Sample Midterm Test 1 (mt1s01)
Covering Chapters 10-13 of *Fundamentals of Signals & Systems*

Problem 1 (25 marks)

Consider the causal LTI state-space system with initial state $x(t_0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$:

$$\dot{x}(t) = Ax(t) + Bu(t),$$
$$y(t) = Cx(t) + Du(t)$$

where $x(t) \in \mathbb{R}^2$, $u(t) \in \mathbb{R}$, $y(t) \in \mathbb{R}$, $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 2 \end{bmatrix}$, $D = 0$

(a) [*5 marks*] Find the state transition matrix $\Phi(t, t_0) = e^{A(t-t_0)}$.

Answer:
We have to diagonalize the *A*-matrix first
The eigenvalues of the A matrix are computed:

$$\det(\lambda I - A) = 0$$
$$\det \begin{bmatrix} \lambda + 1 & -1 \\ 1 & \lambda + 1 \end{bmatrix} = \lambda^2 + 2\lambda + 2 = 0$$
$$\Rightarrow \lambda_1 = -1 + j, \lambda_2 = -1 - j$$

Next, we compute the eigenvectors of A:
Eigenvector v_1 corresponding to $\lambda_1 = -1 + j$:

$$\begin{bmatrix} j & -1 \\ 1 & j \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow v_{11} = 1, v_{12} = j$$

Eigenvector v_2 corresponding to $\lambda_2 = -1 - j$:

$$\begin{bmatrix} -j & -1 \\ 1 & -j \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow v_{21} = 1, v_{22} = -j$$

Diagonalizing matrix $T = [v_1 \quad v_2] = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$

$$\Phi(t,t_0) = e^{A(t-t_0)} = T\text{diag}\{e^{(-1+j)(t-t_0)}, e^{(-1-j)(t-t_0)}\}T^{-1}$$

$$= \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{(-1+j)(t-t_0)} & 0 \\ 0 & e^{(-1-j)(t-t_0)} \end{bmatrix} \frac{j}{2} \begin{bmatrix} -j & -1 \\ -j & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} \frac{1}{2}e^{(-1+j)(t-t_0)} & -\frac{j}{2}e^{(-1+j)(t-t_0)} \\ \frac{1}{2}e^{(-1-j)(t-t_0)} & \frac{j}{2}e^{(-1-j)(t-t_0)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2}e^{(-1+j)(t-t_0)} + \frac{1}{2}e^{(-1-j)(t-t_0)} & \frac{1}{2j}e^{(-1+j)(t-t_0)} - \frac{1}{2j}e^{(-1-j)(t-t_0)} \\ -\frac{1}{2j}e^{(-1+j)(t-t_0)} + \frac{1}{2j}e^{(-1-j)(t-t_0)} & \frac{1}{2}e^{(-1+j)(t-t_0)} + \frac{1}{2}e^{(-1-j)(t-t_0)} \end{bmatrix}$$

$$= \begin{bmatrix} e^{-(t-t_0)}\cos(t-t_0) & e^{-(t-t_0)}\sin(t-t_0) \\ -e^{-(t-t_0)}\sin(t-t_0) & e^{-(t-t_0)}\cos(t-t_0) \end{bmatrix}$$

(b) [*10 marks*] Write down the general formula for the response of the system $y(t)$, identifying the zero-state response and the zero-input response. Then, compute the response $y(t)$ for the above system with $t_0 = 1$ and $u(t) = \delta(t-1)$.

Answer:

General response:

$$y(t) = \underbrace{Ce^{A(t-t_0)}x(t_0)q(t-t_0)}_{y_{zi}(t)} + \underbrace{\int_{t_0}^{t} Ce^{A(\tau-t_0)}Bu(t-\tau)d\tau + Du(t)}_{y_{zs}(t)}$$

where $q(t)$ is the unit step.

