ECSE 306 - Fall 2008

Fundamentals of Signals and Systems

McGill University
Department of Electrical and Computer Engineering

Lecture 27

November 10, 2008

Hui Qun Deng, PhD

Properties of Fourier Series of DT signals

1. Time scaling (down-sampling and up-sampling)
2. Periodic convolution
3. Multiplication
4. Difference
5. Running sum
6. Conjugation and symmetry
7. Parseval' relation for DT FS

Fourier Transform of DT signals

Time Scaling $x[mn]$, m>1 (decimation or down-sampling)

Signal $x[mn]$ is called a *decimated* or *downsampled* version of $x[n]$, i.e., only every m^{th} sample of $x[n]$ is retained.

$x[mn]$ is periodic if $\exists M$ such that

$$x[mn] = x[m(n+M)] = x[mn + mM]$$

The above condition holds if $\exists$p, M $\in$Z such that

$$mM = pN$$

Letting $m=p<N$, $M=N$, then the signal $x[mn]$ is also periodic of period N.

Time Scaling $x[n/m]$, m>1 (up-sampling)

The signal $x_{\uparrow m}[n] := \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$

is sometimes called an *upsampled* version of the periodic signal $x[n]$.

The upsampling operation inserts *m-1* zeros between consecutive samples of the original signal.

The up-sampled signal has a fundamental period *mN*, because

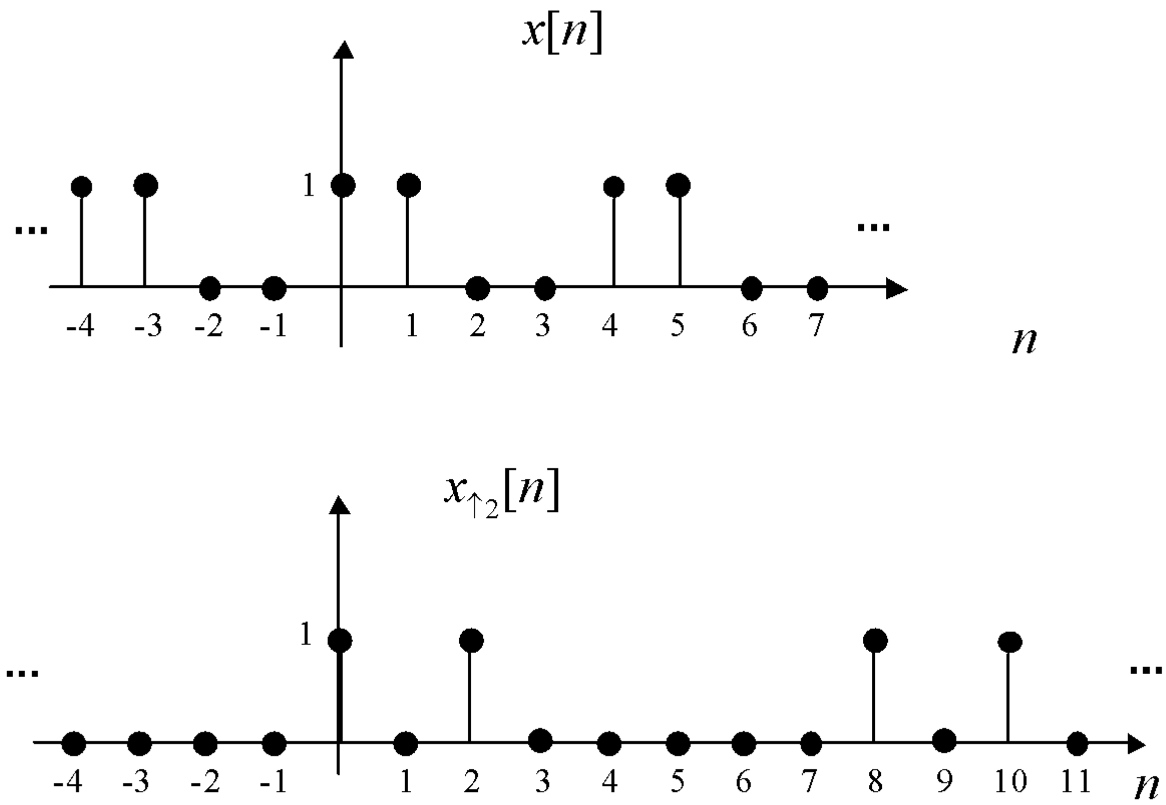
$$x_{\uparrow m}[n+mN] := \begin{cases} x[(n+mN)/m] = x[n/m+N] = x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$$

$$= x_{\uparrow m}[n]$$

Thus, the up-sampled signal has a fundamental frequency $\dfrac{2\pi}{mN}$.

