McGill University
Department of Electrical and Computer Engineering

Lecture 23

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Time and Frequency Analysis of BIBO Stable Continuous-Time LTI Systems

- Analysis of frequency responses using the poles and zeros of transfer functions
- Bode plots

Frequency response as a function of the poles and zeros of a transfer function

Consider a stable transfer function in terms of poles and zeros:

$$H(s) = \frac{K(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)}.$$

Let $s = j\omega$ to obtain the frequency response of the system.

$$|H(j\omega)| = \frac{|K|\prod_{k=1}^{m}|j\omega - z_k|}{\prod_{j=1}^{n}|j\omega - p_j|}$$

$$\angle H(j\omega) = \angle K + \sum_{k=1}^{m}\angle(j\omega - z_k) - \sum_{j=1}^{n}\angle(j\omega - p_j)$$

For a frequency ω, each first-order pole (zero) factor corresponds to ***a vector originating at the pole p_j (zero z_k) and ending at $j\omega$***, contributing to the overall frequency response.

- The magnitude at frequency ω is given by:
  the product of the lengths of the vectors originating at zeros divided by the product of the lengths of the vectors originating at poles.

- The phase at frequency ω is given by:
  the sum of the angles of the vectors originating at zeros minus the sum of the angles of the vectors originating at poles.

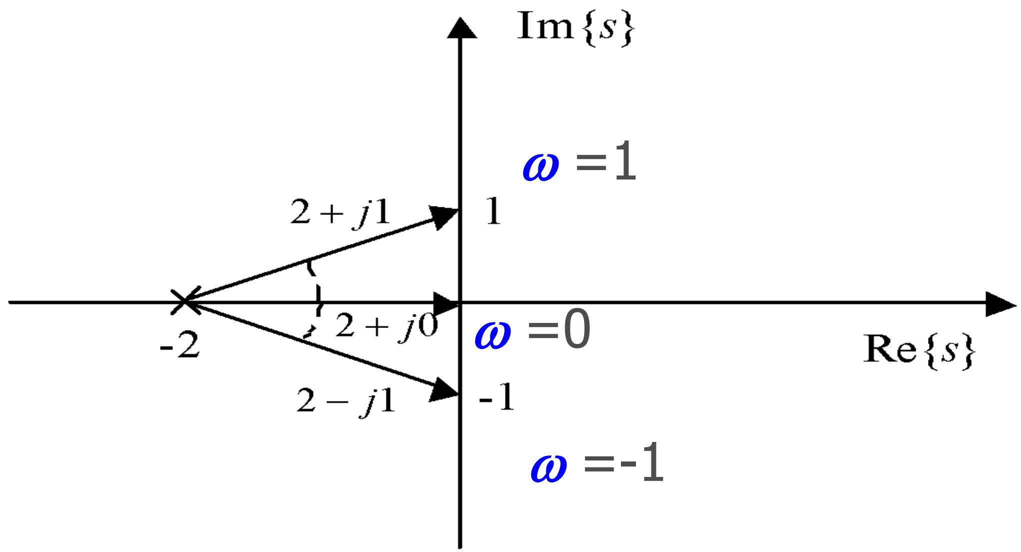
Example

A stable first-order system with transfer function

$$H(s) = \frac{1}{s+2}, \ \mathrm{Re}\{s\} > -2$$

Its *frequency response is* $$H(j\omega) = \frac{1}{j\omega + 2}$$

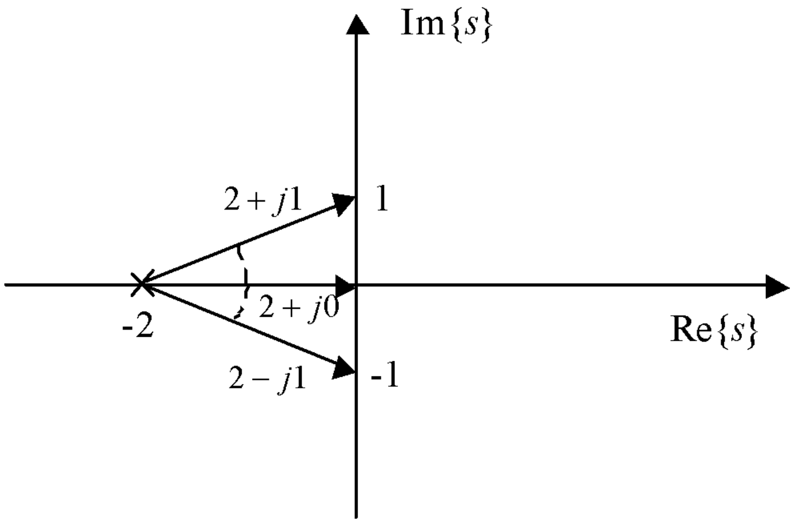
In the s-plane, the denominator $2 + j\omega$ can be viewed as a ***vector, which is a function of ω.***



- <u>as ω goes from $-\infty$ to 0:</u>

the magnitude of $2+j\omega$ (the vector's length) goes from ∞ to 2 while its phase goes from $-\pi/2$ to 0.