Specific case: $t_0 = 1$ and $u(t) = \delta(t-1)$

Time-domain solution:

$$y(t) = \underbrace{Ce^{A(t-1)}x(1)q(t-1)}_{y_{zi}(t)} + \underbrace{\int_{1}^{t} Ce^{A\tau}Bu(t-\tau)d\tau + Du(t)}_{y_{zs}(t)}$$

$$= \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} e^{-(t-1)}\cos(t-1) & e^{-(t-1)}\sin(t-1) \\ -e^{-(t-1)}\sin(t-1) & e^{-(t-1)}\cos(t-1) \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} q(t-1)$$

$$+\int_{1}^{t} \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} e^{-\tau}\cos(\tau) & e^{-\tau}\sin(\tau) \\ -e^{-\tau}\sin(\tau) & e^{-\tau}\cos(\tau) \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \delta(t-\tau-1)d\tau$$

$$= 2e^{-(t-1)}\left[\sin(t-1) + \cos(t-1)\right]q(t-1) - 4e^{-(t-1)}\sin(t-1)q(t-1)$$

$$= 2e^{-(t-1)}\left[-\sin(t-1) + \cos(t-1)\right]q(t-1)$$

Laplace-domain solution also acceptable:

$$\mathcal{Y}(s) = \left[C(sI_n - A)^{-1} B + D \right] \mathcal{U}(s) + C(sI_n - A)^{-1} x_0 e^{-s}$$

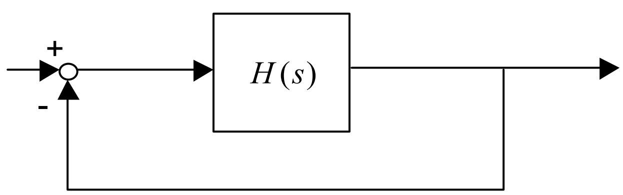
we have $\mathcal{U}(s) = e^{-s}$, thus

$$\begin{aligned}
\mathcal{Y}(s) &= \left(\begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right) e^{-s} + \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-s} \\
&= \frac{1}{s^2+2s+2} \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} e^{-s} + \frac{1}{s^2+2s+2} \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-s} \\
&= \frac{-4e^{-s}}{s^2+2s+2} + \frac{(2s+4)e^{-s}}{s^2+2s+2} = \frac{2(s+1)e^{-s}}{s^2+2s+2} - \frac{2e^{-s}}{s^2+2s+2}, \ \mathrm{Re}\{s\} > -1
\end{aligned}$$

and the inverse Laplace transform is

$$y(t) = 2e^{-(t-1)} \left[-\sin(t-1) + \cos(t-1) \right] q(t-1) .$$

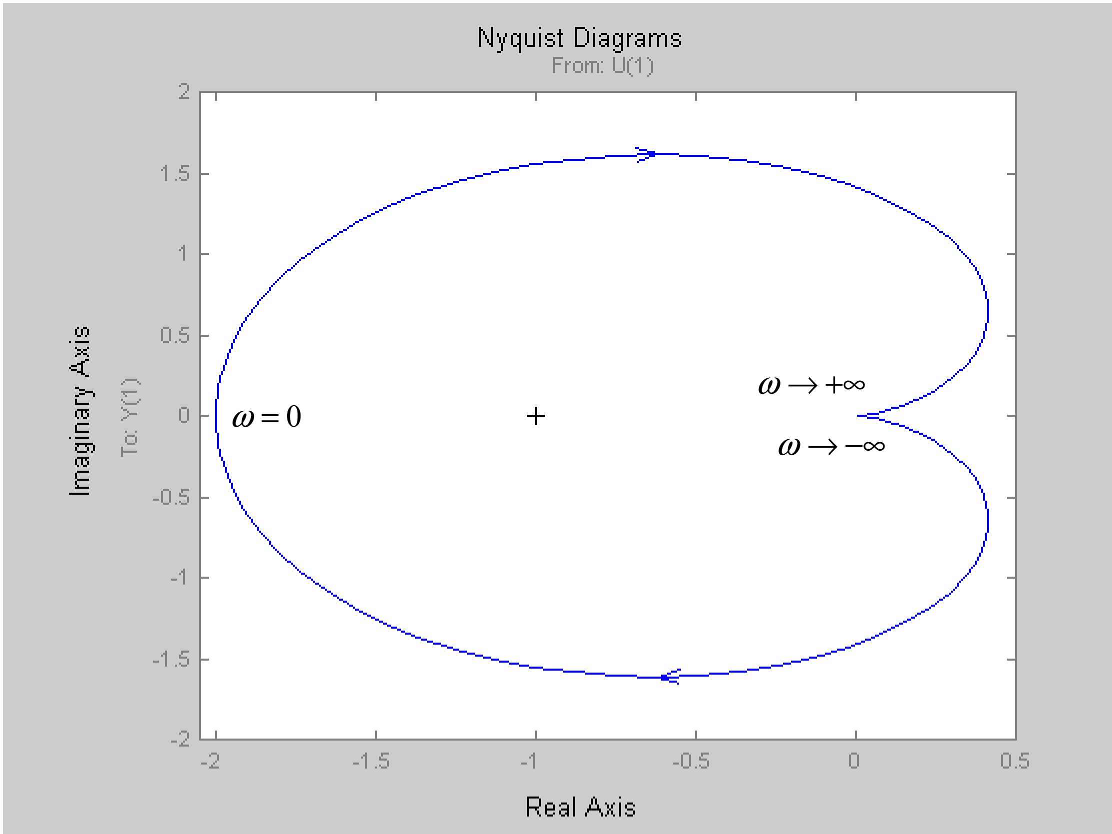
(c) [*10 marks*] Find the transfer function $H(s)$ of the state-space system. Sketch its Nyquist plot. Assess the stability of the unity feedback control system shown below.



