ECSE 306 - Fall 2008

Fundamentals of Signals and Systems

McGill University
Department of Electrical and Computer Engineering

Lecture 26

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Hui Qun Deng, PhD

Discrete-Time Fourier Series

1. Distinct harmonically-related periodic exponentials
2. DT FS pair
3. Properties of FS

Response of Discrete-Time LTI (DTLTI) Systems to Complex Exponentials

The response of a DTLTI system to a complex exponential input Cz^n, $z \in$ Complex, is the convolution:

$$y[n] = x[n] * h[n] = \sum_{k=\infty}^{\infty} h[k]x[n-k] = \sum_{k=\infty}^{\infty} h[k]z^{n-k}$$

$$= z^n \sum_{k=\infty}^{\infty} h[k]z^{-k} = z^n H(z)$$

H(z) is called the Z transform of the impulse response of the system:

$$H(z) := \sum_{k=\infty}^{\infty} h[k]z^{-k}$$

Thus, the response to an complex exponential is the same complex exponential multiplied by a (complex) amplitude H(z):

$$z^n \rightarrow \boxed{DTLTI} \rightarrow z^n H(z)$$

DT periodic complex exponentials

The frequency of a complex exponential $e^{j\omega_1 n}$ is ω_1.

Its fundamental period is

$$N = \arg\min_n \{\omega_1 n = 2\pi m\}, \quad m = 1,2,\ldots$$

Its fundamental frequency is

$$\omega_0 = \frac{2\pi}{N}.$$

The set of all DT complex exponentials with period N is:

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N}n},$$

$$k = \ldots, -2, -1, 0, 1, 2, \ldots$$

Harmonically-related DT exponentials

Consider the set of all harmonically-related DC exponentials with fundamental period N:

$$\phi_k[n] = e^{jk\omega_0 n} = e^{jk\frac{2\pi}{N}n},$$

$$k = \ldots, -2, -1, 0, 1, 2, \ldots$$

In fact, there are only *N* (not infinite) distinct exponentials in this set, because

$$e^{jk\frac{2\pi}{N}n} = e^{j(k+N)\frac{2\pi}{N}n}.$$

In contrast, for continuous time periodic signals, there are infinite number of harmonic exponentials.

Distinct harmonics

N consecutive narmonically-related complex exponentials

$\phi_k[n] = e^{jk\frac{2\pi}{N}n}$ are distinct, and are denoted as a set:

$$\{\phi_k[n]\}_{k=p,p+1,\ldots,p+N-1}\,.$$

The above set is identical to the following set

$$\{\phi_k[n]\}_{k=r,r+1,\ldots,r+N-1}\,.$$

Harmonically-related exponentials are orthogonal:

$$\sum_{n=0}^{N-1}\phi_k[n]\phi_r^*[n] = \begin{cases} N, & k = r \\ 0, & k \neq r \end{cases}$$

Note: the operation of multiplication-and-then-summation sample by sample measures the correlation between two signals.

The fundamental period of N distinct harmonically-related complex exponentials is not necessarily N for all.

For example, the case N=6 is considered.

Frequencies of discrete-time complex harmonics for N=6

k	**0**	**1**	**2**	**3**	**4**	**5**
$\phi_k[n]$	$e^{j0\frac{2\pi}{6}n}=1$	$e^{j\frac{2\pi}{6}n}$	$e^{j2\frac{2\pi}{6}n}$	$e^{j3\frac{2\pi}{6}n}$	$e^{j4\frac{2\pi}{6}n}$	$e^{j5\frac{2\pi}{6}n}$
Frequency ω_1	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{4\pi}{3}$	$\frac{5\pi}{3}$
Fundamental frequency ω_0	2π (or 0)	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π	$\frac{2\pi}{3}$	$\frac{\pi}{3}$
Fundamental period N	1	6	3	2	3	6

$$N = \arg\min_n \{\omega_1 n = 2\pi m\}, \quad m = 1,2,\ldots$$

Fourier Series Representation of DT Periodic Signals

Consider a periodic DT signal of period *N* that can be represented using a linear combination of the exponentials in the set $\{\phi_k[n]\}_{k=p,p+1,\ldots,p+N-1}$.

