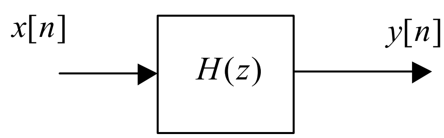
Sample Midterm Test 2 (mt2s03)
Covering Chapters 13-16 of *Fundamentals of Signals & Systems*

Problem 1 (30 marks)

Consider the DLTI system



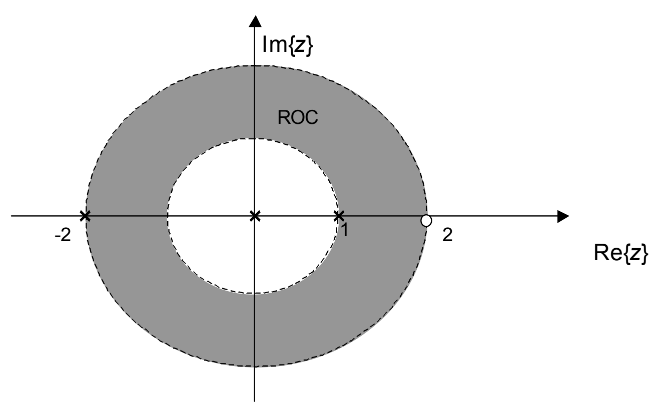
with transfer function $H(z)=\dfrac{1-2z^{-1}}{(z-1)(z+2)},\ 1<|z|<2$.

(a) [5 marks] Sketch the pole-zero plot of the system.

Answer:

$$H(z)=\frac{1-2z^{-1}}{(z-1)(z+2)}=\frac{z-2}{z(z-1)(z+2)},\ 1<|z|<2$$. The poles and zero are

$p_1=0,\ p_2=1,\ p_3=-2,\ z_1=2$.

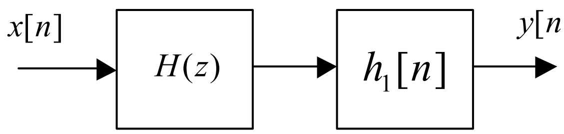


(b) [5 marks] Is this system stable? Is it causal? Justify your answers.

Answer:

No, the system is not stable since the unit circle is excluded from the ROC (open boundary).
No, the system is not causal since the ROC is not the exterior of a disk extending to infinity.

(c) [15 marks] Compute the impulse response $h_0[n]$ of the new system formed by cascading the system $H(z)$ above with a second system of impulse response $h_1[n] = \delta[n] - \delta[n-1]$.



Answer:

$$H_1(z) = 1 - z^{-1},\ |z| > 0.$$

Overall system:

$$H_0(z) = H(z)H_1(z) = \frac{(1-2z^{-1})(1-z^{-1})}{(z-1)(z+2)}, 1 < |z| < 2$$

$$= \frac{(1-2z^{-1})}{z(z+2)}, 0 < |z| < 2$$

$$= \frac{z^{-2}(1-2z^{-1})}{1+2z^{-1}}, 0 < |z| < 2$$

Impulse response: $X(z) = 1$, so the z-transform of the output is

$$H_0(z) = \frac{z^{-2}(1-2z^{-1})}{1+2z^{-1}}, 0 < |z| < 2$$

$$= \underbrace{\frac{z^{-2}}{1+2z^{-1}}}_{0<|z|<2} - \underbrace{\frac{2z^{-3}}{1+2z^{-1}}}_{0<|z|<2}$$