Answer:

$$\begin{aligned}
H(s) &= C(sI_n - A)^{-1} B + D \\
&= \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
&= \frac{1}{s^2+2s+2} \begin{bmatrix} 0 & 2 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} \\
&= \frac{-4}{s^2+2s+2}, \ \mathrm{Re}\{s\} > -1
\end{aligned}$$

Nyquist plot

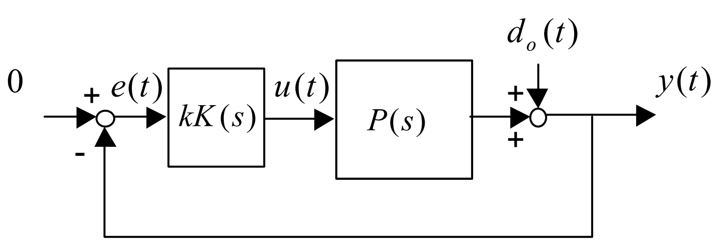


The Nyquist plot encircles the critical point -1 once, but H(s) has no RHP pole.

Therefore, the closed-loop system is unstable.

Problem 2 (25 marks)

Consider the LTI unity feedback regulator



Where $P(s)=\dfrac{s-10}{s^2+10s+50}$, and $K(s)$ is given in the form of a differential system: $\dfrac{dy}{dt}+100y=\dfrac{dx}{dt}+10x$.

(a) [*5 marks*] Compute the sensitivity function $S(s)$ and the complementary sensitivity function $T(s)$ with $k = 10$.

Answer:

Sensitivity:

$$S(s) = \frac{1}{1+10K(s)P(s)} = \frac{1}{1+10\left(\dfrac{s+10}{s+100}\right)\left(\dfrac{s-10}{s^2+10s+50}\right)}$$
$$= \frac{s^3+110s^2+1050s+5000}{s^3+110s^2+1050s+5000+10s^2-1000}$$
$$\frac{s^3+110s^2+1050s+5000}{s^3+120s^2+1050s+4000}$$

Complementary sensitivity

$$T(s) = 1 - S(s) = 1 - \frac{s^3+110s^2+1050s+5000}{s^3+120s^2+1050s+4000} = \frac{10s^2-1000}{s^3+120s^2+1050s+4000}$$

(b) [*5 marks*] Give the steady-state error in the output for a step disturbance $d_o(t) = 100u(t)$.

Answer:

Steady-state error is $100S(0) = 100\left.\dfrac{s^3+110s^2+1050s+5000}{s^3+120s^2+1050s+4000}\right|_{s=0} = \dfrac{500}{4} = 125$

(c) [*10 marks*] Sketch the root locus of this feedback control system for $k \in [0, +\infty)$.

