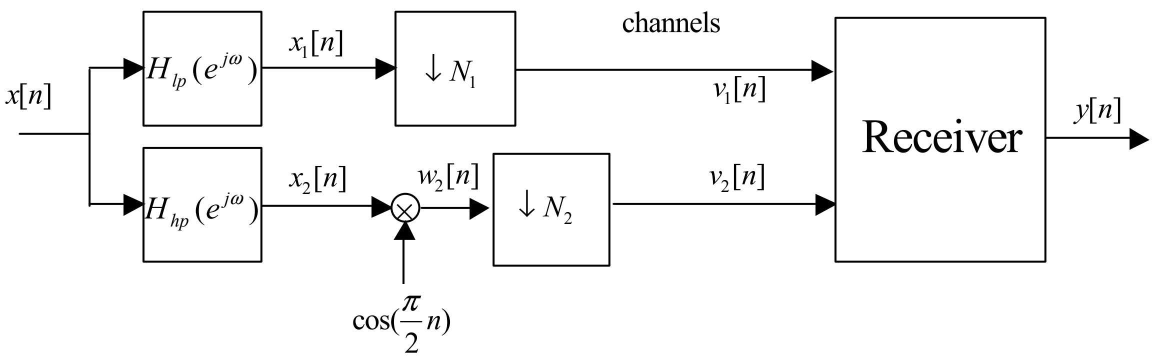
Sample Final Exam (finals03)
Covering Chapters 10-17 of *Fundamentals of Signals & Systems*

Problem 1 (25 marks)

Consider the discrete-time system shown below, where $\downarrow N$ represents decimation by N. This system transmits a signal $x[n]$ coming in at 1000 samples/s over two low-bit-rate channels.



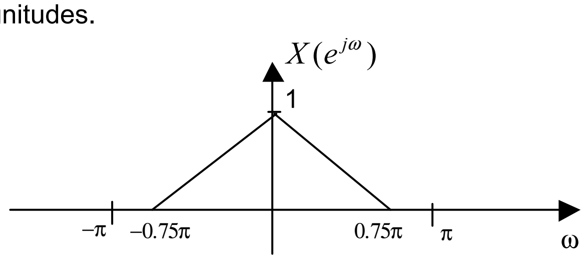
Numerical values:

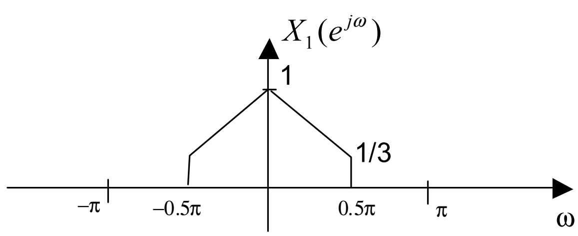
lowpass filters' cutoff frequencies $\omega_{clp1} = \omega_{clp2} = \dfrac{\pi}{2}$,

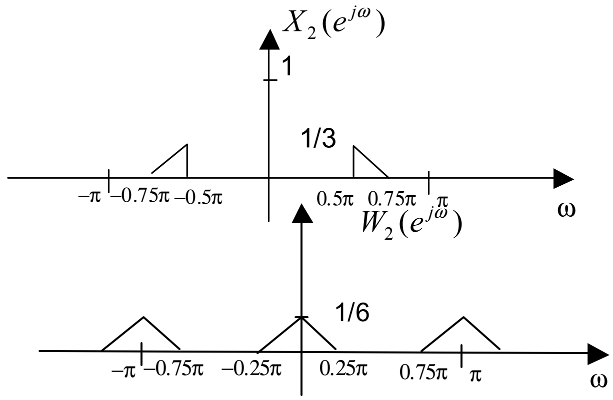
highpass filter's cutoff frequency $\omega_{chp} = \dfrac{\pi}{2}$,

signal's spectrum over $[-\pi,\pi]$: $X(e^{j\omega}) = \begin{cases} 1 - \dfrac{4}{3\pi}|\omega|, & |\omega| \le \dfrac{3\pi}{4} \\ 0, & \dfrac{3\pi}{4} < |\omega| < \pi \end{cases}$.

(a) [7 marks] Sketch the spectra $X(e^{j\omega})$, $X_1(e^{j\omega})$, $X_2(e^{j\omega})$, $W_2(e^{j\omega})$, indicating the important frequencies and magnitudes.

