Sample Midterm Test 1 (mt1s03)
Covering Chapters 10-12 of *Fundamentals of Signals & Systems*

Problem 1 (25 marks)

Consider the causal LTI state-space system initially at rest:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $A = \begin{bmatrix} 17 & -9 \\ 54 & -28 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$, $D = 0$

(a) [*8 marks*] Find the state transition matrix $\Phi(t,t_0) = e^{A(t-t_0)}$.

Answer:

We have to diagonalize the A-matrix first

The eigenvalues of the A matrix are:

$$\det(\lambda I - A) = 0$$
$$\lambda_1 = -1, \lambda_2 = -10$$

Next, we compute the eigenvectors of A:

Eigenvector v_1 corresponding to $\lambda_1 = -1$:

$$\begin{bmatrix} -18 & 9 \\ -54 & 27 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow v_{11} = 1, v_{12} = 2$$

Eigenvector v_2 corresponding to $\lambda_2 = -10$:

$$\begin{bmatrix} -27 & 9 \\ -54 & 18 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow v_{21} = 1, v_{22} = 3$$

Diagonalizing matrix $T = [v_1 \quad v_2] = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}$

$$\Phi(t,t_0) = e^{A(t-t_0)} = T\mathrm{diag}\{e^{-(t-t_0)}, e^{-10(t-t_0)}\}T^{-1}$$
$$= \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} e^{-(t-t_0)} & 0 \\ 0 & e^{-10(t-t_0)} \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 3e^{-(t-t_0)} & -e^{-(t-t_0)} \\ -2e^{-10(t-t_0)} & e^{-10(t-t_0)} \end{bmatrix}$$
$$= \begin{bmatrix} 3e^{-(t-t_0)} - 2e^{-10(t-t_0)} & -e^{-(t-t_0)} + e^{-10(t-t_0)} \\ 6e^{-(t-t_0)} - 6e^{-10(t-t_0)} & -2e^{-(t-t_0)} + 3e^{-10(t-t_0)} \end{bmatrix}$$

Laplace transform approach also acceptable: $e^{At} \leftrightarrow (sI - A)^{-1}$

(b) [*10 marks*] Compute the impulse response $h(t)$ of the state-space system.

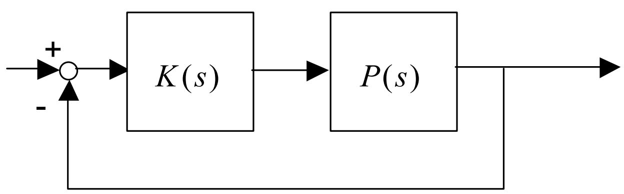
Answer:
Impulse response:

$$h(t) = Ce^{At}Bq(t) = \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}3e^{-t}-2e^{-10t} & -e^{-t}+e^{-10t} \\ 6e^{-t}-6e^{-10t} & -2e^{-t}+3e^{-10t}\end{bmatrix}\begin{bmatrix}0 \\ -1\end{bmatrix}$$

$$= \begin{bmatrix}1 & 1\end{bmatrix}\begin{bmatrix}e^{-t}-e^{-10t} \\ 2e^{-t}-3e^{-10t}\end{bmatrix} = (3e^{-t}-4e^{-10t})q(t)$$

where $q(t)$ is the unit step.

(c) [*12 marks*] Find the transfer function $P(s)$ of the above state-space system which is the plant in the feedback control system shown below. Suppose that a pure gain controller $K(s)=1$ is used. Sketch the Bode plot abd the Nyquist plot of the loop gain. Assess the stability of the unity feedback control system using the Nyquist criterion and justify your conclusion. Find the phase and gain margins of the closed-loop system. Determine whether the closed-loop system would be stable with the controller $K(s)=100$.