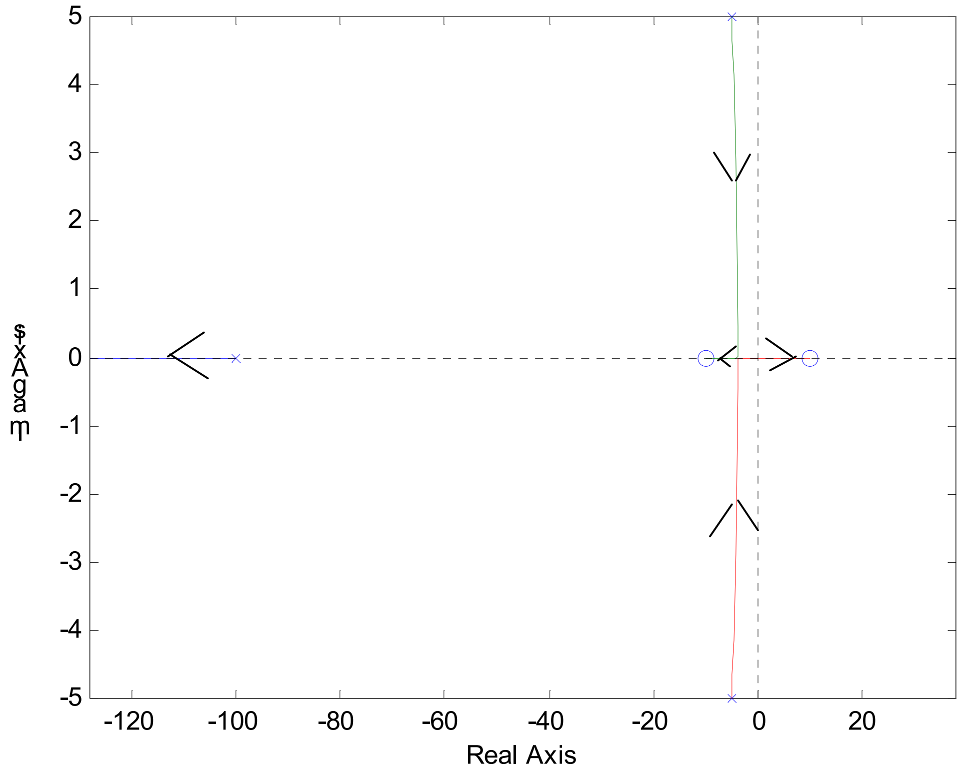
The closed-loop poles are the zeros of the characteristic polynomial:

$$\begin{aligned} p(s) &= (s+100)(s^2+10s+50) + k(s+10)(s-10) \\ &= s^3+110s^2+1050s+5000+ks^2-k100 \\ &= s^3+(110+k)s^2+1050s+(5000-k100) \end{aligned}.$$

the loop gain is $L(s) = kP(s)K(s) = k\left(\dfrac{s+10}{s+100}\right)\left(\dfrac{s-10}{s^2+10s+50}\right)$.

- The root locus starts at the (open-loop) poles of $L(s)$: $-100, -5\pm 5j$ for $k = 0$ and it ends at the zeros of $L(s)$: $10, -10, \infty$ for $k = +\infty$.
- On the real line, the root locus will have one branch between the zero -10 and the zero +10 (Rule 4) and to the right of the pole at -100 extending to infinity.

Root locus:



(d) [*5 marks*] Find the value of the gain k for which the system becomes unstable.

Answer:

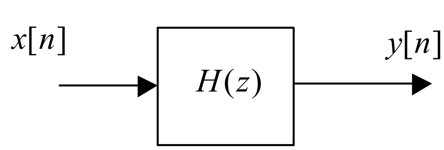
The only branch going into the RHP is on the real line, hence it crosses the imaginary axis at the origin: $p(0)=0$

$$p(s)=s^3+(110+k)s^2+1050s+(5000-k100)$$

$$p(0)=0 \Rightarrow \quad 5000-k100=0 \Leftrightarrow k=50$$

Problem 3 (20 marks)

Consider the DLTI system



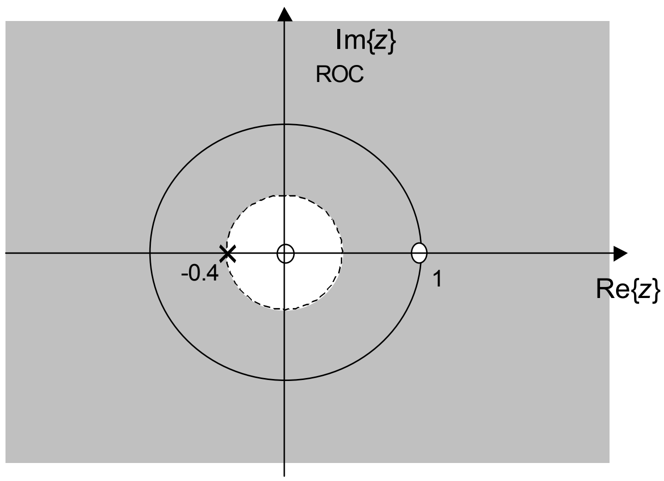
with transfer function $H(z)=\dfrac{z-1}{z^{-1}(z+0.4)}$.