Using the table, we find

$$h_0[n] = -(-2)^{n-2}u[-n+1] - 2(-2)^{n-3}u[-n+2].$$

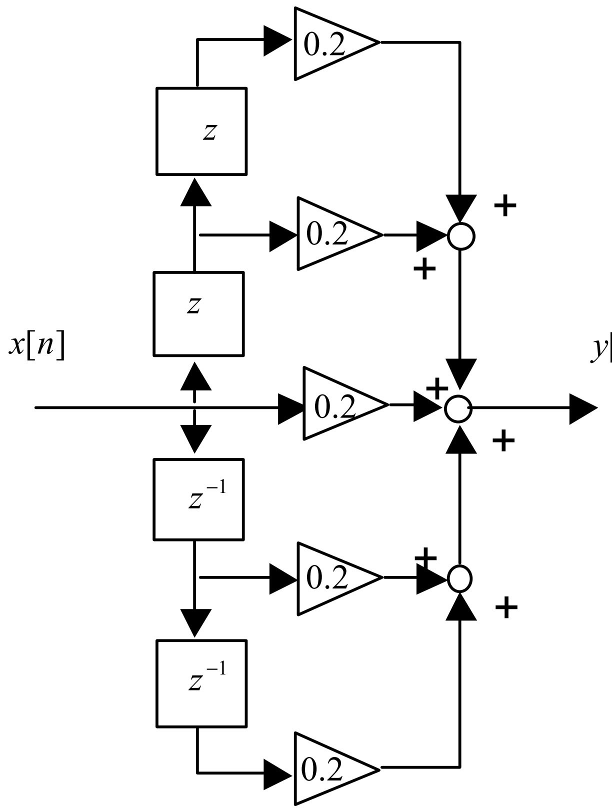
(d) [5 marks] Is this new system stable? Is it causal? Justify your answers.

Answer:

New system is stable as its ROC now includes the unit circle. The system is not causal because its impulse response is different from zero at negative times. Alternatively: the system is not causal since the ROC is not the exterior of a disk extending to infinity.

Problem 2 (30 marks)

Consider the FIR filter described by its block diagram given below.

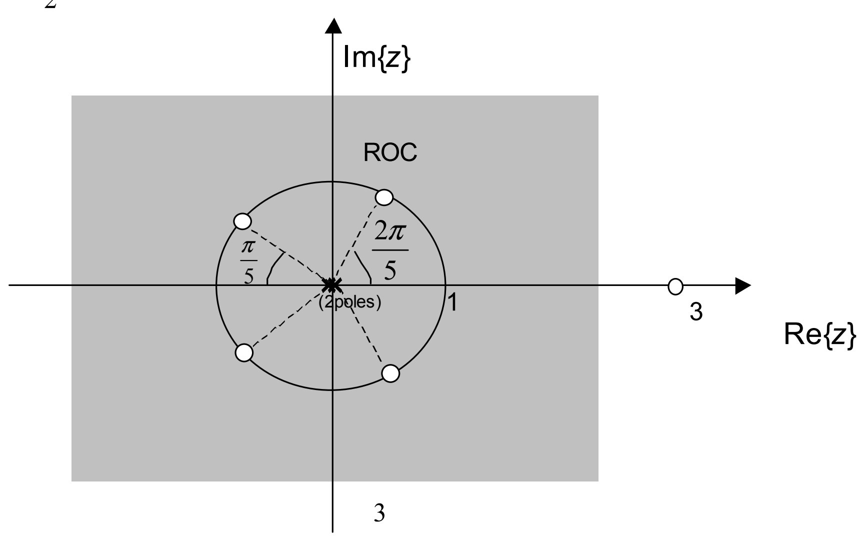


(a) [10 marks] Write the transfer function $H(z)$ of the filter and specify its ROC. Sketch its pole-zero plot. Is the filter causal? Justify your answer.

Answer:

$$H(z) = \frac{1}{5}\left(z^2 + z^1 + 1 + z^{-1} + z^{-2}\right),\ 0 < |z| < +\infty$$

$$= \frac{1}{5}\frac{z^4 + z^3 + z^2 + z^1 + 1}{z^2},\ 0 < |z| < +\infty$$



(b) [10 marks] Find the frequency response $H(e^{j\omega})$ of the filter, give and provide rough sketches of its magnitude and phase. What type of filter is it? (low-pass, band-pass or high-pass?)