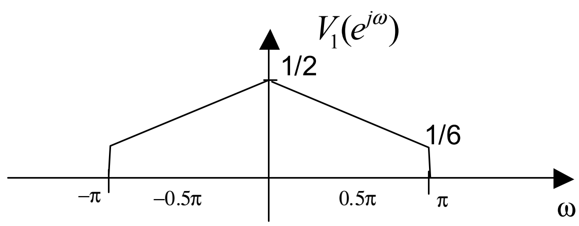




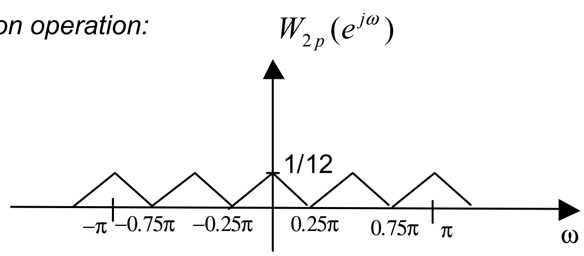


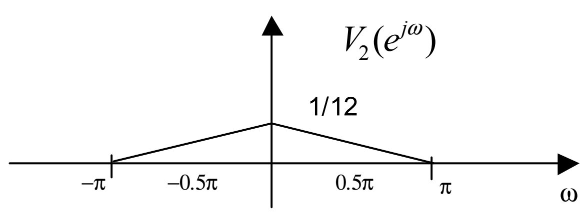
(b) [10 marks] Let the decimation factors be $N_1 = 2$ and $N_2 = 4$. Sketch the corresponding spectra $V_1(e^{j\omega})$, $X_2(e^{j\omega})$, $V_2(e^{j\omega})$, indicating the important frequencies and magnitudes. Assuming for the moment that $v_1[n], v_2[n]$ are quantized using 16-bit quantizers, find the bit rate of each channel, and the total bit rate. How would this compare to the bit rate for of a direct transmission of $x[n]$ using a 16-bit quantizer?

Answer:



After sampling in decimation operation:





With $N_1 = 2$, the first channel transmits at a bit rate of:

$$\frac{1000}{2}\text{samples/s}\times 16\text{bits/sample} = 8000 \text{ bits/s}$$

And with $N_2 = 4$, the second channel transmits at a bit rate of:

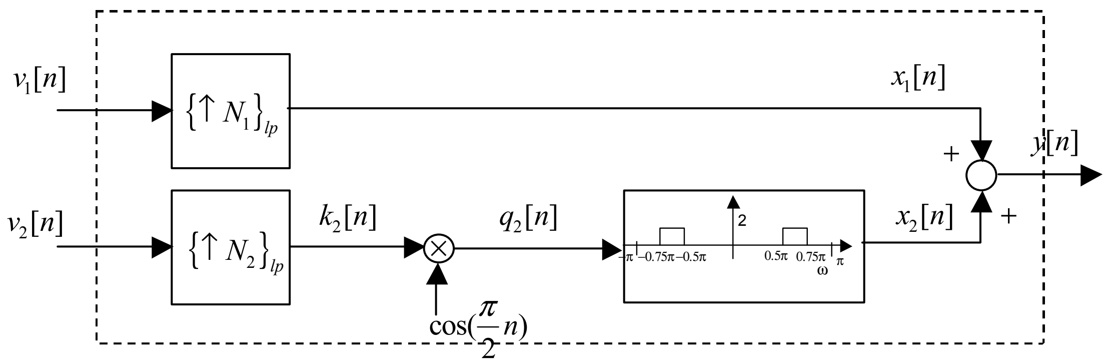
$$\frac{1000}{4}\text{samples/s}\times 16\text{bits/sample} = 4000 \text{ bits/s}$$

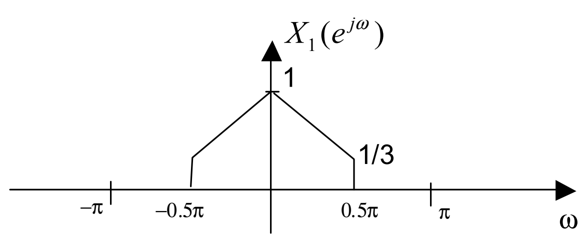
Thus, the total bit rate is 12000 bits/s.
A direct transmission of the signal would require a bit rate of

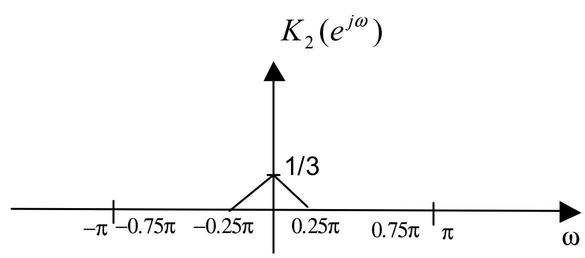
$$1000\text{samples/s}\times 16\text{bits/sample} = 16000 \text{ bits/s}$$