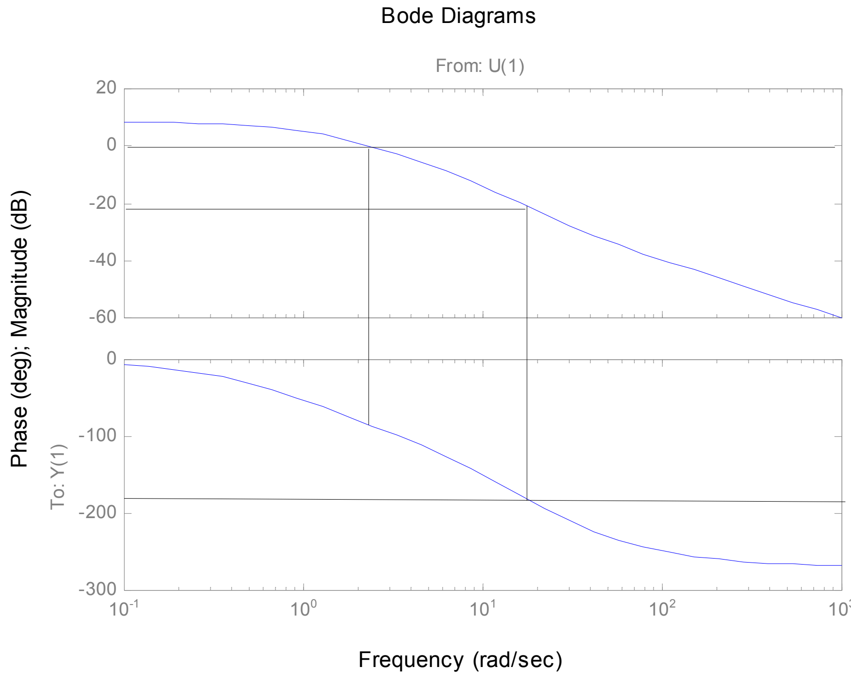
Answer:

Taking the Laplace transform of the impulse response, we obtain:

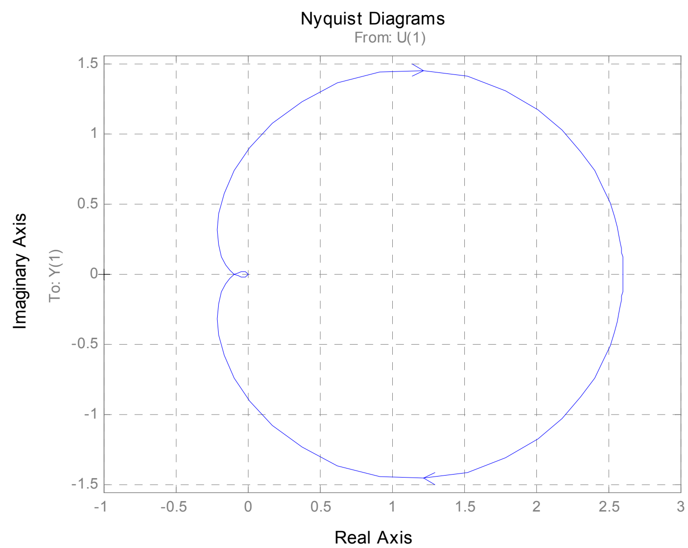
$$P(s) = \frac{3}{\underbrace{s+1}_{\mathrm{Re}\{s\}>-1}} - \frac{4}{\underbrace{s+10}_{\mathrm{Re}\{s\}>-10}} = \frac{3s+30-4s-4}{s^2+11s+10} = \frac{-s+26}{(s+1)(s+10)},\ \mathrm{Re}\{s\}>-1$$

Loop gain: $L(s) = \dfrac{-s+26}{(s+1)(s+10)},\ \mathrm{Re}\{s\}>-1$

From the Nyquist plot below we can see that this feedback system is stable. The Bode plot shows a crossover frequency of $\omega_{co} \approx 2.3\text{rd/s}$, and the phase margin is found: $\phi_m \approx 180-85^\circ = 95^\circ$. The gain margin is found to be $k_m \approx 20\text{dB}$ at frequency $\omega_{-\pi} = 18\text{rd/s}$. Hence, a controller gain of 100 (40dB) would destabilize the system.