Answer:

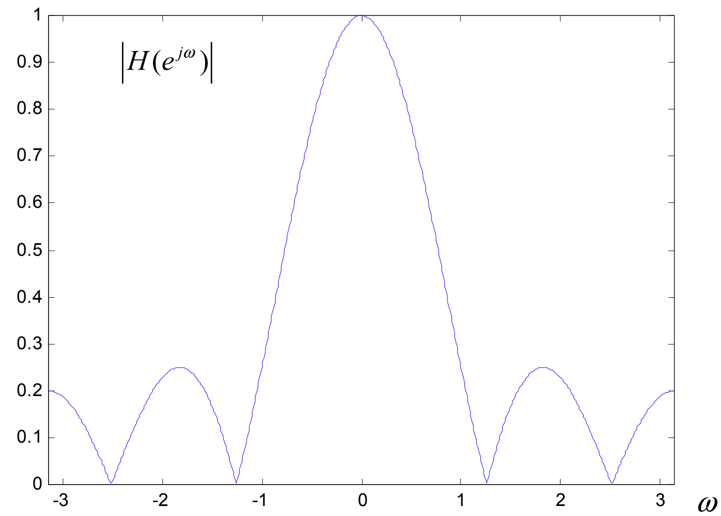
$$H(e^{j\omega}) = 0.2e^{2j\omega} + 0.2e^{j\omega} + 0.2 + 0.2e^{-j\omega} + 0.2e^{-2j\omega}$$
$$= 0.4\cos(2\omega) + 0.4\cos(\omega) + 0.2$$

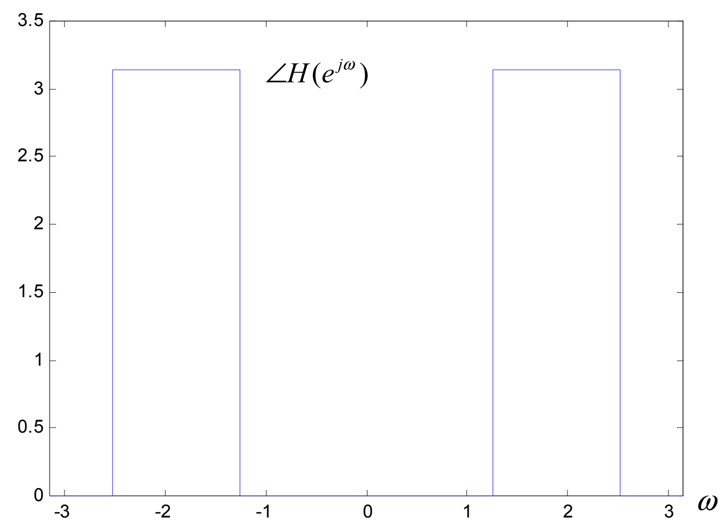
magnitude:

$\left|H(e^{j\omega})\right| = \left|0.4\cos(2\omega) + 0.4\cos(\omega) + 0.2\right|$. This is a lowpass filter.

phase:

$$\angle H(e^{j\omega}) = \begin{cases} 0, & |\omega| \le \frac{2\pi}{5} \text{ and } \frac{4\pi}{5} \le |\omega| \le \pi \\ \pi, & \frac{2\pi}{5} < |\omega| < \frac{4\pi}{5} \end{cases}$$



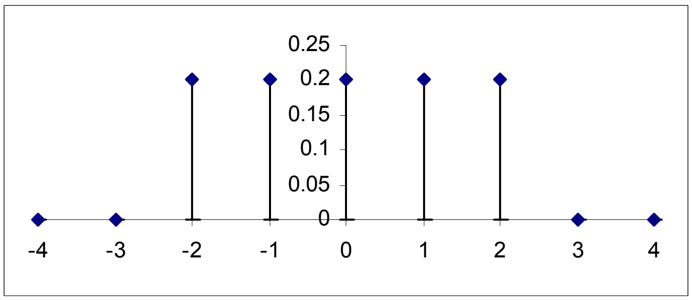


(c) [5 marks] Find and sketch the impulse response of the filter.