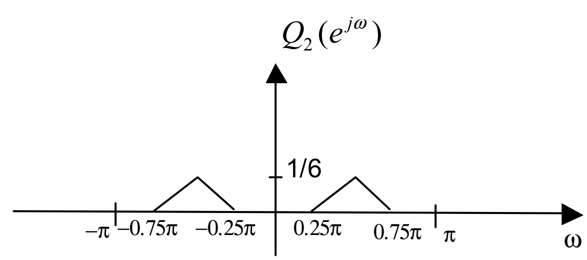
(c) [8 marks] Design the receiver system (draw its block diagram) such that $y[n] = x[n]$ (assume that there is no quantization of the signals.) You can use upsamplers (symbol $\{\uparrow N\}_{lp}$, with embedded ideal lowpass filters of cutoff frequency $\frac{\pi}{N}$ and gain N), synchronous demodulators, ideal filters and summing junctions. Sketch the spectra of all signals in your receiver.

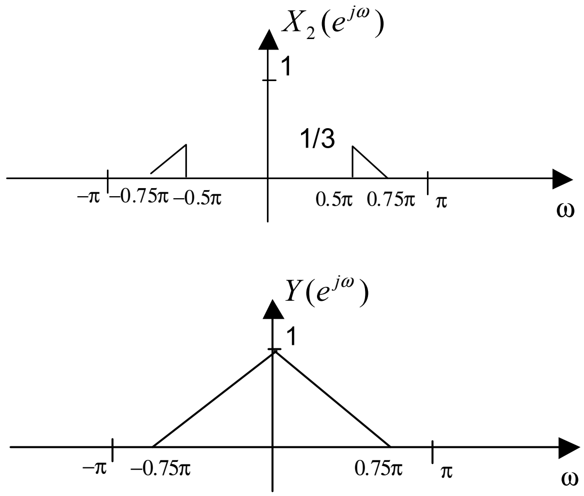
Answer:





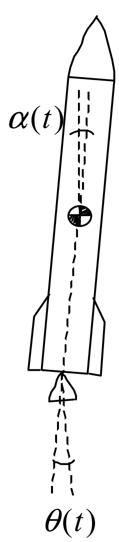


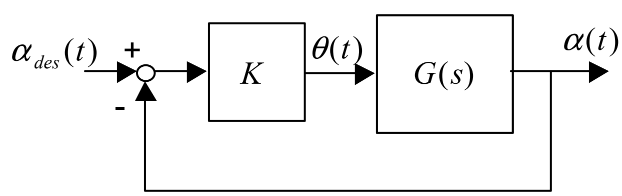




Problem 2 (20 marks)

You are the engineer in charge of the design of a rocket's guidance control system so that the rocket can track a desired pitch angle trajectory $\alpha_{des}(t)$ in a vertical plane during the take-off phase. The transfer function from the rocket's thrust vector angle command with respect to its longitudinal axis, call it $\theta(t)$, to the angle between the rocket's pitch angle $\alpha(t)$ (angle between the longitudinal axis and the inertial vertical axis), is given by $G(s) := \dfrac{\hat{\alpha}(s)}{\hat{\theta}(s)} = \dfrac{1}{s(\frac{1}{9}s^2 + \frac{2}{3}s + 1)}$.