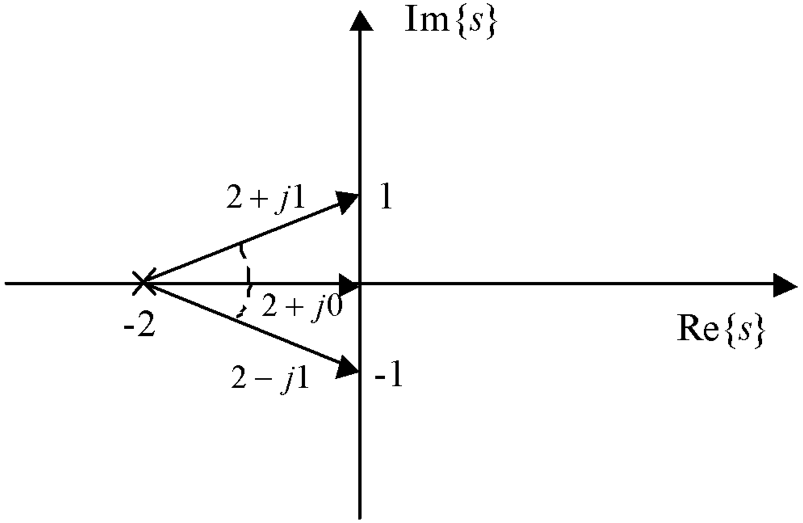
$\Rightarrow$ magnitude of $H(j\omega)=(2+j\omega)^{-1}$ varies from 0 to 0.5 while its phase goes from $\pi/2$ to 0.