Example

Example (upsampling)



FS of up-sampled DT signals

The Fourier series coefficients of the upsampled signal $x_{\uparrow m}[n]$ are given by:

$$x_{\uparrow m}[n] \overset{\text{FS}}{\longleftrightarrow} \frac{1}{m} a_k \,,$$

where $\{\frac{1}{m} a_k\}$ is viewed as a periodic sequence of period mN .

Proof:

$$b_k = \frac{1}{mN} \sum_{n=\langle mN \rangle} x_{\uparrow m}[n] e^{-jk\frac{2\pi}{mN}n} = \frac{1}{mN} \sum_{n=0,m,..,m(N-1)} x[n/m] e^{-jk\frac{2\pi}{mN}n}$$

$$= \frac{1}{mN} \sum_{p=\langle N \rangle} x[p] e^{-jk\frac{2\pi}{N}p} = \frac{1}{m} a_k$$

Periodic Convolution of Two Signals

Suppose that $x[n]$ and $y[n]$ are both periodic with period N.

For $x[n] \overset{\mathcal{FS}}{\leftrightarrow} a_k$, $y[n] \overset{\mathcal{FS}}{\leftrightarrow} b_k$, we have

$$\sum_{m=<N>} x[m]y[n-m] \overset{FS}{\leftrightarrow} Na_k b_k \,.$$

Remarks

- The periodic convolution is itself periodic of period N (Show it as an exercise)

- Periodic convolution is useful in periodic signal filtering. The DTFS coefficients of the input signal are a_k's. b_k's are designed to attenuate or amplify certain frequencies. The resulting DT output signal is given by the periodic convolution above.

Multiplication of two periodic signals

With the two periodic signals as defined above, we have:

$$x[n]y[n] \overset{\mathcal{FS}}{\longleftrightarrow} \sum_{l=\langle N \rangle} a_l b_{k-l} \ ,$$

i.e., multiplication in the time domain corresponds to a periodic convolution of the spectral (FS) coefficient sequences. (Recall: the spectral (FS) coefficient sequence is also periodic.)

This property is used in the discrete-time modulations of a periodic signal.

First Difference

The first difference of a periodic signal is often used as an approximation to the continuous-time derivative.

It has the following spectral (FS) coefficients:

$$x[n] - x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} (1 - e^{-jk\frac{2\pi}{N}})a_k$$

Running Sum

The running sum of a signal is the inverse of the first difference.

Note: the running sum of a periodic signal is periodic only if $a_0 = 0$, i.e., only if the DC component of the signal is 0.

$$\sum_{m=-\infty}^{n} x[m] \overset{\mathcal{FS}}{\longleftrightarrow} \frac{1}{(1-e^{-jk\frac{2\pi}{N}})} a_k$$

Conjugation and Conjugate Symmetry

Taking the conjugate of a periodic signal has the effect of conjugation and frequency reversal on the spectral coefficients.

$$x^*[n] \overset{FS}{\leftrightarrow} a^*_{-k}$$

Consequently:

- For $x[n]$ real, the sequence of coefficients is *conjugate symmetric* ($a_{-k} = a_k^*$). This implies

$$\left|a_{-k}\right| = \left|a_k\right|, \ \angle(a_{-k}) = -\angle(a_k), \ a_0 \in \mathbb{R}, \ \mathrm{Re}\{a_{-k}\} = \mathrm{Re}\{a_k\},$$
$$\mathrm{Im}\{a_{-k}\} = -\mathrm{Im}\{a_k\}.$$