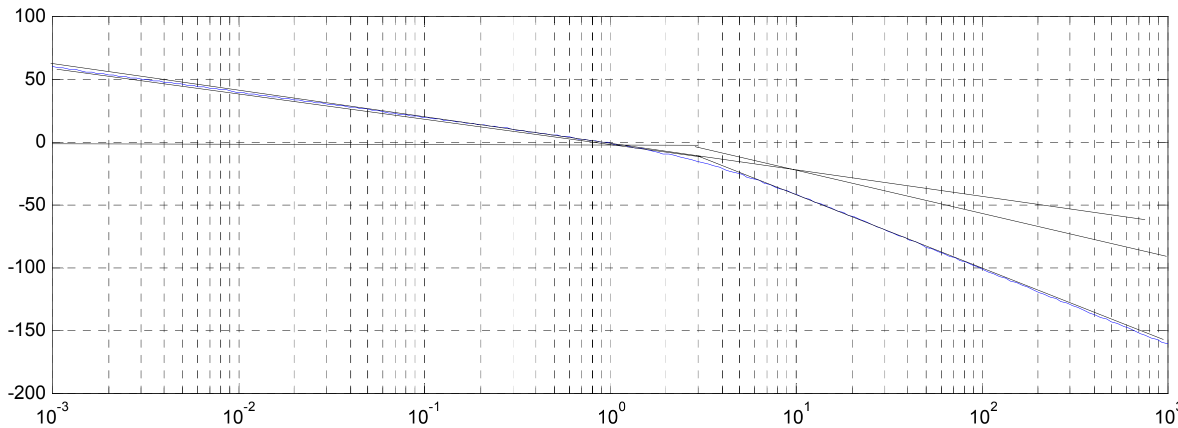


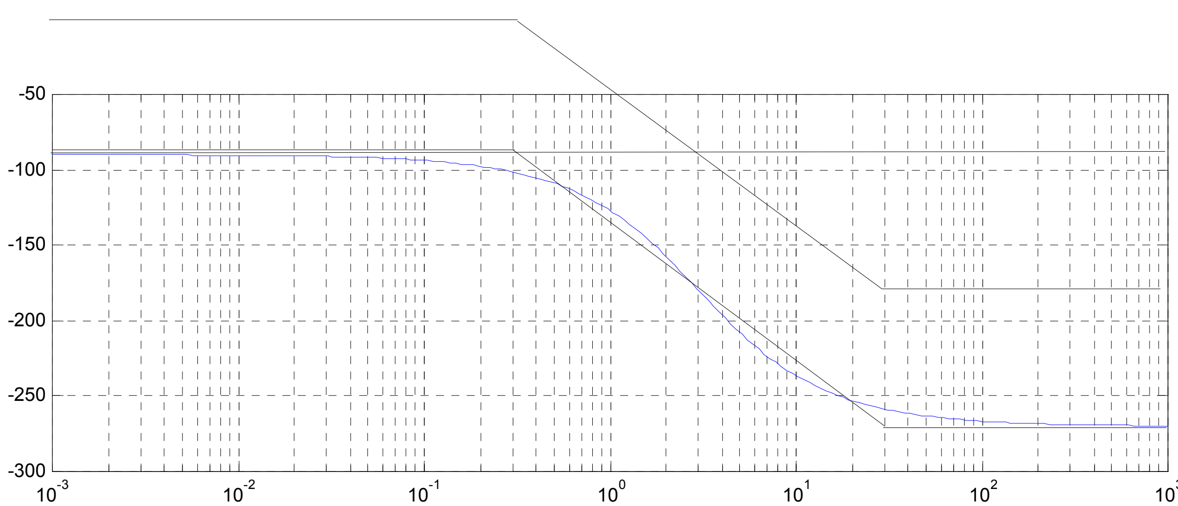
(a) [8 marks] Suppose that you decide to use a pure gain $K \in \mathbb{R}$ as a feedback controller. Sketch the Bode plot of the loop gain $L(s)$ and its Nyquist plot for $K = 1$, and use the Nyquist criterion to determine the range of $K > 0$ for which the rocket will be stable.

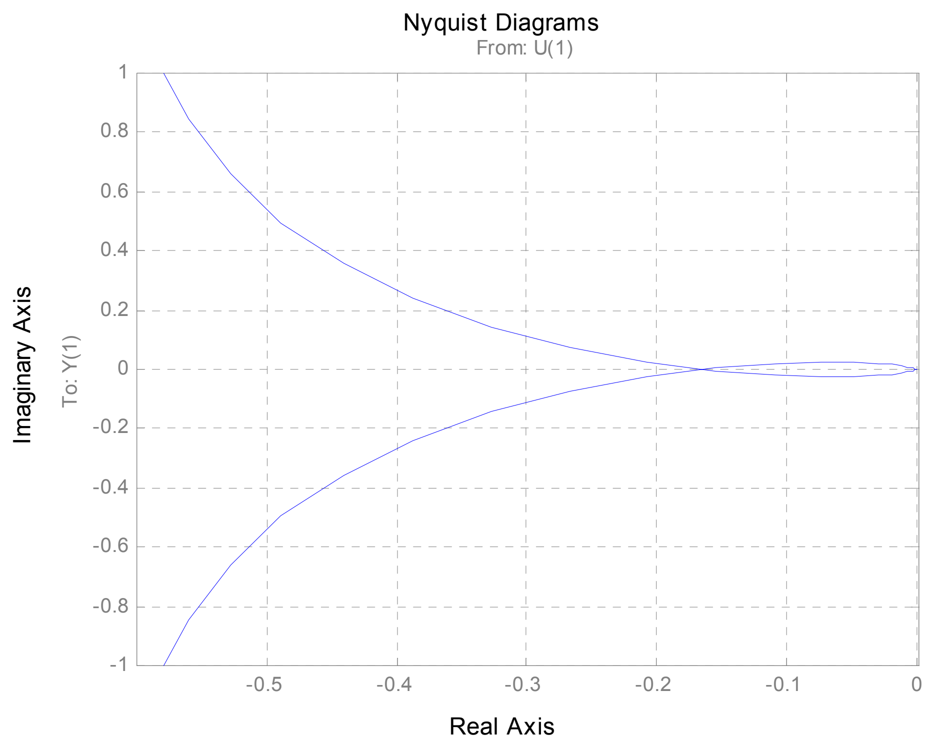
(b) [8 marks] Compute the phase margin of the closed-loop system for $K = 1$. Assuming that the controller would be implemented on earth, what would be the longest communication delay that would not destabilize the control system?