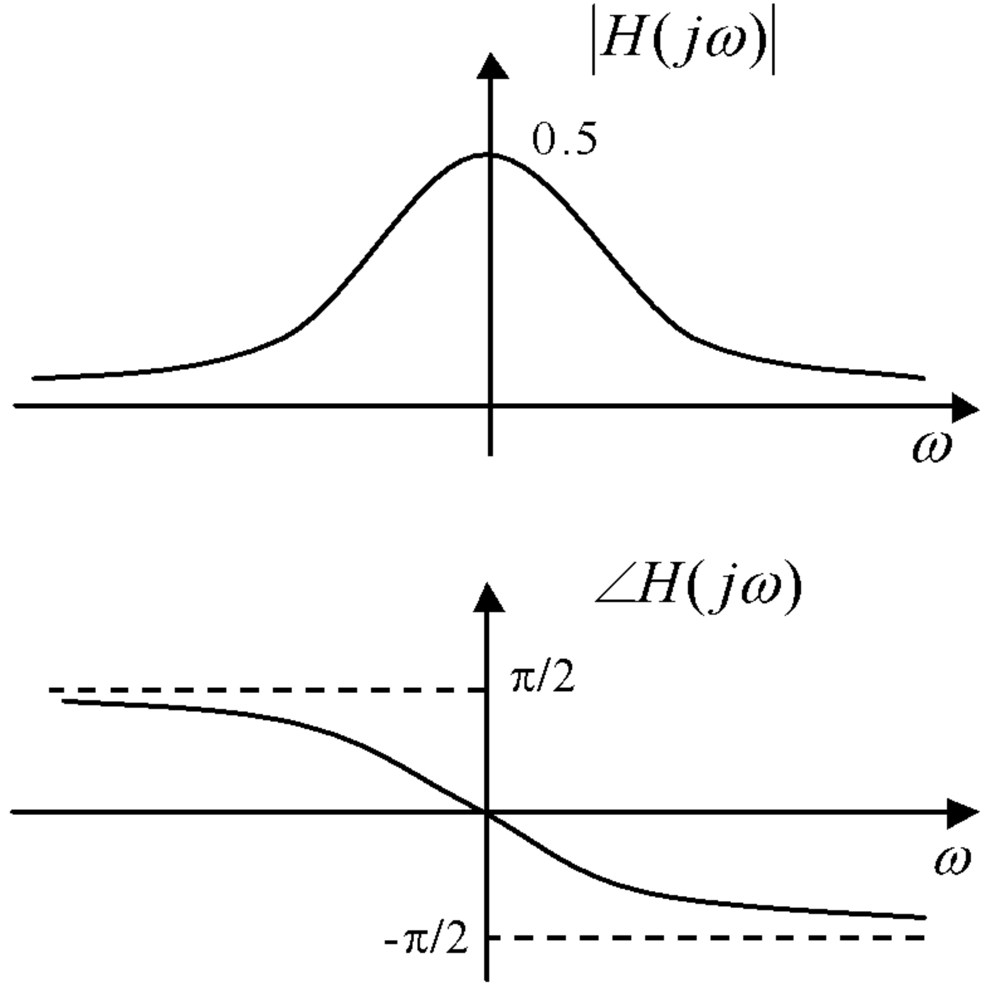


- As ω goes from 0 to ∞ :

the magnitude of $2 + j\omega$ (the vector's length) goes from 2 to ∞ while its phase goes from 0 to $\pi/2$ radians.

The magnitude of $H(j\omega) = (2 + j\omega)^{-1}$ varies from 0.5 to 0 while its phase goes from 0 to $-\pi/2$ radians.





|H(jω)|
0.5
ω
∠H(jω)
π/2
ω
-π/2

The frequency response of a ***first-order pole system*** is

$$H(j\omega) = \frac{1}{j\omega + 2} = \frac{e^{j\arctan(\frac{-\omega}{2})}}{\sqrt{4+\omega^2}}$$

For a higher-order system, its overall frequency response can be obtained from multiple ***first-order pole and zero systems.***

Example of third-order system

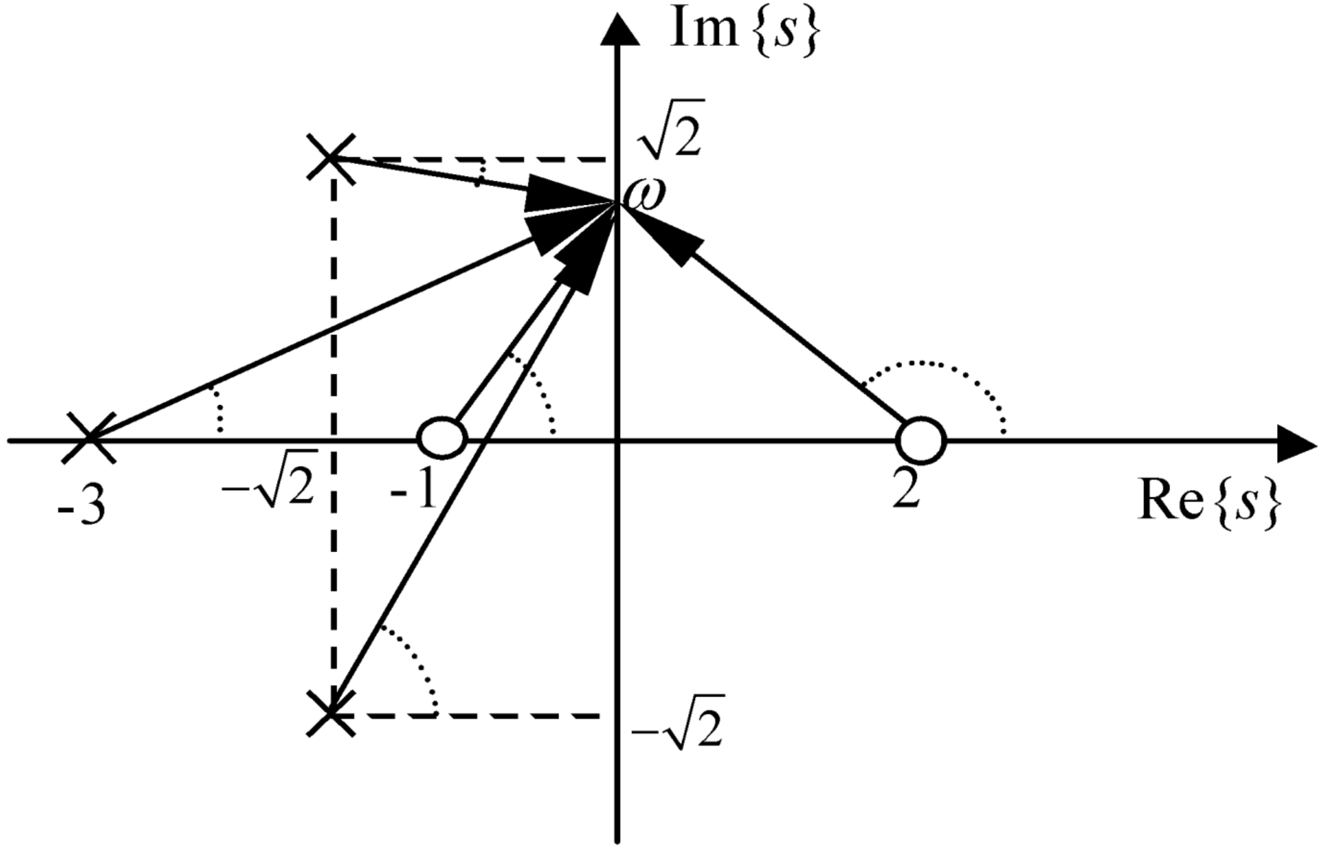
A stable third-order system with transfer function