- For $x[n]$ real and even, the sequence of coefficients is also real and even ($a_k = a_{-k} \in \text{Real}$)
- For $x[n]$ real and odd, the sequence of coefficients is purely imaginary and odd ($a_{-k} = -a_k$ purely imaginary)
- For even-odd decomposition of the signal $x[n] = x_e[n] + x_o[n]$, $x_e[n] \overset{FS}{\leftrightarrow} \text{Re}\{a_k\}$, $x_o[n] \overset{FS}{\leftrightarrow} j\,\text{Im}\{a_k\}$

Parseval's Relation for Discrete Time Fourier Series

$$\frac{1}{N}\sum_{n=<N>}|x[n]|^2 = \sum_{k=<N>}|a_k|^2$$

The above Eq. says:

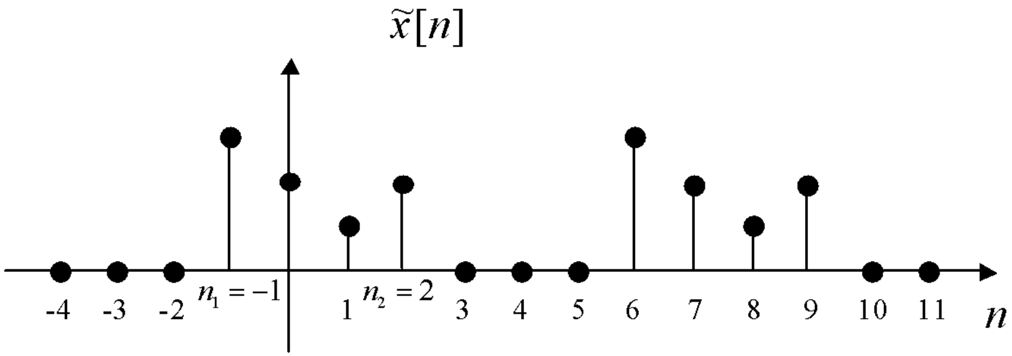
The average power in one period

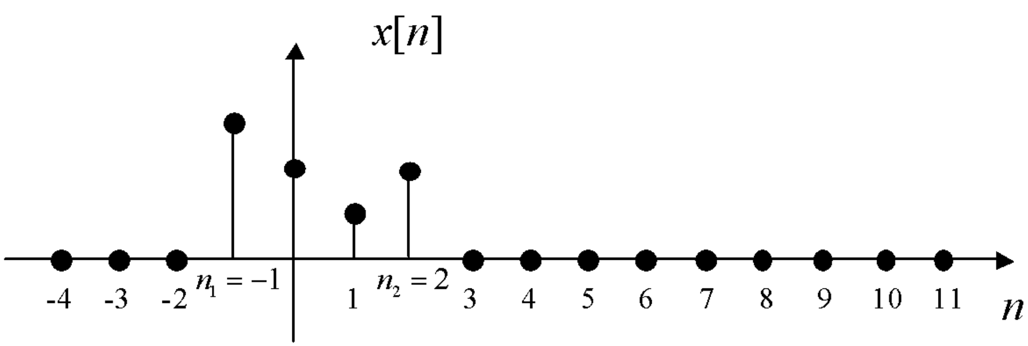
= The sum of the average powers in all (N) harmonic components of x[n].

From periodic signals to aperiodic signals

Consider the following periodic signal $\tilde{x}[n]$, and $x[n]$ equal to $\tilde{x}[n]$ over one period.

$$x[n] = \begin{cases} \tilde{x}[n], & n = 0,1,\ldots N-1 \\ 0, & otherwise \end{cases}$$





Let's examine the DTFS pair of $\tilde{x}[n]$ given by

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n},$$

$$a_k = \frac{1}{N}\sum_{n=<N>} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n} = \frac{1}{N}\sum_{n=n_1}^{n_2} \tilde{x}[n] e^{-jk\frac{2\pi}{N}n}$$

$$= \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n] e^{-jk\frac{2\pi}{N}n}$$

Definition of FT of DT signals

Define the function,

$$X(e^{j\omega}) := \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n},$$

we see that the DTFS coefficients a_k are scaled *samples* of this continuous function of frequency ω. That is,

$$a_k = \frac{1}{N} X(e^{j\frac{2\pi}{N}k}) = \frac{1}{N} X(e^{j\omega_0 k}).$$ Using this expression,

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi} \underbrace{\sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0}_{\text{This is a Riemann Sum.}}.$$

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n} = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0$$

Apply the concept of Riemann Sum to the above.