Answer:

Loop gain: $L(s) = \dfrac{1}{s(\frac{1}{9}s^2 + \frac{2}{3}s + 1)}$







Assuming that the Nyquist contour is indented so as to include the pole at 0, then the Nyquist criterion states that the number of counterclockwise encirclements of the critical point $-1/K$ should be equal to 1. Note that there is already one counterclockwise encirclement at infinity due to the indentation around the pole s=0, so the "visible" part of the Nyquist plot of $L(j\omega)$ should leave the critical point to its left. On the Bode plot, this means that the gain should be smaller than 0dB at frequencies where the phase crosses the –180deg line. We can see on the Bode plot above that the gain is approximately –15dB at that point, therefore the closed-loop system

Phase margin:

The crossover frequency is approximately $\omega_{co} = 1\text{rd/s}$. The phase margin is given by:

$\angle L(j\omega_{co}) \cong -130^\circ \Rightarrow \phi_m = 50^\circ$. From the broken line approximation, we find a phase margin of $\angle L(j\omega_{co}) \cong -140^\circ \Rightarrow \phi_m = 40^\circ$. Using the latter, we compute the maximum time-delay that can be tolerated:

$$\omega_{co}\tau = \frac{\pi}{180}\phi_m$$

$$\Rightarrow \tau = \frac{\pi}{180\omega_{co}}\phi_m = \frac{40\pi}{180 \cdot 10} = 0.0698\,\text{s}$$

(c) [4 marks] Compute the sensitivity function of the system and give the steady-state pitch angle error to a desired pitch angle step of 30 degrees on the output.

Answer:

$$S(s)=\frac{1}{1+L(s)}=\frac{1}{1+\dfrac{1}{s(\frac{1}{9}s^2+\frac{2}{3}s+1)}}=\frac{s(\frac{1}{9}s^2+\frac{2}{3}s+1)}{s(\frac{1}{9}s^2+\frac{2}{3}s+1)+1}$$

The Laplace transform of the error signal is

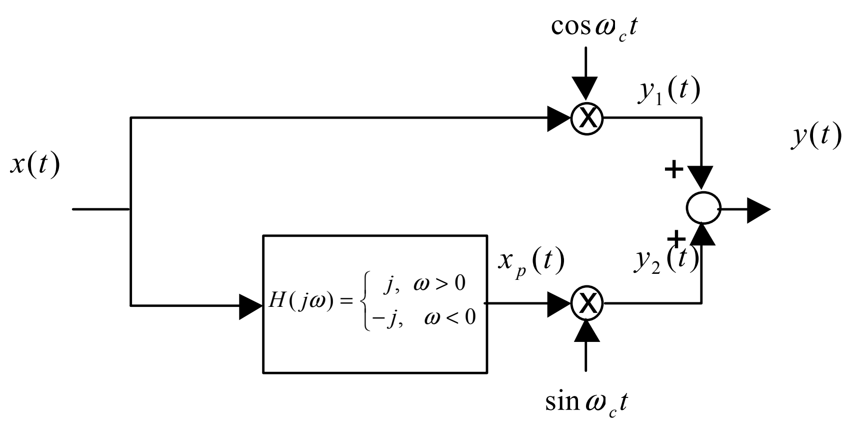
$\hat{e}(s)=\frac{1}{s}S(s)$, and from the final value theorem, we have $\lim_{t\to+\infty} e(t)=S(0)=0$.

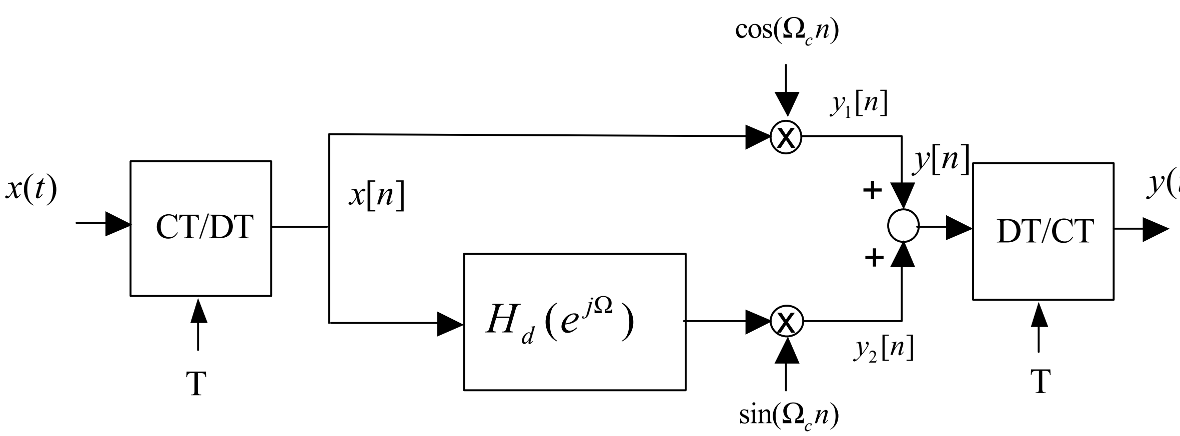
Therefore, the steady-state error to a 30deg step is 0deg, i.e., the rocket tracks the trajectory perfectly.