(a) [5 marks] Sketch the pole-zero plot of the system. Find the ROC that makes this system stable and indicate it on the pole-zero plot.

Answer:

$$H(z) = \frac{z-1}{z^{-1}(z+0.4)} = \frac{z(z-1)}{z+0.4}.$$

The poles and zero are $p_1 = -0.4,\ z_1 = 0,\ z_2 = 1$.
Pole-zero plot:



ROC is: $|z| > 0.4 / \{\infty\}$

(b) [5 marks] Is the system causal with the ROC that you found in (b)? Justify your answer.

Answer:
No, the system is not causal because the ROC is not the exterior of a disk extending to infinity.

(c) [10 marks] Suppose that $H(e^{j\omega})$ is bounded for all frequencies. Calculate the response of the system $y[n]$ to the input $x[n] = \delta[n-2]$.
Answer:

$$X(z) = z^{-2},\ |z| > 0,$$

so the z-transform of the output is

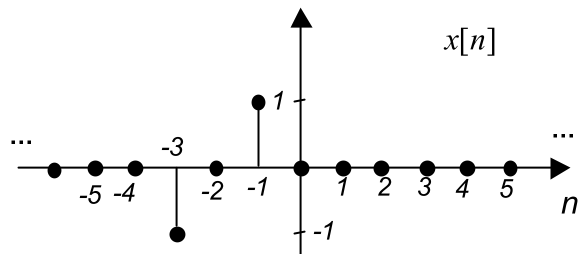
$$Y(z) = \frac{z^{-2}(z-1)}{z^{-1}(z+0.4)} = \frac{z-1}{z(z+0.4)} = \frac{z^{-1} - z^{-2}}{(1+0.4z^{-1})},\ 0.4 < |z|$$

Using the table, we find

$$y[n] = (-0.4)^{n-1} u[n-1] - (-0.4)^{n-2} u[n-2].$$

Problem 4 (20 marks)

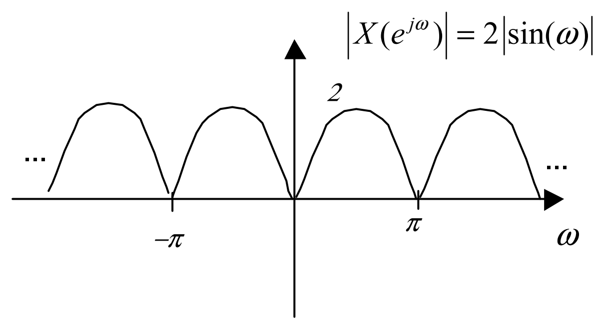
(a) [*8 marks*] Compute the Fourier transform $X(e^{j\omega})$ of the signal $x[n]$ shown below and sketch its magnitude and phase over the interval $[-\pi, \pi]$.

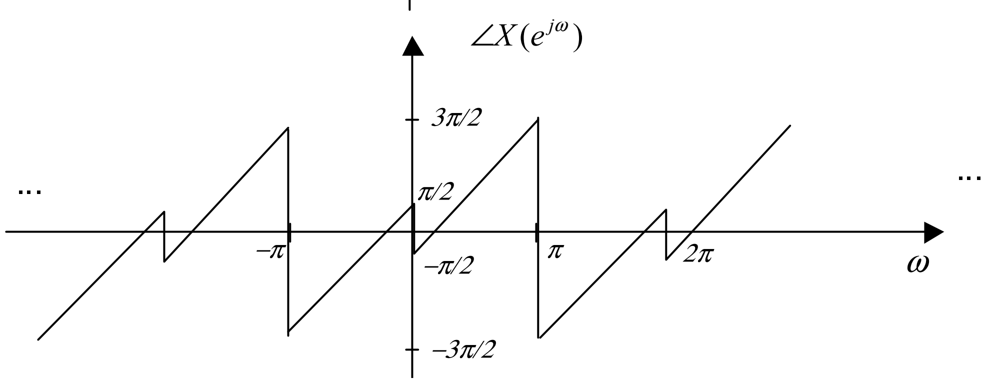


Answer:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = -e^{j3\omega} + e^{j\omega} = e^{j2\omega}\left(-e^{j\omega} + e^{-j\omega}\right)$$

$$= -2je^{j2\omega}\sin(\omega) = 2e^{j(2\omega - \frac{\pi}{2})}\sin(\omega)$$





(b) [*6 marks*] Find the DTFT $W(e^{j\omega})$ of the periodic signal $w[n]=\sum_{k=-\infty}^{+\infty} x[n-2-k4]$.