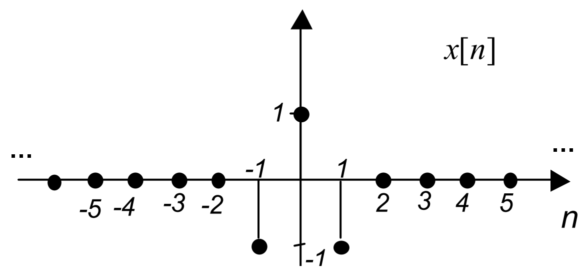
Nyquist plot:



We have no open-loop pole encircled by the Nyquist contour. Therefore, the Nyquist plot should not encircle the critical point –1, and it does not. Therefore, the closed-loop system is stable.

Problem 2 (30 marks)

(a) [*10 marks*] Compute the Fourier transform $X(e^{j\omega})$ of the signal $x[n]$ shown below and sketch its magnitude and phase over the interval $[-\pi,\pi]$.



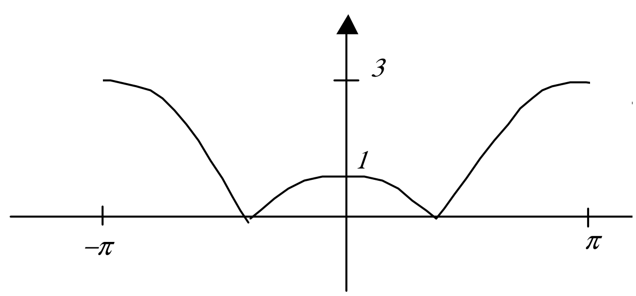
Answer:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = 1 - e^{j\omega} - e^{-j\omega} = 1 - 2\cos(\omega)$$

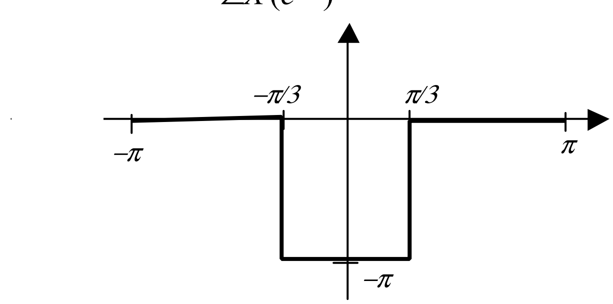
$$\left|X(e^{j\omega})\right| = \left|[1 - 2\cos(\omega)]\right|$$

$$\angle X(e^{j\omega}) = \begin{cases} -\pi, & |\omega| \le \dfrac{\pi}{3} \\ 0 & \dfrac{\pi}{3} < |\omega| \le \pi \end{cases}$$

$\left|X(e^{j\omega})\right|$



$\angle X(e^{j\omega})$



(b) Find the DTFT $W(e^{j\omega})$ of the periodic signal $w[n] = x[n-1] * \sum_{k=-\infty}^{+\infty} \delta[n-k6]$ and the coefficients of its Fourier series. Write the signal $w[n]$ as a Fourier series and compute its values at $n = 0,1,2$.