Problem 3 (15 marks)

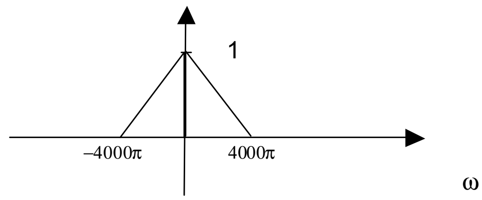
Discrete-time AM-SSB modulator

You have to design an upper SSB modulator in discrete-time as an implementation of the continuous-time modulator shown below.





The modulating signal has a Fourier transform as shown below, and the carrier frequency is $\omega_c = 2\,000\,000\pi$ rd/s.



Design the modulation system (with ideal components) for the slowest sampling rate. Find the ideal phase-shift filter $H_d(e^{j\Omega})$ and compute its impulse response $h_d[n]$. Compute the frequency Ω_c. Sketch the spectra $X(e^{j\Omega})$, $Y_1(e^{j\Omega})$, $Y_2(e^{j\Omega})$, $Y(e^{j\Omega})$. Explain how you could implement an approximation to this ideal filter, and describe modifications to the overall system so that it would work in practice.

Answer:
First, the ideal phase-shift filter should have the following frequency response:

$$H_d(e^{j\Omega}) = \begin{cases} j, & 0 < \Omega < \pi \\ -j, & -\pi < \Omega < 0 \end{cases}$$

The inverse FT yields:

$$\begin{aligned} h_d[n] &= \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\Omega})e^{j\Omega n}d\Omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{0} -je^{j\Omega n}d\Omega + \frac{1}{2\pi}\int_{0}^{\pi} je^{j\Omega n}d\Omega \\ &= \frac{-j}{2\pi}\frac{1}{jn}\left[e^{j\Omega n}\right]_{-\pi}^{0} + \frac{j}{2\pi}\frac{1}{jn}\left[e^{j\Omega n}\right]_{0}^{\pi}. \\ &= \frac{-1}{2\pi n}\left[1 - e^{-j\pi n}\right] + \frac{1}{2\pi n}\left[e^{j\pi n} - 1\right] \\ &= \frac{1}{\pi n}[(-1)^n - 1] \end{aligned}$$

To compute the slowest sampling frequency, we start from the desired upper SSB modulated signal at the output, whose bandwidth is $\omega_M = 2\,004\,000\pi$ which should correspond to $\Omega = \pi$. Thus, we can compute the sampling period from the relationship $\frac{\Omega}{T} = \omega \quad \Rightarrow \quad \frac{\pi}{\omega_M} = T$

So $T = 4.99 \times 10^{-7}\,s$. The DT carrier frequency is then:

$$\Omega_c = \omega_c T \quad \Rightarrow \quad \omega_c \frac{\pi}{\omega_M} = 2000000\pi \frac{\pi}{2004000\pi} = \frac{500\pi}{501} = 3.1353$$

This system could be implemented in practice using an FIR approximation to the ideal phase shift filter. Starting from $h_d[n]$, a windowed impulse response of length M+1 time delayed by M/2 to make it causal would work. However, the resulting delay of M/2 samples introduced in the lower path of the modulator should be balanced out by the introduction of an equivalent delay $z^{-M/2}$ in the upper path.

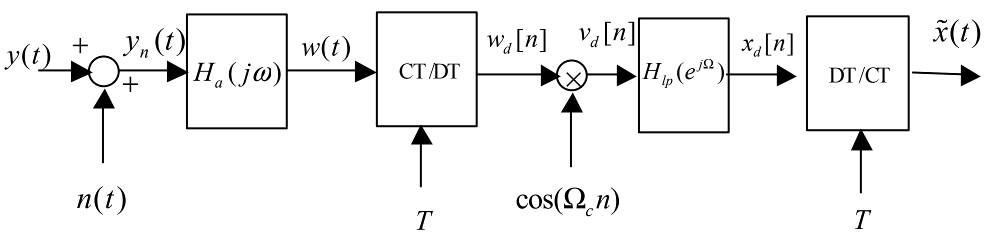
Problem 4 (20 marks)

The system shown below demodulates the noisy modulated continuous-time signal $y_n(t) = y(t) + n(t)$ composed of the sum of:

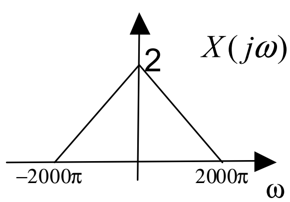
- signal $y(t)$ which is an upper single-sideband, suppressed carrier amplitude modulation (SSB/SC-AM) of $x(t)$ (assume for $Y(j\omega)$ a magnitude of one half that of $X(j\omega)$),
- a noise signal $n(t)$.