$$H(s) = \frac{(s+1)(s-2)}{(s+3)(s+\sqrt{2}-j\sqrt{2})(s+\sqrt{2}+j\sqrt{2})}, \ \mathrm{Re}\{s\} > -\sqrt{2}$$

has the frequency response

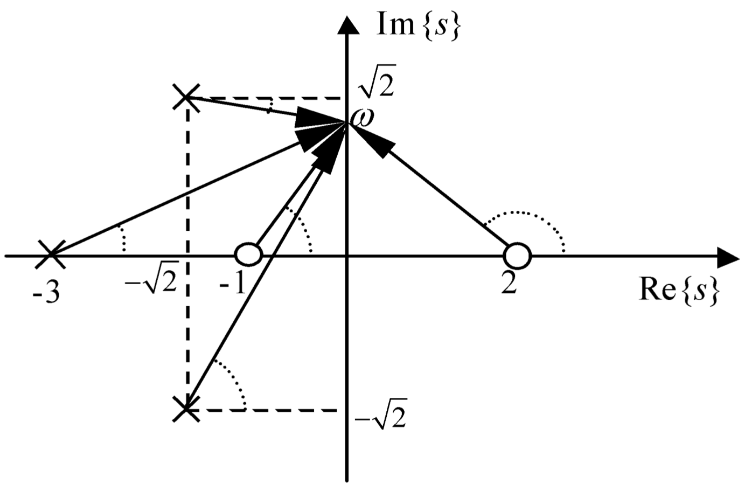
$$H(j\omega) = \frac{(j\omega+1)(j\omega-2)}{(j\omega+3)(j\omega+\sqrt{2}-j\sqrt{2})(j\omega+\sqrt{2}+j\sqrt{2})} .$$

Frequency response as a function of zeros and poles



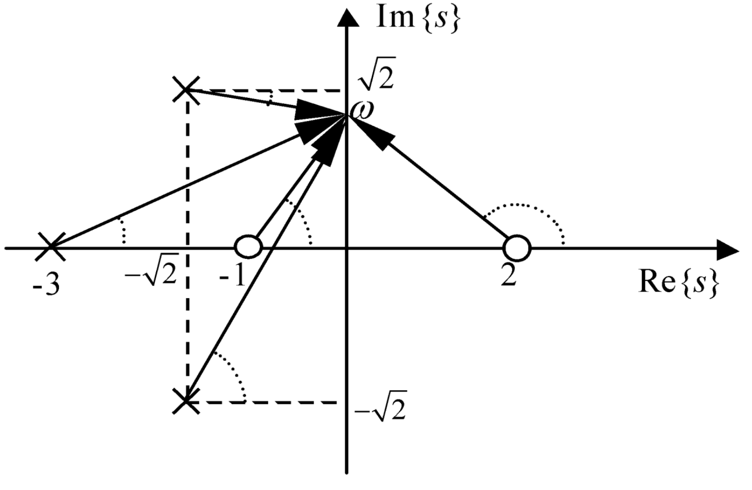
For the above example, we can make the following qualitative observations.

- At $\omega = 0$, the lengths of the vectors originating from the two zeros are minimized, and the output level is low.
- At $\omega = \pm\sqrt{2}$, the lengths of the vectors originating from the poles $p_{1,2} = -\sqrt{2} \pm j\sqrt{2}$ are minimized, and the output level is high.