Answer:
First consider the DTFT of the signal in (a): $X(e^{j\omega}) = 1 - 2\cos(\omega)$

$$\begin{aligned}
W(e^{j\omega}) &= e^{-j\omega} X(e^{j\omega}) \frac{2\pi}{3} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{3}) \\
&= e^{-j\omega}[1-2\cos(\omega)] \frac{2\pi}{3} \sum_{k=-\infty}^{\infty} \delta(\omega - k\frac{2\pi}{3}) \\
&= \frac{2\pi}{3} \sum_{k=-\infty}^{\infty} e^{-jk\frac{2\pi}{3}} [1 - 2\cos(k\frac{2\pi}{3})] \delta(\omega - k\frac{2\pi}{3})
\end{aligned}$$

The spectral coefficients are simply the impulse areas divided by 2π :

$$a_k = \frac{1}{3} e^{-jk\frac{2\pi}{3}} [1 - 2\cos(k\frac{2\pi}{3})], \quad k = 0,1,2$$

$$a_0 = -\frac{1}{3},\ a_1 = \frac{2}{3} e^{-j\frac{2\pi}{3}},\ a_2 = \frac{2}{3} e^{-j\frac{4\pi}{3}}$$

Fourier series:

$$w[n] = \sum_{k=0}^{2} a_k e^{jk\frac{2\pi}{3}n} = -\frac{1}{3} + \frac{2}{3} e^{-j\frac{2\pi}{3}} e^{j\frac{2\pi}{3}n} + \frac{2}{3} e^{-j\frac{4\pi}{3}} e^{j\frac{4\pi}{3}n}$$

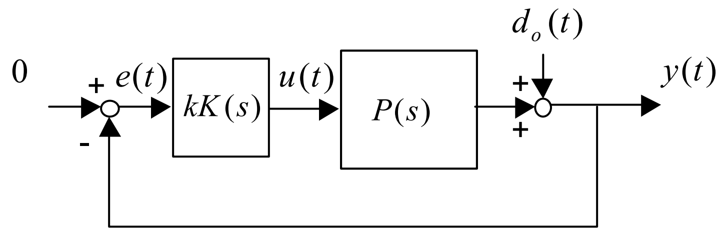
$$\begin{aligned}
w[0] &= -\frac{1}{3} + \frac{2}{3} e^{-j\frac{2\pi}{3}} + \frac{2}{3} e^{-j\frac{4\pi}{3}} = -\frac{1}{3} + \frac{2}{3}(\cos\frac{2\pi}{3} - j\sin\frac{2\pi}{3}) + \frac{2}{3}(\cos\frac{4\pi}{3} - j\sin\frac{4\pi}{3}) \\
&= -\frac{1}{3} + \frac{2}{3}(-\frac{1}{2} - j\frac{\sqrt{3}}{2}) + \frac{2}{3}(-\frac{1}{2} + j\frac{\sqrt{3}}{2}) = -1
\end{aligned}$$

$$w[1] = -\frac{1}{3} + \frac{2}{3} e^{-j\frac{2\pi}{3}} e^{j\frac{2\pi}{3}} + \frac{2}{3} e^{j-\frac{4\pi}{3}} e^{j\frac{4\pi}{3}} = -\frac{1}{3} + \frac{2}{3} + \frac{2}{3} = 1$$

$$\begin{aligned}
w[2] &= -\frac{1}{3} + \frac{2}{3} e^{-j\frac{2\pi}{3}} e^{j\frac{4\pi}{3}} + \frac{2}{3} e^{j-\frac{4\pi}{3}} e^{j\frac{8\pi}{3}} = -\frac{1}{3} + \frac{2}{3} e^{j\frac{2\pi}{3}} + \frac{2}{3} e^{j\frac{4\pi}{3}} \\
&= -\frac{1}{3} + \frac{2}{3}(\cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3}) + \frac{2}{3}(\cos\frac{4\pi}{3} + j\sin\frac{4\pi}{3}) \\
&= -\frac{1}{3} + \frac{2}{3}(-\frac{1}{2} + j\frac{\sqrt{3}}{2}) + \frac{2}{3}(-\frac{1}{2} - j\frac{\sqrt{3}}{2}) = -1
\end{aligned}$$

Problem 3 (25 marks)

Consider the LTI unity feedback regulator



Where $P(s) = \dfrac{s-1}{s^2+6s+18}$, $K(s) = \dfrac{s+3}{s+5}$ and $k \in [0, +\infty)$.