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k \phi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk\frac{2\pi}{N}n}$$

$$\tilde{x}[n] = \sum_{k=0}^{N-1} a_k e^{jk\frac{2\pi}{N}n}$$

DT FS coefficients

We can compute the coefficients a_k by multiplying the Fourier series by $\phi_k[n]^* = e^{-jk\frac{2\pi}{N}n}$ and summing over N :

$$\sum_{n=\langle N\rangle} \tilde{x}[n]\phi_k[n]^* = \sum_{n=\langle N\rangle}\sum_{p=\langle N\rangle} a_p e^{jp\frac{2\pi}{N}n} e^{-jk\frac{2\pi}{N}n} = Na_k$$

Hence,

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle}\tilde{x}[n]e^{-jk\frac{2\pi}{N}n}.$$

Without losing generality, we have:

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1}\tilde{x}[n]e^{-jk\frac{2\pi}{N}n}$$

DT Fourier series coefficients

$$a_0 = \frac{1}{N}\left(x[0] + x[1] + \cdots + x[N-1]\right)$$

$$a_1 = \frac{1}{N}\left(x[0] + x[1]e^{-j\frac{2\pi}{N}} + \cdots + x[N-1]e^{-j\frac{2\pi}{N}(N-1)}\right)$$

$$\vdots$$

$$a_{N-1} = \frac{1}{N}\left(x[0] + x[1]e^{-j(N-1)\frac{2\pi}{N}} + \cdots + x[N-1]e^{-j(N-1)\frac{2\pi}{N}(N-1)}\right)$$

which can be written in matrix-vector form as

$$\begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{N-1} \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & \cdots & e^{-j\frac{2\pi}{N}(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & e^{-j(N-1)\frac{2\pi}{N}} & \cdots & e^{-j(N-1)\frac{2\pi}{N}(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

.

The matrix in this equation can be shown to be invertible, hence to each $x[n]$ of period N there corresponds a unique set of coefficients, and vice-versa.

The coefficients a_k are called the *discrete-time Fourier series coefficients* of x[n].

The *discrete-time Fourier series pair* is given by

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N \rangle} a_k e^{jk\frac{2\pi}{N}n},$$

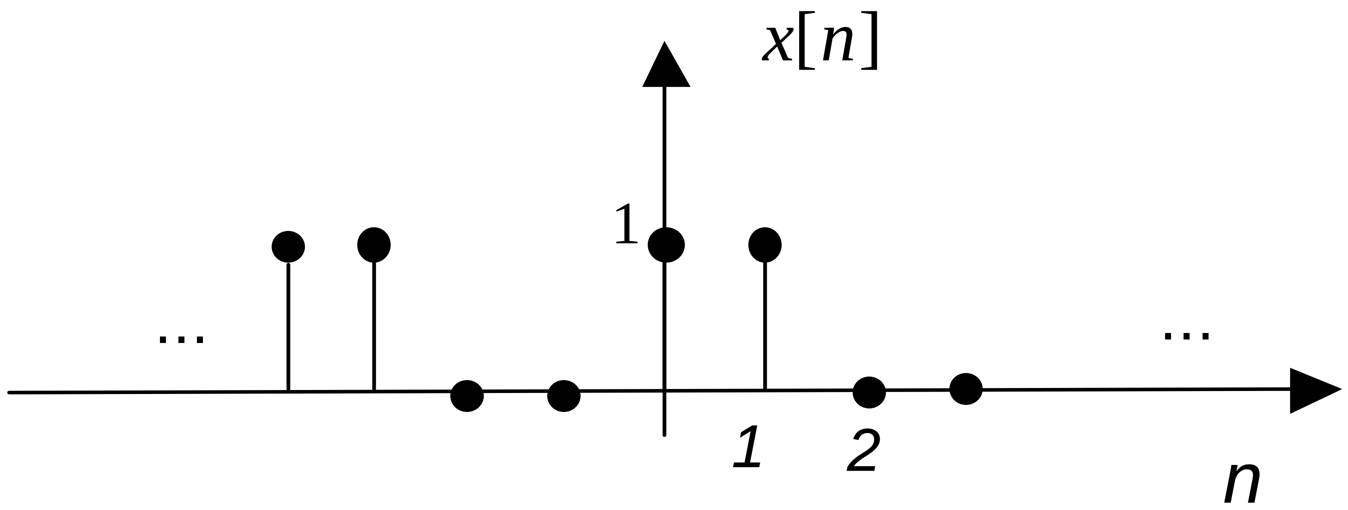
$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\omega_0 n} = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\frac{2\pi}{N}n}.$$