Taking the limit of the above equation as $N \to +\infty$, we get

- $k\omega_0 \to \omega$
- the summation over $N \to \infty$ intervals of width $\omega_0 = \frac{2\pi}{N} \to d\omega$ tends to an integral over an interval of width 2π
- $\tilde{x}[n] \to x[n]$

Thus,
$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Fourier transform pair of discrete-time signals

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

FT of DT signals is periodic

Fourier transform of $x[n]$ is periodic of period 2π:

$$X(e^{j(\omega+2\pi)}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j(\omega+2\pi)n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} e^{-j2\pi n}$$

$$= \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n} = X(e^{j\omega})$$

Convergence of the DT Fourier Transform

Sufficient conditions for convergence of the infinite summation of the DTFT.

The DTFT $\sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ will converge either if:

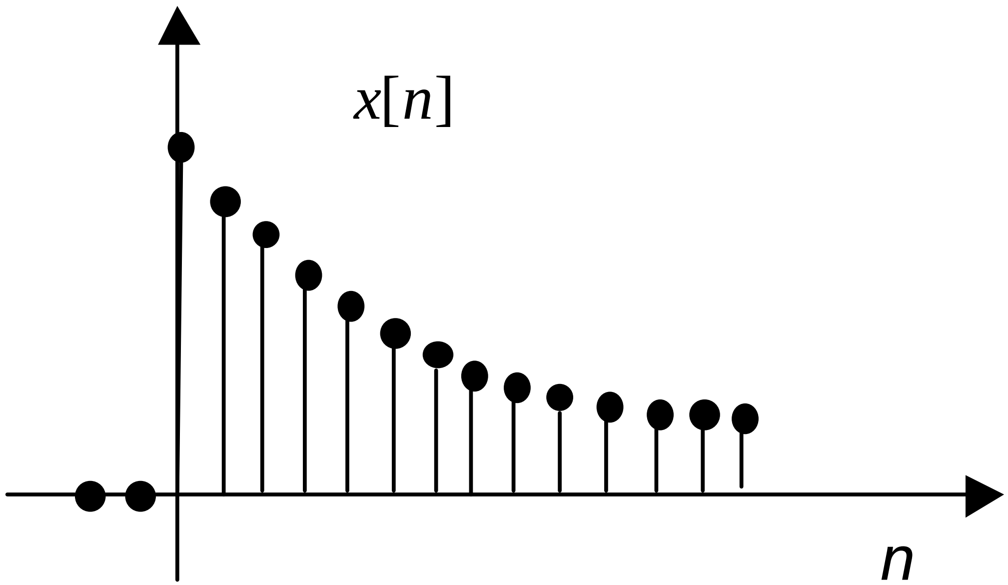
the signal is *absolutely summable*, i.e., $\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$,

or if the sequence has *finite energy*, i.e., $\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$.

In contrast, the finite integral in the synthesis equation always converges.

Example 1: The FT of DT exponentials

Consider the exponential signal $x[n] = a^n u[n],\ a \in \mathbb{R},\ 0 < a < 1$.