(a) [*8 marks*] Let $k = 1$. Use any of the four theorems to assess the stability of the closed-loop system. (hint: there is a closed-loop pole at –6.197.)

Answer:

<u>Using Theorem I</u>:

We have to show that $T(s)$, $P^{-1}(s)T(s)$ and $P(s)S(s)$ are all stable.

$$T(s) = \frac{P(s)K(s)}{1+P(s)K(s)} = \frac{\dfrac{(s-1)(s+3)}{(s^2+6s+18)(s+5)}}{1+\dfrac{(s-1)(s+3)}{(s^2+6s+18)(s+5)}} = \frac{(s-1)(s+3)}{s^3+12s^2+50s+87}$$

Using the hint, we have:

$$\begin{aligned} s^3+12s^2+50s+87 &= (s^2+as+b)(s+6.197) \\ &= s^3+(6.197+a)s^2+(6.197a+b)s+6.197b \end{aligned}$$

From which we find the coefficients $a = 5.803,\ b = 14.039$ leading to the remaining two closed-loop poles: -2.901+j2.371, -2.901-j2.371 which are stable.

is stable (all three poles in LHP, strictly proper)

poles are: -6.197, -2.1, -0.2

$$P^{-1}(s)T(s) = \frac{\dfrac{10(0.5s+1)}{0.1s+1}}{1+\dfrac{10(4s+1)(0.5s+1)}{(s-1)(0.1s+1)^2}} = \frac{(0.1s^2+0.9s-1)(5s+10)}{0.01s^3+20.19s^2+45.8s+9}$$

is also stable (same poles as above, proper)

$$P(s)S(s)=\frac{P(s)}{1+P(s)K(s)}=\frac{\dfrac{4s+1}{(s-1)(0.1s+1)}}{1+\dfrac{10(4s+1)(0.5s+1)}{(s-1)(0.1s+1)^2}}=\frac{0.4s^2+4.1s+1}{0.01s^3+20.19s^2+45.8s+9}$$

is stable (same poles as above, strictly proper)

Therefore the feedback system is stable.

<u>Using Theorem II</u>:

We have to show that either $T(s)$ or $S(s)$ is stable, and that no pole-zero cancellation occurs in the closed RHP in forming the loop gain. The latter condition holds, but

$$T(s)=\frac{P(s)K(s)}{1+P(s)K(s)}=\frac{\dfrac{10(4s+1)(0.5s+1)}{(s-1)(0.1s+1)^2}}{1+\dfrac{10(4s+1)(0.5s+1)}{(s-1)(0.1s+1)^2}}=\frac{20s^2+45s+10}{0.01s^3+20.19s^2+45.8s+9}$$

is stable (all three poles in LHP, strictly proper)

poles are: -2017, -2.1, -0.2

Therefore the feedback system is stable.

<u>Using Theorem III</u>:

We have to show that the closed-loop poles, i.e., the zeros of the characteristic polynomial $p(s)$, are all in the open LHP. The plant and the controller are already expressed as ratios of coprime polynomials.

$$p(s)=n_K n_P+d_K d_P=0.01s^3+20.19s^2+45.8s+9$$

All three closed-loop poles -2017, -2.1, -0.2 lie in the open LHP and therefore the feedback system is stable.

<u>Using Theorem IV</u>:

We have to show that $1+K(s)P(s)$ has no zero in the closed RHP, and that no pole-zero cancellation occurs in the closed RHP in forming the loop gain. The latter condition obviously holds, and

$$1+K(s)P(s)=\frac{1}{S(s)}=1+\frac{10(4s+1)(0.5s+1)}{(s-1)(0.1s+1)^2}=\frac{0.01s^3+20.19s^2+45.8s+9}{0.01s^3+0.19s^2+0.8s-1}$$

All three zeros of this TF -2017, -2.1, -0.2 lie in the open LHP and therefore the feedback system is stable.