Remarks

- The coefficients a_k can be seen as a periodic sequence, as they repeat with period N.
- All summations are *finite*, which means that the sums *always converge*

Example

Consider the following DT periodic signal $x[n]$ of period $N = 4$:



We can compute its 4 distinct Fourier series coefficients

$$\omega_0 = \frac{2\pi}{N} = \frac{\pi}{2}$$

DTFS coefficients

$$a_0 = \frac{1}{4}\sum_{n=0}^{3} x[n] = \frac{1}{2}$$

$$a_1 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j\frac{2\pi}{4}n} = \frac{1}{4}\left(1+e^{-j\frac{\pi}{2}}\right) = \frac{1}{4}(1-j) = \frac{1}{2\sqrt{2}}e^{-j\frac{\pi}{4}}$$

$$a_2 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j2\frac{2\pi}{4}n} = \frac{1}{4}\left(1+e^{-j\pi}\right) = \frac{1}{4}(1-1) = 0$$

$$a_3 = \frac{1}{4}\sum_{n=0}^{3} x[n]e^{-j3\frac{2\pi}{4}n} = \frac{1}{4}\left(1+e^{-j\frac{3\pi}{2}}\right) = \frac{1}{4}(1+j) = \frac{1}{2\sqrt{2}}e^{+j\frac{\pi}{4}}$$

Let's see if we can recover $x[1]$:

$$x[1] = \sum_{k=0}^{3} a_k e^{jk\frac{2\pi}{4}}$$

$$= \frac{1}{2} + \frac{1}{2\sqrt{2}} e^{-j\frac{\pi}{4}} e^{j\frac{\pi}{2}} + 0 + \frac{1}{2\sqrt{2}} e^{j\frac{\pi}{4}} e^{-j\frac{\pi}{2}}$$

$$= \frac{1}{2} + \frac{1}{\sqrt{2}} \mathrm{Re}\{e^{j\frac{\pi}{4}}\} = 1$$

Properties of DT Fourier Series

Notation: $x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k$ represents a discrete-time Fourier series pair.

The properties of DT Fourier series are similar to those of CT Fourier series.

All signals are assumed to be periodic with fundamental period *N* and fundamental frequency $\omega_0 = \dfrac{2\pi}{N}$, unless otherwise specified.

The DTFS coefficients are often called *spectral coefficients*.

Linearity

The operation of calculating the DTFS of a periodic signal is linear.

For $x[n] \overset{\mathcal{FS}}{\leftrightarrow} a_k$, $y[n] \overset{\mathcal{FS}}{\leftrightarrow} b_k$, if we form the linear combination $z[n] = Ax[n] + By[n]$, then we have

$$z[n] \overset{\mathcal{FS}}{\leftrightarrow} Aa_k + Bb_k .$$

Time Shifting

Time shifting leads to a multiplication by a complex exponential.

For $x[n] \overset{\mathcal{FS}}{\leftrightarrow} a_k$,

$$x[n-n_0] \overset{\mathcal{FS}}{\leftrightarrow} e^{-jk\frac{2\pi}{N}n_0} a_k \; .$$

Remarks:

The magnitudes of the Fourier series coefficients are not changed, only their phases.

A time shift by an integer number of periods, i.e., of $n_0 = pN,\; p = \ldots,-2,-1,0,1,2,\ldots$ does not change the DTFS coefficients, as expected.

Time Reversal

Time reversal leads to a "frequency reversal" of the corresponding sequence of Fourier series coefficients:

$$x[-n] \overset{\mathcal{FS}}{\leftrightarrow} a_{-k}\ .$$

Interesting consequences:

- For x[n] even, the sequence of coefficients is also even ($a_{-k} = a_k$)
- For x[n] odd, the sequence of coefficients is also odd ($a_{-k} = -a_k$)