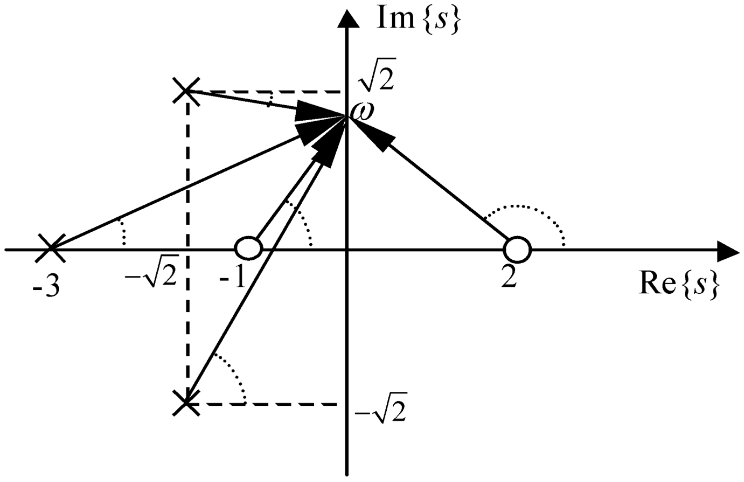


- The phase at $\omega = 0$ is $-\pi$ and has a net contribution of $-\pi$ only from the zero $z_1 = 2$.

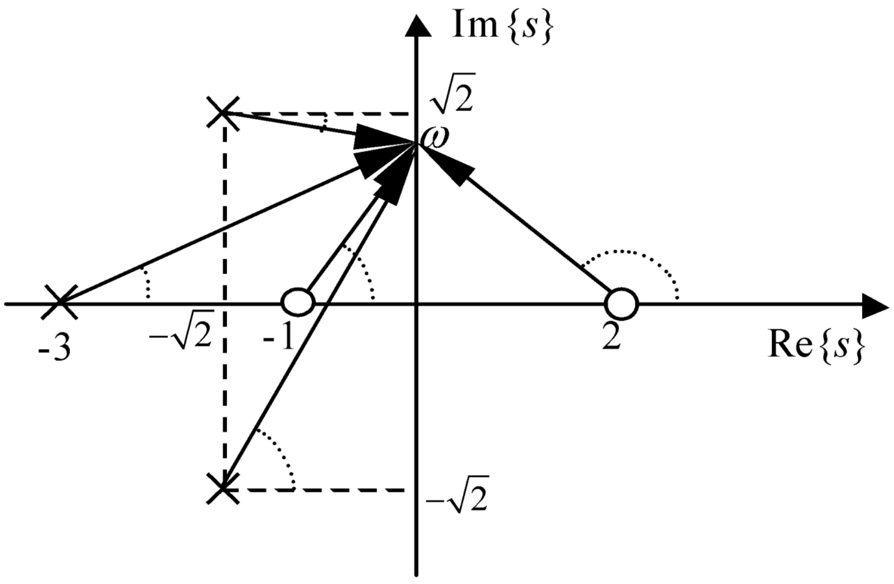
- The phase around $\omega = \pm\sqrt{2}$ should be more sensitive to a small change in ω than elsewhere. This is even more noticeable when the complex poles are closer to the imaginary axis.



- At $\omega = +\infty$, the phase is $-\pi/2$. This comes from a contribution of $-\pi/2$ from the three pole vectors, a contribution of $\pi/2$ from the RHP zero vector, and a contribution of $\pi/2$ from the LHP zero vector.

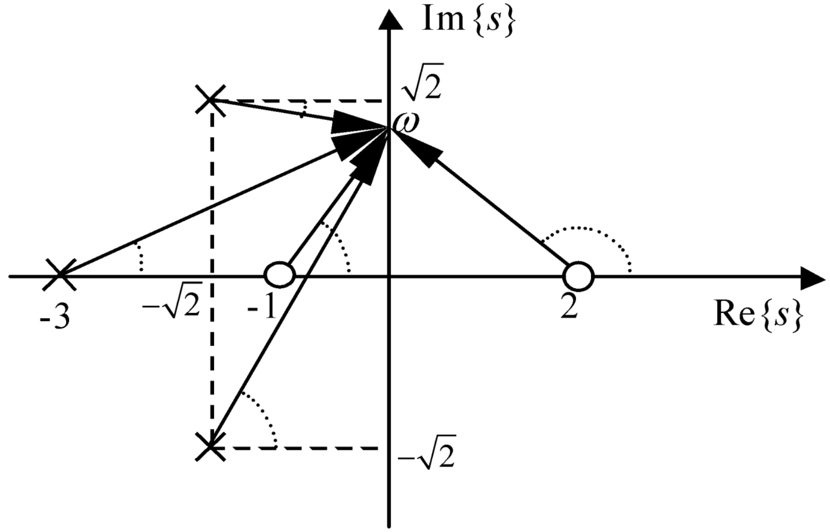


- At $\omega = -\infty$, the phase is $\pi/2$. This comes from a contribution of $\pi/2$ from the three pole vectors, a contribution of $-\pi/2$ from the RHP zero vector, and a contribution of $-\pi/2$ from the LHP zero vector.