(b) [*12 marks*] Sketch the root locus of this feedback control system for $k \in [0, +\infty)$.

Answer:

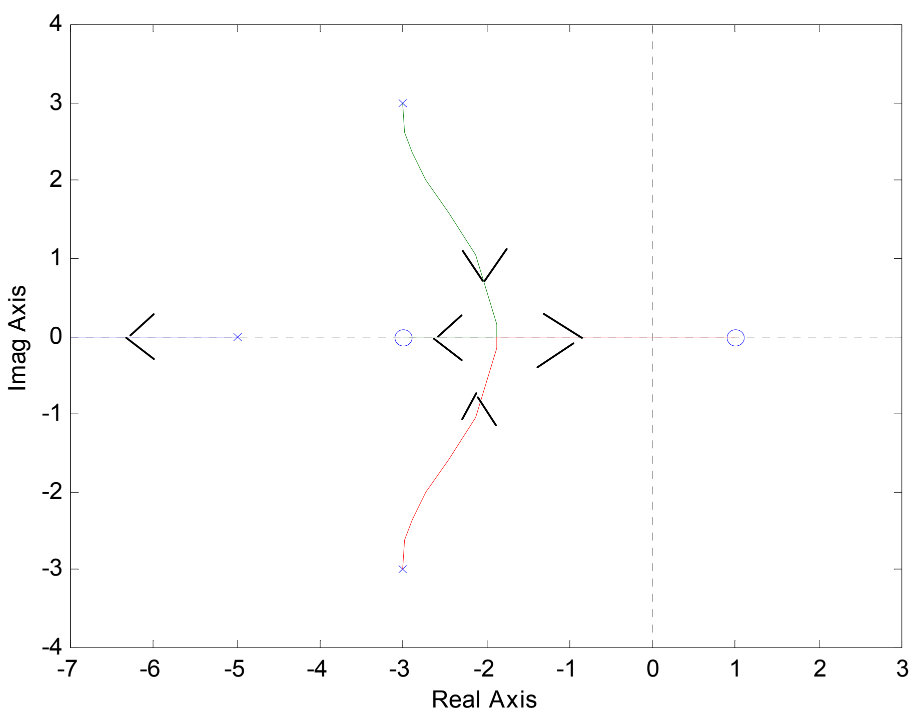
the loop gain is $L(s) = P(s)K(s) = \dfrac{k(s-1)(s+3)}{(s^2+6s+18)(s+5)}$.

- The root locus starts at the (open-loop) poles of $L(s)$: $-3 \pm j3, -5$ for $k = 0$ and it ends at the zeros of $L(s)$: $1, -3, \infty$ for $k = +\infty$.
- On the real line, the root locus will have one segment between the zero 1 and the pole –3, and a second segment between the pole at -5 and $-\infty$. (this is our asymptote going to infinity) (Rule 4)
- For the one branch of the root locus going to infinity, the asymptote is described by

$$\text{Centre of asymptotes} = \frac{\sum \text{poles of } L(s) - \sum \text{zeros of } L(s)}{\nu - \mu} = \frac{-1-2-3-(1-4)}{1} = -3$$

$$\text{Angle of asymptote} = \frac{2k+1}{\nu-\mu}\pi, \quad k = 0 = \pi$$

Root locus:



(c) [*5 marks*] Find the value of the gain k for which the system becomes unstable.

Answer:

The only branch going into the RHP is on the real line, hence it crosses the imaginary axis at the origin: $p(0)=0$

$$\begin{aligned} p(s) &= (s^2+6s+18)(s+5)+k(s-1)(s+3) \\ &= s^3+11s^2+48s+90+k(s^2+2s-3) \\ &= s^3+(11+k)s^2+(48+2k)s+90-3k \end{aligned}$$

$$p(0)=0 \Rightarrow \quad 90-3k=0 \Leftrightarrow k=30$$