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Lecture 28

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Properties of DT Fourier Transform

Linearity

The operation of calculating the DTFT of a signal is linear:

If $x[n] \overset{\mathcal{F}}{\leftrightarrow} X(e^{j\omega})$, $y[n] \overset{\mathcal{F}}{\leftrightarrow} X(e^{j\omega})$,

and if $z[n] = Ax[n] + By[n]$,

then $z[n] \overset{\mathcal{F}}{\leftrightarrow} AX(e^{j\omega}) + BY(e^{j\omega})$.

Time shifting and frequency shifting

Time Shifting

Time shifting leads to a multiplication by a complex exponential.

$$x[n-n_0] \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega}) \,.$$

Remark: Only the phase of the DTFT is changed.

Frequency Shifting

Frequency shifting leads to a multiplication of $x[n]$ by a complex exponential.

$$e^{j\omega_0 n} x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}) \,.$$

Time reversal

Time reversal corresponds to the frequency reversal of the DTFT:

$$x[-n] \overset{\mathcal{F}}{\leftrightarrow} X(e^{-j\omega}) .$$

Proof: $$\sum_{n=-\infty}^{+\infty} x[-n] e^{-j\omega n} = \sum_{m=-\infty}^{\infty} x[m] e^{j\omega m} = X(e^{-j\omega}) .$$

Note:

- For $x[n]$ even, $X(e^{j\omega})$ is also even,
  for $x[n]$ odd, $X(e^{j\omega})$ is also odd

Time scaling

Upsampling (time expansion)

The signal $$x_{(m)}[n] := \begin{cases} x[n/m], & n = 0, m, 2m, 3m, \ldots.. \\ 0, & otherwise \end{cases}$$

is an *upsampled* version of the original signal $x[n]$. The *upsampling operation* inserts *m-1* zeros between consecutive samples of the original signal. Spectrum is compressed around DC:

$$x_{(m)}[n] \overset{\mathcal{F}}{\leftrightarrow} X(e^{jm\omega}).$$

Down Sampling

Downsampling (decimation)

The signal $x[mn]$ is called a *decimated* or *downsampled* version of $x[n]$, that is, only every m^{th} sample of $x[n]$ is retained.

Since aliasing may occur, we will postpone this analysis

Differentiation in frequency

Differentiation in Frequency

Differentiation of the DTFT with respect to frequency yields

$$nx[n] \stackrel{\mathcal{F}}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$$

Convolution in time domain

Convolution of Two Signals

For $x[n] \overset{\mathcal{F}}{\leftrightarrow} X(e^{j\omega})$, $y[n] \overset{\mathcal{F}}{\leftrightarrow} Y(e^{j\omega})$, we have

$$\sum_{m=-\infty}^{\infty} x[m]y[n-m] \overset{\mathcal{F}}{\leftrightarrow} X(e^{j\omega})Y(e^{j\omega}) \ ,$$

Proof: (under the appropriate assumption of convergence to interchange the order of summations)

$$\sum_{n=-\infty}^{+\infty}\sum_{m=-\infty}^{\infty} x[m]y[n-m]e^{-j\omega n} = \sum_{m=-\infty}^{+\infty} x[m]\sum_{n=-\infty}^{\infty} y[n-m]e^{-j\omega n}$$

$$= \sum_{m=-\infty}^{+\infty} x[m]\sum_{p=-\infty}^{\infty} y[p]e^{-j\omega(p+m)}$$

$$= \sum_{m=-\infty}^{+\infty} x[m]e^{-j\omega m}\sum_{p=-\infty}^{\infty} y[p]e^{-j\omega p}$$

$$= X(e^{j\omega})Y(e^{j\omega})$$

Remarks

- The basic use of this property is to compute the output signal of a system for a particular input signal, given its impulse response or DTFT.

- The convolution property is also useful in DT filter design and feedback control system design.

Calculating DT convolution using FT

Example:

Given a system with $h[n] = \alpha^n u[n],\ |\alpha| < 1$, and an input $x[n] = \beta^n u[n],\ |\beta| < 1$, determine the output signal.