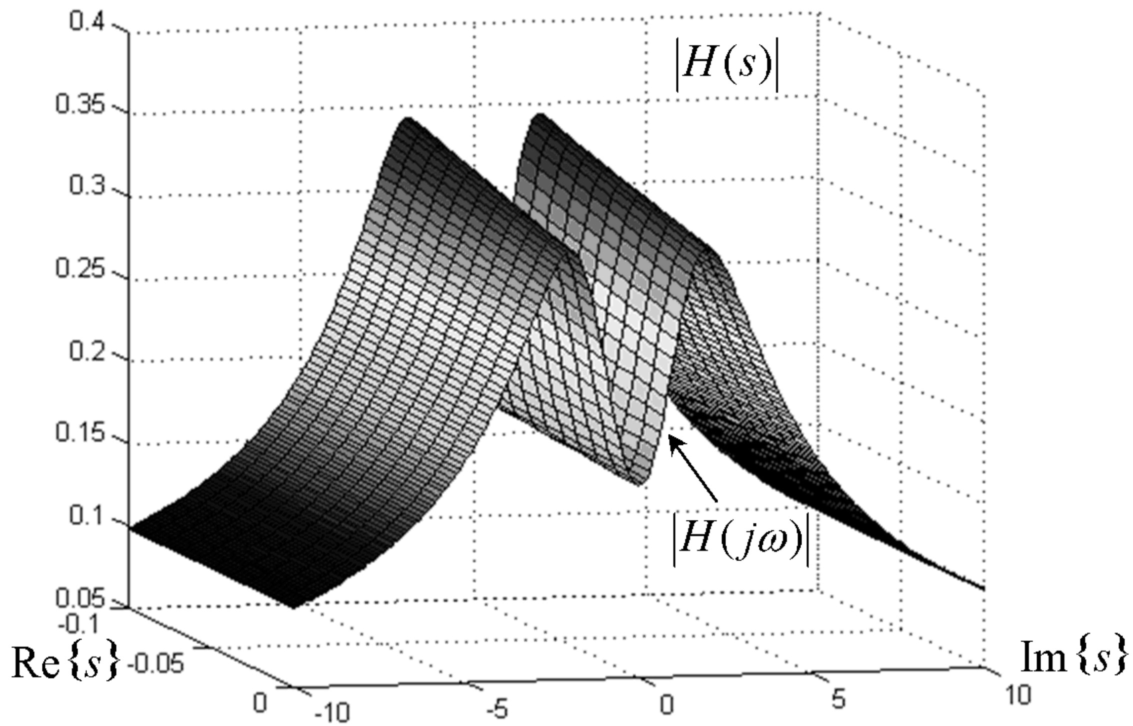


Remark:

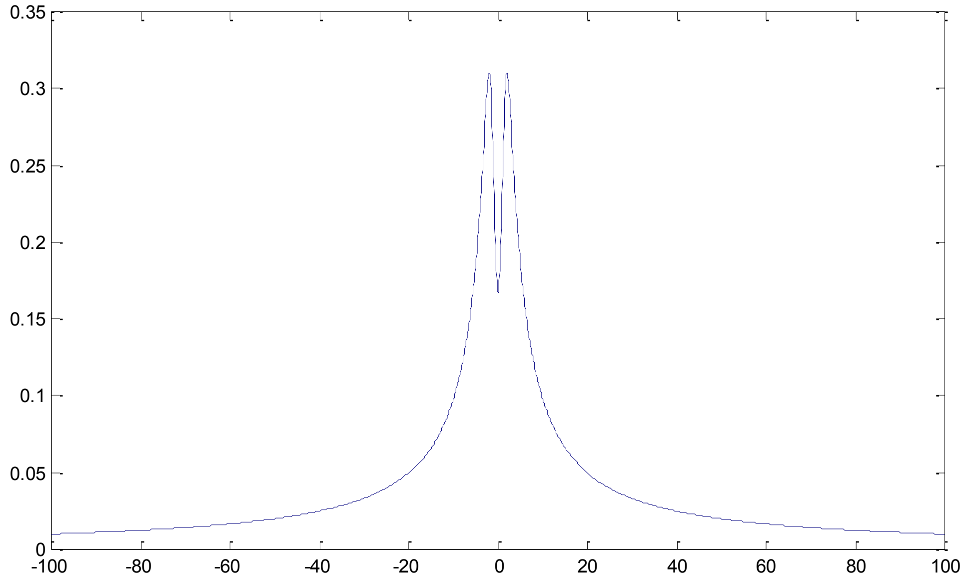
The definition of an angle is sometimes ambiguous. For most purposes the angle $\pi/2$ can be considered the same as $-3\pi/2$