Answer:

Inverse z-transform of $H(z)=\frac{1}{5}\left(z^2+z^1+1+z^{-1}+z^{-2}\right),\ 0<|z|<+\infty$ yields:

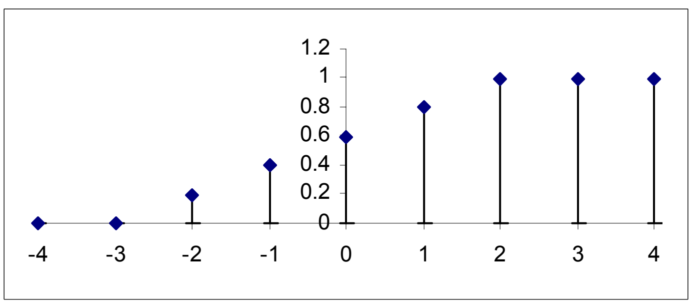
$$h[n]=0.2\delta[n+2]+0.2\delta[n+1]+0.2\delta[n]+0.2\delta[n-1]+0.2\delta[n-2]$$



(d) [5 marks] Find and sketch the unit step response of the filter.

Answer:

$$s[n]=0.2\delta[n+2]+0.4\delta[n+1]+0.6\delta[n]+0.8\delta[n-1]+u[n-2]$$



Problem 3 (25 marks)

Suppose we want to design a causal, stable, first-order high-pass filter of the type

$$H(z)=\frac{B(1-z^{-1})}{1-az^{-1}},\ |z|>|a|$$

with -3dB cutoff frequency $\omega_c=\frac{2\pi}{3}$ (i.e., frequency where the magnitude of the frequency response of the filter is $\frac{1}{\sqrt{2}}$) and a real.

(a) [5 marks] Express the real constant B in terms of the pole a to obtain unity gain at the highest frequency $\omega=\pi$.

Answer:
The gain at $\omega = \pi$ is

$$H(e^{j\pi}) = H(-1) = \frac{2B}{1+a},$$

and unity gain is obtained for $B = \dfrac{1+a}{2}$.

(b) [20 marks] "Design" the filter, i.e., find the numerical values of the pole a and the constant B. Sketch the magnitude of the frequency response of the filter.

Answer:

$$\frac{1}{2} = \left|H(e^{j\omega_c})\right|^2 = \frac{(1+a)^2[(1-\cos\omega_c)^2 + \sin^2\omega_c]}{4\left[(1-a\cos\omega_c)^2 + a^2\sin^2\omega_c\right]}$$

$$= \frac{(1+a)^2(1-\cos\omega_c)}{2\left[(1-a\cos\omega_c)^2 + a^2\sin^2\omega_c\right]}$$

$$= \frac{0.75(1+a)^2}{(1+0.5a)^2 + 0.75a^2}$$

This yields the quadratic equation

$$1.5(1+a)^2 = (1+0.5a)^2 + 0.75a^2$$

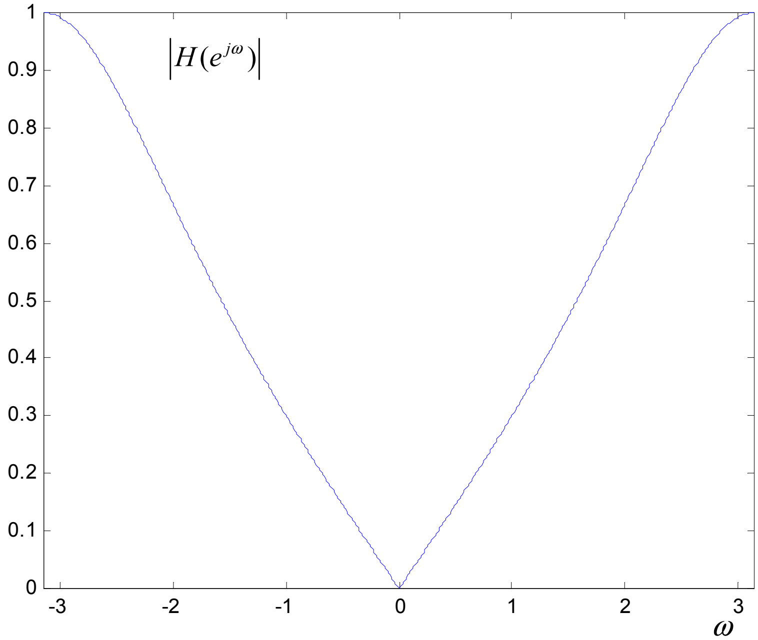
$$\Leftrightarrow$$

$$1.5 + 3a + 1.5a^2 = 1 + a + a^2$$

$$a^2 + 4a + 1 = 0$$

whose solutions are $a_{1,2} = \dfrac{-4 \pm \sqrt{12}}{2} = -2 \pm \sqrt{3}$. But for the filter to be stable, we select the pole inside the unit circle: $a = -2 + \sqrt{3} = -0.268$. Finally $B = \dfrac{1-0.268}{2} = 0.366$, and the high-pass filter is