Answer:

First consider the DTFT of the signal in Problem 4(a): $X(e^{j\omega})=2e^{j(2\omega-\frac{\pi}{2})}\sin(\omega)$

$$\begin{aligned}
W(e^{j\omega}) &= e^{-j2\omega}X(e^{j\omega})\frac{2\pi}{4}\sum_{k=-\infty}^{\infty}\delta(\omega-k\frac{2\pi}{4})\\
&= 2e^{-j\frac{\pi}{2}}\sin(\omega)\frac{2\pi}{4}\sum_{k=-\infty}^{\infty}\delta(\omega-k\frac{2\pi}{4})\\
&= -j\pi\sum_{k=-\infty}^{\infty}\sin(k\frac{2\pi}{4})\delta(\omega-k\frac{2\pi}{4})\\
&= -j\pi\sum_{k=-\infty}^{\infty}\sin(k\frac{\pi}{2})\delta(\omega-k\frac{2\pi}{4})
\end{aligned}$$

(c) [*6 marks*] Suppose the periodic signal $w[n]$ is upsampled to obtain $y[n]=w_{(2)}[n]$. Sketch the magnitude of its Fourier series coefficients.

Answer:

The DTFS coefficients of $y[n]$ are identified by first using the above expression:

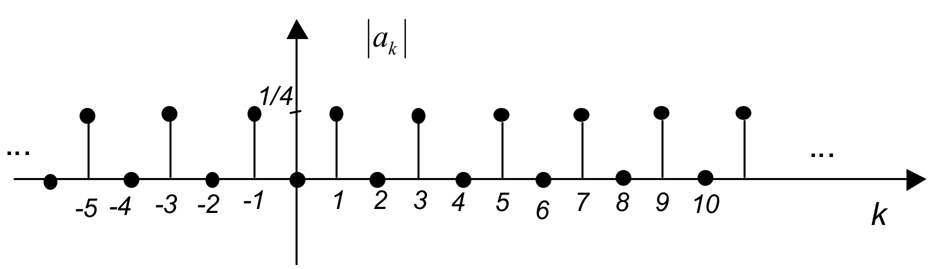
$$W(e^{j\omega})=-j\pi\sum_{k=-\infty}^{\infty}\sin(k\frac{\pi}{2})\delta(\omega-k\frac{2\pi}{4})=2\pi\sum_{k=-\infty}^{\infty}a_k\delta(\omega-k\frac{2\pi}{4})$$

$$a_k=-\frac{j}{2}\sin(k\frac{\pi}{2}),\ \ k=0,1,2,3$$

$$a_0=0,\ a_1=-\frac{j}{2},\ a_2=0,\ a_3=\frac{j}{2}$$

The upsampling operation makes the fundamental period equal to 8, compresses the line spectrum around dc by a factor 2 and multiplies the coefficients by 1/2, and therefore the DTFS are:

$$a_0=0,\ a_1=-\frac{j}{4},\ a_2=0,\ a_3=\frac{j}{4},\ a_4=0,\ a_5=-\frac{j}{4},\ a_6=0,\ a_7=\frac{j}{4}$$



Problem 5 (10 marks)

Just answer *True* or *False.*

(a) The ROC of the transfer function of a discrete-time LTI system with a two-sided impulse response extending to $n = \pm\infty$ is either an open ring in the z-plane, or the empty set.

Answer: True

(b) Let $\{a_k\}$ be the Fourier series coefficients of the periodic signal $x[n] = (-1)^{n-1}$. Then $a_0 = 0,\ a_1 = -1,\ a_3 = -1$.

Answer: True.

(c) For stability of a feedback control system, the Nyquist plot of the loop gain $L(s)$ must encircle the critical point counterclockwise a number of times equal to the number of closed right half-plane zeros of $L(s)$.

Answer: False

(d) The DTFT of a real, odd signal can be expressed as $\frac{2}{j}\sum_{k=0}^{+\infty} x[n]\sin\omega n$.

Answer: True

(e) The transient component of the step response of a causal, stable discrete-time system contains only terms associated with the system's poles.

Answer: True