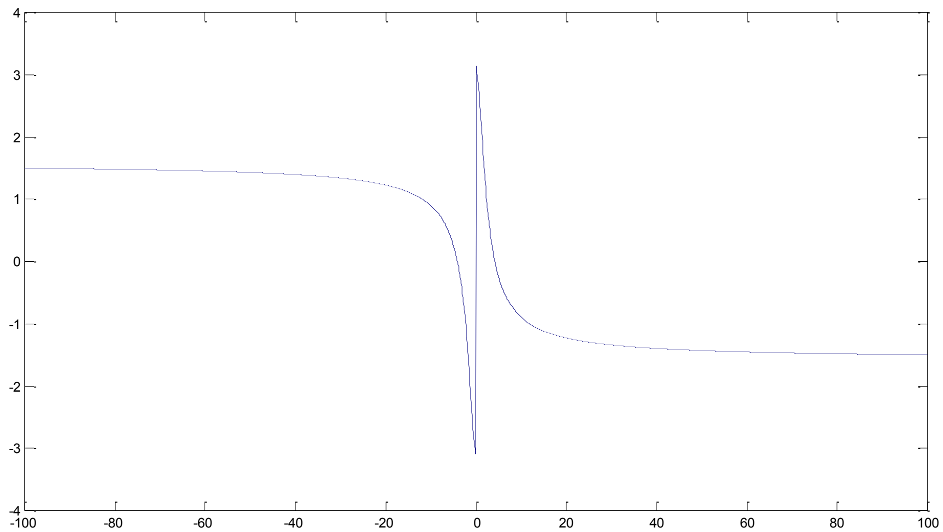


Im{s}
√2
ω
-3
−√2
-1
2
Re{s}
−√2



|H(s)|
|H(jω)|
0.4
0.35
0.3
0.25
0.2
0.15
0.1
0.05
-0.1
Re{s} -0.05
0
-10
-5
0
5
10
Im{s}


0.35
0.3
0.25
0.2
0.15
0.1
0.05
0
-100
-80
-60
-40
-20
0
20
40
60
80
100


4
3
2
1
0
-1
-2
-3
-4
-100
-80
-60
-40
-20
0
20
40
60
80
100

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Bode plots

The plots of $20\log_{10}|H(j\omega)|$ and $\angle H(j\omega)$ versus $\log_{10}(\omega)$ are referred to as Bode plots.

The benefits of using a log scale to plot the frequency response:

- To cover a wide dynamic range of the magnitude
- To cover a wide range of frequency
- *To add* rather than multiply the magnitudes of product of frequency responses, which is easier to do graphically:

$$\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$$

dB (Decibels)

It is customary to use *decibels (dB)* as the magnitude units.

$$20\log_{10}|H(j\omega)| \qquad (dB)$$

Magnitude gains expressed in dB

Gain	Gain (dB)
0	$-\infty$ dB
0.01	-40 dB
0.1	-20 dB
1	0 dB
$\sqrt{2}$	3 dB
2	6 dB
10	20 dB
100	40 dB
1000	60 dB

Power gain in dB

The power gain is defined as:

$$10\log_{10}\left|H(j\omega)\right|^2 dB = 20\log_{10}\left|H(j\omega)\right| dB$$

i.e., the power gain is identical to the amplitude gain in dB.

Example of a first-order system

Consider again the first-order system with frequency response

$$H(j\omega) = \frac{1}{j\omega + 2}.$$

It is convenient to write it as the product of a gain and a first-order transfer function with unity gain at DC:

$$H(j\omega) = \frac{1}{2}\frac{1}{j\omega/2 + 1}.$$

The Bode plot of the magnitude

The Bode plot of the magnitude is the graph of

$$20\log_{10}|H(j\omega)| = 20\log_{10}\left|\frac{1}{2}\right| + 20\log_{10}\left|\frac{1}{\frac{j\omega}{2}+1}\right| dB$$

$$= -20\log_{10} 2 - 20\log_{10}\left|\tfrac{j\omega}{2}+1\right| dB$$

$$= -6\,dB - 20\log_{10}\left|\tfrac{j\omega}{2}+1\right| dB$$

The low- and high-frequency asymptotes of a first-order system

For low frequencies ($\omega << 2$),

$$20\log_{10}|H(j\omega)| \approx -6\,dB - 20\log_{10}|1|\,dB = -6\,dB\,.$$

i.e., for low frequencies, the frequency response approximates a straight line –6 dB.

