the signal is periodic of fundamental period $N = 21$.

Write the signal in terms of harmonics of the fundamental component :

$$x[n] = 1 + \frac{1}{2j}\left(e^{j\frac{2\pi 7}{21}n} - e^{-j\frac{2\pi 7}{21}n}\right) - \frac{1}{2}\left(e^{j\frac{2\cdot 3\pi}{21}n} + e^{-j\frac{2\cdot 3\pi}{21}n}\right).$$

The nonzero DTFS coefficients of the signal are

$$a_0 = 1, a_{-7} = j\frac{1}{2}, a_7 = -j\frac{1}{2}, a_{-3} = a_3 = \frac{1}{2}.$$

Hence, the FT of the signal is

$$X(e^{j\omega}) = \sum_{k=0}^{20} 2\pi a_k \sum_{l=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{21} - 2\pi l)$$

$$= 1 + \pi \sum_{l=-\infty}^{\infty} \delta(\omega + \frac{6\pi}{21} - 2\pi l) + \pi \sum_{l=-\infty}^{\infty} \delta(\omega - \frac{6\pi}{21} - 2\pi l)$$

$$+ j\pi \sum_{l=-\infty}^{\infty} \delta(\omega + \frac{14\pi}{21} - 2\pi l) - j\pi \sum_{l=-\infty}^{\infty} \delta(\omega - \frac{14\pi}{21} - 2\pi l)$$

$$= 1 + \sum_{l=-\infty}^{\infty} \left[\pi\delta(\omega + \frac{6\pi}{21} - 2\pi l) + \pi\delta(\omega - \frac{6\pi}{21} - 2\pi l) + j\pi\delta(\omega + \frac{14\pi}{21} - 2\pi l) - j\pi\delta(\omega - \frac{14\pi}{21} - 2\pi l) \right]$$

Example: FT of DT impulse train

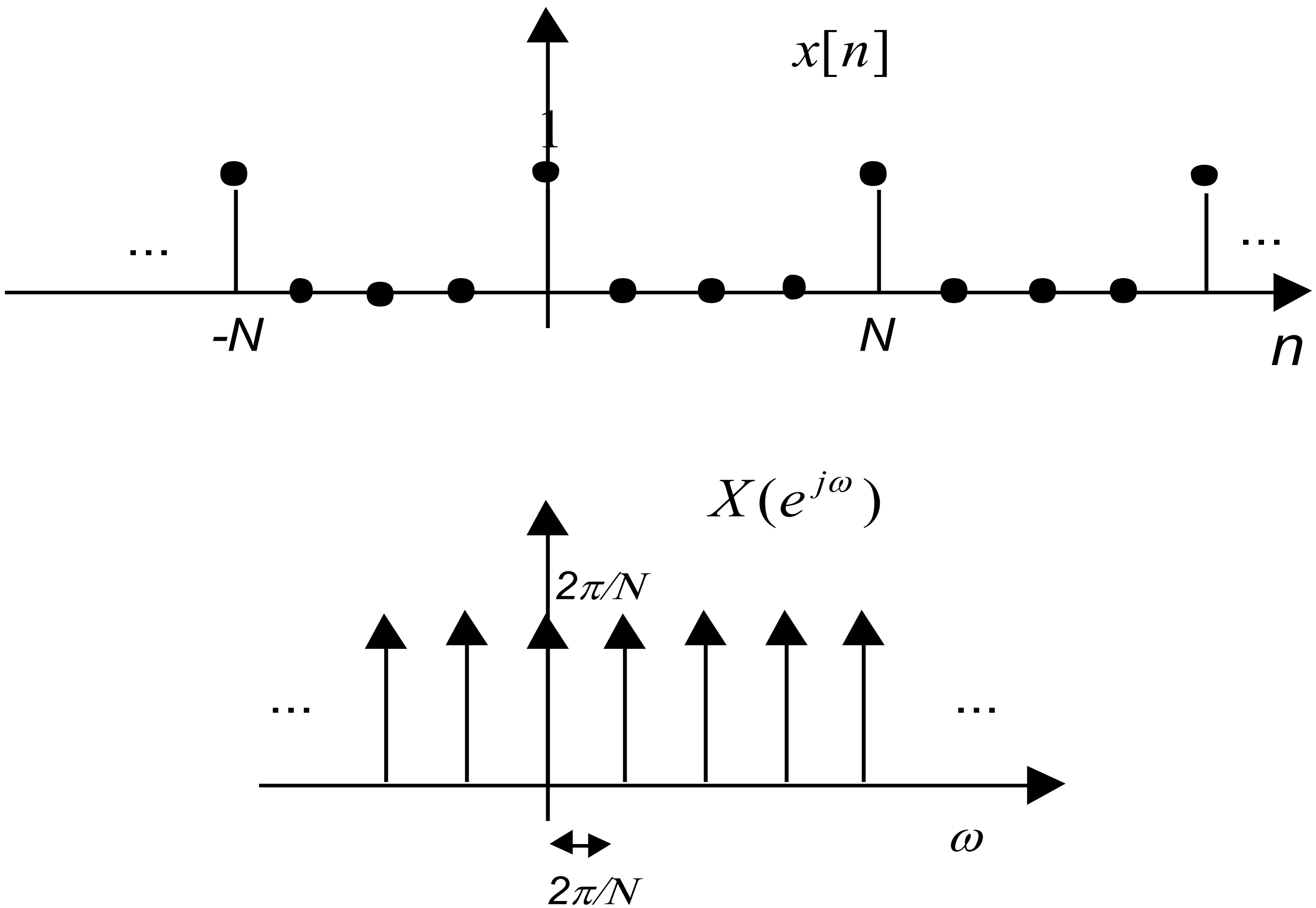
Find the DTFT of $x[n] = \sum_{k=-\infty}^{\infty} \delta[n - kN]$. This signal is periodic of period N. Its DTFS coefficients are given by:

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=0}^{N-1} \delta[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}$$

and hence, the DTFT of the impulse train is simply:

$$X(e^{j\omega}) = \sum_{k=0}^{N-1} \frac{2\pi}{N} \sum_{l=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{N} - 2\pi l)$$

$$= \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{N})$$



$x[n]$
1
...
...
-N
N
n
$X(e^{j\omega})$
$2\pi/N$
...
...
ω
$2\pi/N$