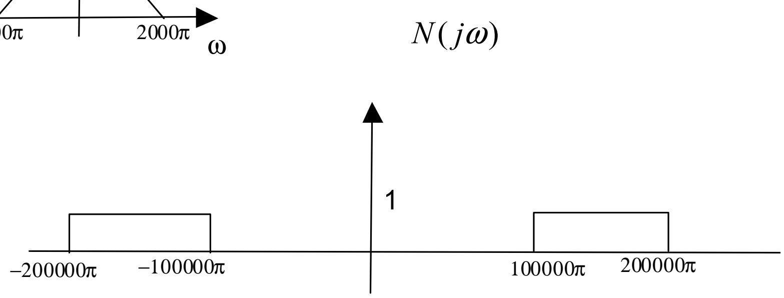
The carrier signal is $\cos(\omega_c t)$ where $\omega_c = 98000\pi$ rd/s.

The antialiasing filter $H_a(j\omega)$ is a perfect unity-gain lowpass filter with cutoff frequency ω_a.



The modulating (or message) signal $x(t)$ has spectrum $X(j\omega)$ as shown below. The spectrum $N(j\omega)$ of the noise signal is also shown.

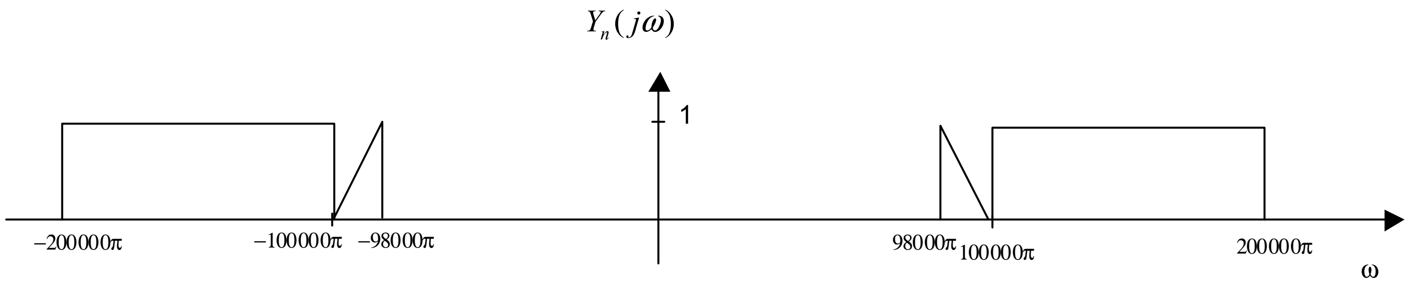


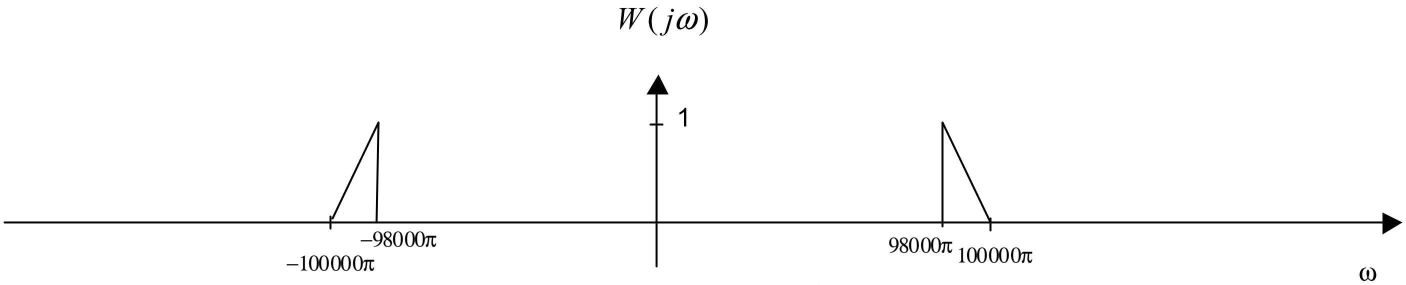


(a) [6 marks] Find the minimum antialiasing filter's cutoff frequency ω_a that will avoid any unrepairable distortion of the modulated signals due to the additive noise $n(t)$. Sketch the spectra $Y_n(j\omega)$ and $W(j\omega)$ of signals $y_n(t)$ and $w(t)$ for the frequency ω_a that you found. Indicate the important frequencies and magnitudes on your sketch.