The DTFT of the output is given by

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}}\frac{1}{1-\beta e^{-j\omega}}$$

We perform a partial fraction expansion of $Y(e^{j\omega})$ to be able to use the table of DTFT pairs to obtain $y[n]$. Let $z = e^{j\omega}$ for convenience.

$$\frac{1}{(1-\alpha z^{-1})(1-\beta z^{-1})} = \frac{A}{1-\alpha z^{-1}} + \frac{B}{1-\beta z^{-1}}$$

$$\left.\frac{1}{(1-\beta z^{-1})}\right|_{z=\alpha} = A + \left.\frac{B(1-\alpha z^{-1})}{1-\beta z^{-1}}\right|_{z=\alpha} \Rightarrow \quad A = \frac{\alpha}{(\alpha-\beta)}, \quad \alpha \neq \beta$$

$$\left.\frac{1}{(1-\alpha z^{-1})}\right|_{z=\beta} = B + \left.\frac{B(1-\beta z^{-1})}{1-\alpha z^{-1}}\right|_{z=\beta} \Rightarrow \quad B = \frac{\beta}{(\beta-\alpha)}, \quad \alpha \neq \beta$$

For $\alpha \neq \beta$, we use the table to get

$$y[n] = \frac{\alpha}{\alpha - \beta}\alpha^n u[n] + \frac{\beta}{\beta - \alpha}\beta^n u[n], \ \ \alpha \neq \beta$$

For the case $\alpha = \beta$, we have

$$Y(e^{j\omega}) = \frac{1}{(1-\alpha e^{j\omega})^2} = \frac{j}{\alpha}\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{j\omega}}\right)$$

The derivative times $\frac{j}{\alpha}$ yields $w[n] = n\alpha^{n-1}u[n]$,

and the multiplication by $e^{j\omega}$ is a unit time advance, so finally

$$y[n] = (n+1)\alpha^{n}u[n+1] = (n+1)\alpha^{n}u[n].$$

Multiplication of Two Signals

With the two signals as defined above:

$$x[n]y[n] \leftrightarrow \frac{1}{2\pi}\int_{2\pi} Y(e^{j\theta})X(e^{j(\omega-\theta)})d\theta$$

Remarks

- Note that the resulting DTFT is a periodic convolution of the two DTFTs.
- This property is used in discrete-time modulation and sampling.

Proof:

$$\sum_{n=-\infty}^{\infty} x[n]y[n]e^{-j\omega n} = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty} x[n]\{\int_{2\pi} Y(e^{j\theta})e^{j\theta n}d\theta\}e^{-j\omega n}$$

$$= \frac{1}{2\pi}\int_{2\pi} Y(e^{j\theta})\{\sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega-\theta)n}\}d\theta$$

$$= \frac{1}{2\pi}\int_{2\pi} Y(e^{j\theta})X(e^{j(\omega-\theta)})d\theta$$

First difference and running sum

First Difference

The first difference of a signal has the following spectrum:

$$x[n]-x[n-1] \overset{\mathcal{F}}{\leftrightarrow} (1-e^{-j\omega})X(e^{j\omega})$$

Running Sum (accumulation)

The running sum of a signal is the inverse of the first difference.

$$\sum_{m=-\infty}^{n} x[m] \overset{\mathcal{F}}{\leftrightarrow} \frac{1}{(1-e^{-j\omega})} X(e^{j\omega})$$

Conjugation and Conjugate Symmetry

Taking the conjugate of a signal has the effect of conjugation and frequency reversal of the DTFT.

$$x^*[n] \overset{F}{\leftrightarrow} X^*(e^{-j\omega})$$

Real and even $x[n]$

For $x[n]$ real, the DTFT is *conjugate symmetric:*

$$X(e^{j\omega}) = X^*(e^{-j\omega}).$$

This implies

$$|X(e^{j\omega})| = |X(e^{-j\omega})|,$$

$$\angle X(e^{-j\omega}) = -\angle X(e^{j\omega}),$$

$$X(1) = real,$$

$$\mathrm{Re}\{X(e^{-j\omega})\} = \mathrm{Re}\{X(e^{j\omega})\},$$

$$\mathrm{Im}\{X(e^{-j\omega})\} = -\mathrm{Im}\{X(e^{j\omega})\}$$

For x[n] real and even, the DTFT is also real and even

$$X(e^{j\omega}) = X(e^{-j\omega}) = real$$

Real-odd and even-odd x[n]

For $x[n]$ real and odd, the DTFT is purely imaginary and odd

$$X(e^{j\omega}) = -X(e^{-j\omega}) = imaginary$$

For even-odd decomposition of the signal

$$x[n] = x_e[n] + x_o[n],$$

$$x_e[n] \overset{\mathcal{F}}{\leftrightarrow} \mathrm{Re}\{X(e^{j\omega})\}, \quad x_o[n] \overset{\mathcal{F}}{\leftrightarrow} j\,\mathrm{Im}\{X(e^{j\omega})\}$$

Parseval's Relation

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 \, d\omega$$

the energy of the signal = the energy in its spectrum.

The squared magnitude of the DTFT $\left|X(e^{j\omega})\right|^2$ is referred to as the *energy-density spectrum* of the signal $x[n]$.

Proof by yourself.