For high frequencies ($\omega >> 2$),

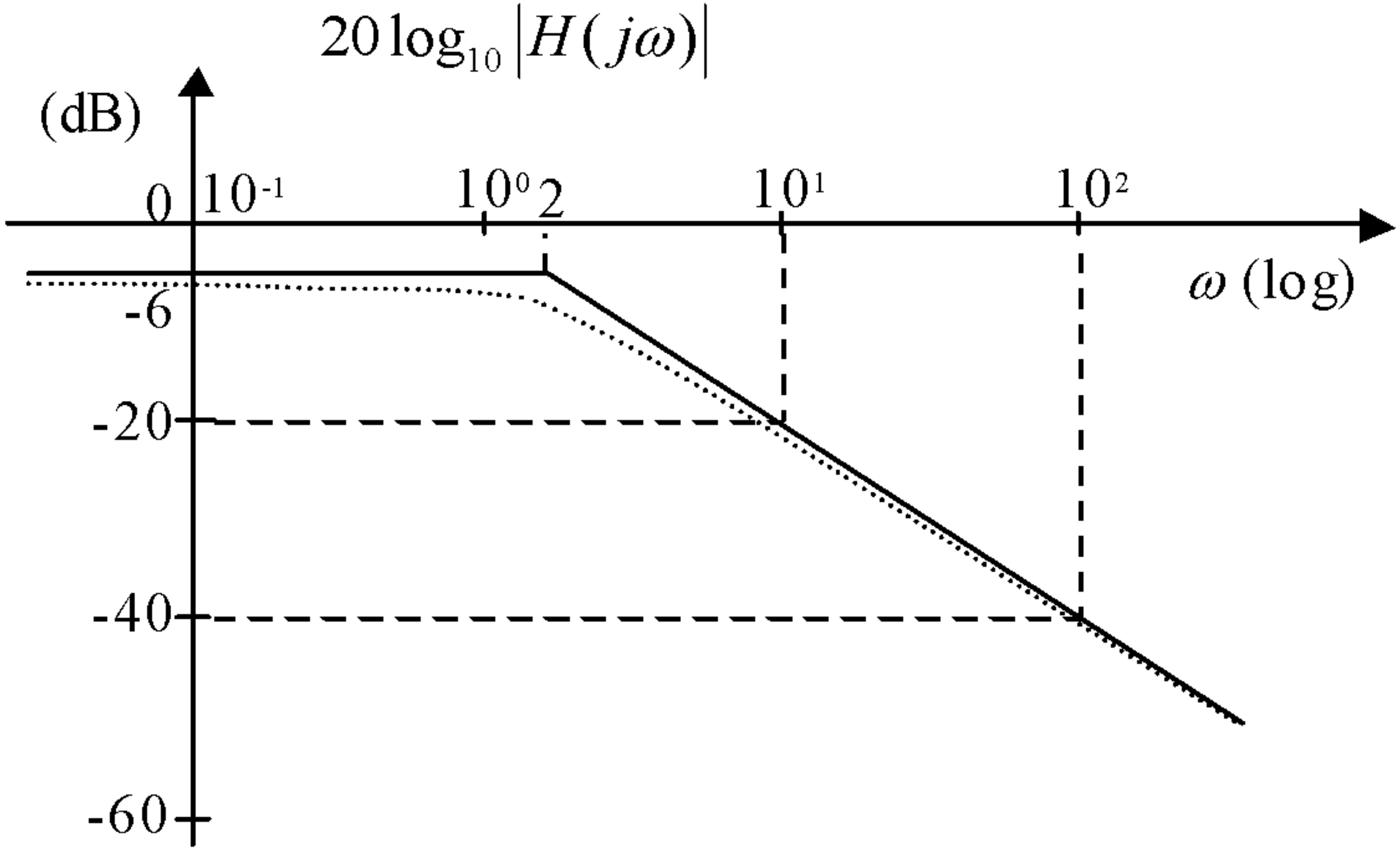
$$\begin{aligned}20\log_{10}|H(j\omega)| &\approx -6\,dB - 20\log_{10}\left|\frac{\omega}{2}\right| dB \\ &= -6\,dB - 20\log_{10}|\omega|\,dB + 20\log_{10}2\,dB \\ &= -20\log_{10}|\omega|\,dB\end{aligned}$$

i.e., for high frequencies, the frequency response approximates a straight line with a slop –20 dB/decade.

For $\omega = 10$, we get -20 dB; for $\omega = 100$, we get -40 dB, etc. The slope of the asymptote is therefore -20dB/decade as $\omega \rightarrow +\infty$.

The break frequency is the frequency at which the two asymptotes meet.

The two asymptotes meet at the break frequency = 2 radians/s, we can sketch the magnitude Bode plot as follows (dashed line: actual magnitude):



The break frequency is also the frequency at which the magnitude drops from the DC gain by 3 dB .