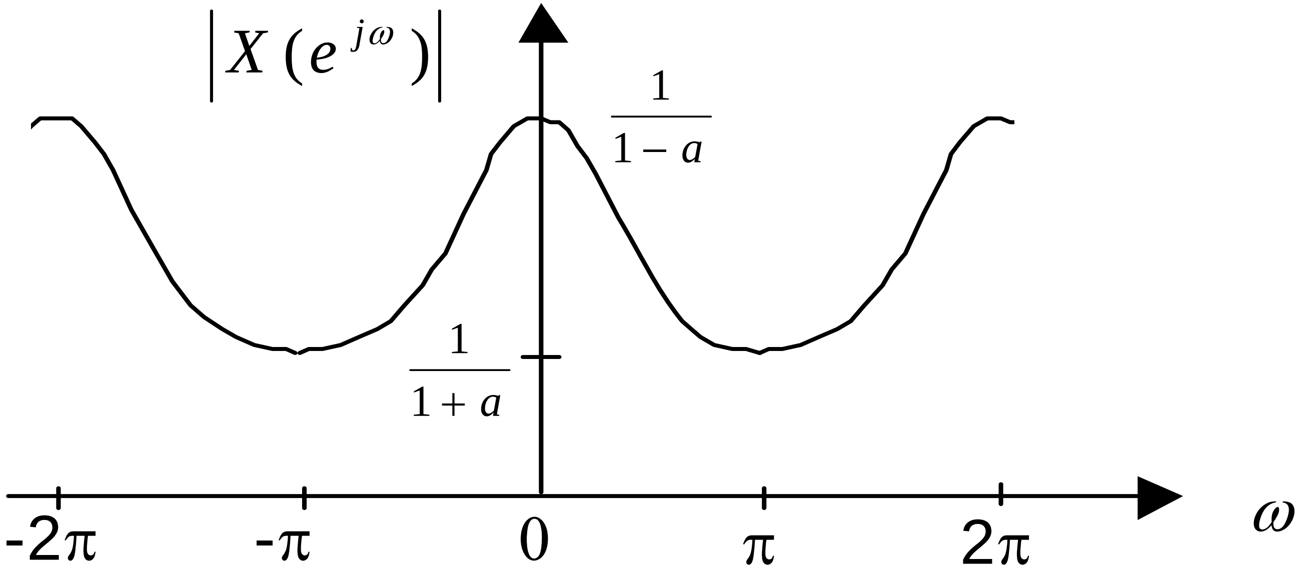


Its FT is:

$$X(e^{j\omega}) = \sum_{n=0}^{+\infty} a^n e^{-j\omega n} = \sum_{n=0}^{+\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

Note that this infinite sum converges because $\left| ae^{-j\omega} \right| = |a| < 1$.

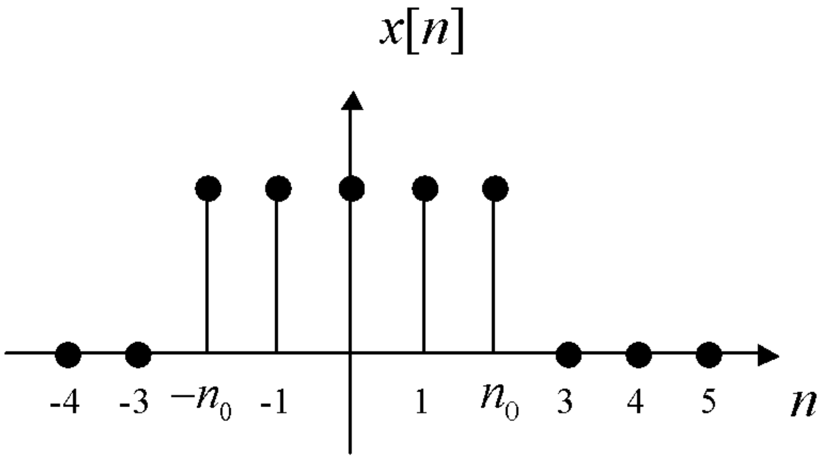
The magnitude of $X(e^{j\omega})$ is plotted below:



Can you see the periodicity of the spectrum?

Example 2: The FT of DT rectangular function

Consider the rectangular pulse $x[n] = \begin{cases} 1, & |n| \le n_0 \\ 0, & |n| > n_0 \end{cases}$.



$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n} = \sum_{n=-n_0}^{n_0} e^{-j\omega n} = e^{j\omega n_0} \sum_{m=0}^{2n_0} e^{-j\omega m}$$

$$= e^{j\omega n_0} \frac{1-e^{-j\omega(2n_0+1)}}{1-e^{-j\omega}}$$

$$= \frac{e^{j\omega n_0} - e^{-j\omega(n_0+1)}}{1-e^{-j\omega}}$$

$$= \frac{e^{-j\omega/2}}{e^{-j\omega/2}} \frac{e^{j\omega(n_0+1/2)} - e^{-j\omega(n_0+1/2)}}{e^{j\omega/2} - e^{-j\omega/2}}$$

$$= \frac{\sin\omega(n_0+1/2)}{\sin(\omega/2)}$$

Example 3

The Fourier transform of a pulse with $n_0 = 2$:

$$X(e^{j\omega}) = \frac{\sin \omega(2 + 1/2)}{\sin(\omega/2)}$$