$$H(z) = \frac{0.366(1-z^{-1})}{1+0.268z^{-1}}, |z| > 0.268$$



Problem 4 (15 marks)

Compute the inverse *z*-transform of

$$X(z) = \frac{z^{-1}}{z - 0.9}, \quad |z| < 0.9$$

by expanding it in a power series.

Solution

$$X(z) = \frac{z^{-1}}{z - 0.9} = \frac{z^{-1}}{-0.9(1 - \frac{1}{0.9}z)} = -\frac{10}{9}\frac{z^{-1}}{(1 - \frac{1}{0.9}z)}$$

long division yields

$$
\begin{array}{r|l}
 & z^{-1} + \frac{10}{9} + \left(\frac{10}{9}\right)^2 z + \ldots \\
\hline
1 - \frac{10}{9} z & z^{-1} \\
 & z^{-1} - \frac{10}{9} \\
\hline
 & \frac{10}{9} \\
 & \frac{10}{9} - \left(\frac{10}{9}\right)^2 z \\
\hline
 & \left(\frac{10}{9}\right)^2 z
\end{array}
$$

We obtain:

$$X(z) = -\frac{10}{9} \frac{z^{-1}}{(1 - \frac{1}{0.9} z)} = -\frac{10}{9}\left(z^{-1} + \frac{10}{9} + \left(\frac{10}{9}\right)^2 z + \left(\frac{10}{9}\right)^3 z^2 + \ldots \right)$$

$$= -\frac{10}{9} z^{-1} - \left(\frac{10}{9}\right)^2 - \left(\frac{10}{9}\right)^3 z - \left(\frac{10}{9}\right)^4 z^2 - \ldots - \left(\frac{10}{9}\right)^{n+2} z^n - \ldots$$

Note that the resulting power series converges because the ROC implies $\left|\frac{10}{9} z\right| < 1$. The signal is

$$X(z) = -\frac{10}{9} z^{-1} - \left(\frac{10}{9}\right)^2 - \left(\frac{10}{9}\right)^3 z - \left(\frac{10}{9}\right)^4 z^2 - \ldots - \left(\frac{10}{9}\right)^{n+2} z^n - \ldots$$

$$x[n] = -\frac{10}{9} \delta[n-1] - \left(\frac{10}{9}\right)^2 \delta[n] - \left(\frac{10}{9}\right)^3 \delta[n+1] - \ldots$$

$$= -(\frac{10}{9})^{-n+2} u[-n+1] = -(0.9)^{n-2} u[-n+1].$$

ALSO ACCEPTABLE:

$$X(z) = \frac{z^{-1}}{z-0.9} = \frac{z^{-1}}{-0.9(1-\frac{1}{0.9}z)} = -\frac{10}{9}\frac{z^{-1}}{(1-\frac{1}{0.9}z)}$$

$$= -\frac{10}{9}z^{-1}\sum_{k=0}^{+\infty}(\frac{z}{0.9})^k$$

$$= -\frac{10}{9}z^{-1}\left(1+(0.9)^{-1}z+(0.9)^{-2}z^2+(0.9)^{-3}z^3+\cdots+(0.9)^{-k}z^k+\cdots\right)$$

$$= -(0.9)^{-1}z^{-1}-(0.9)^{-2}-(0.9)^{-3}z-(0.9)^{-4}z^2-\cdots-(0.9)^{-k-1}z^{k-1}+\cdots$$

$$\Rightarrow$$

$$x[n] = -(0.9)^{n-2}u[-n+1]$$