Answer:

Minimum $\omega_a = 100000\pi$.



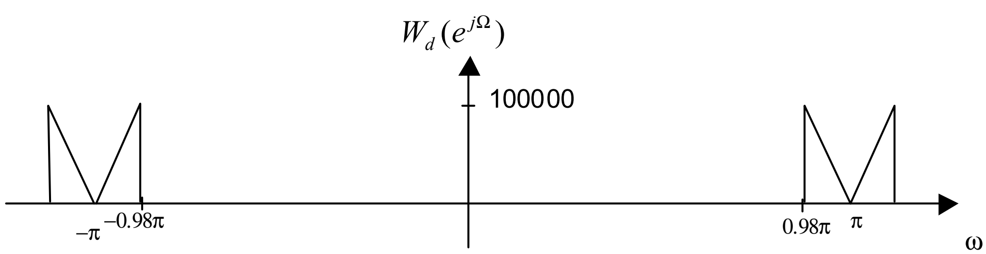


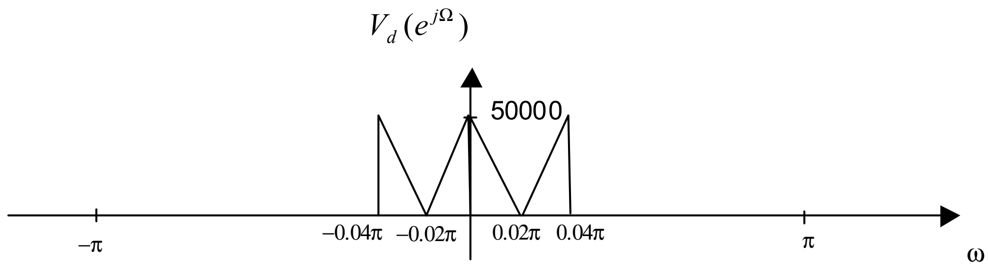
(b) [14 marks] Find the minimum sampling frequency $\omega_s = \dfrac{2\pi}{T}$ and its corresponding sampling period T that would satisfy the sampling theorem, i.e., that would allow perfect reconstruction of the modulated signal. Give the corresponding cutoff frequency Ω_1 of the perfect lowpass filter and its gain K_1 so that $\tilde{x}(t) = x(t)$, and find the demodulation frequency Ω_c. Using these frequencies, sketch the spectra $W_d(e^{j\Omega}), V_d(e^{j\Omega}), X_d(e^{j\Omega})$, and $\tilde{X}(j\omega)$.

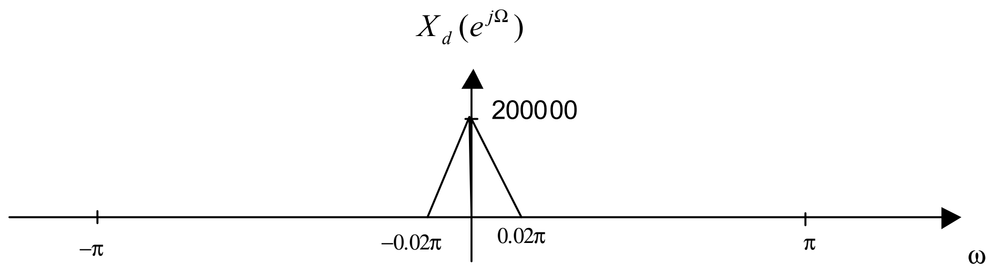
Answer:

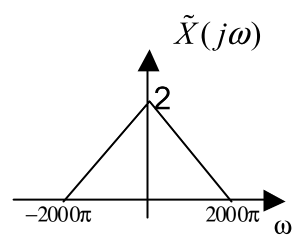
$$\omega_s = 2\omega_a = 200000\pi,\ T = \frac{2\pi}{\omega_s} = \frac{1}{100000} = 10^{-5}$$

Demodulation frequency: $\Omega_c = 98000\pi T = 0.98\pi$. Cutoff frequencies: $\Omega_1 = 2000\pi T = 0.02\pi$.
The lowpass filter must have a gain of $K_1 = 4$:









Problem 5 (20 marks)

The causal continuous-time LTI system given by its transfer function $G(s) = \dfrac{s}{(s+1)(s+2)}$ is discretized with sampling period $T = 0.1$ s for simulation purposes. The "c2d" transformation is used.

(a) [4 marks] Find a state-space realization of the system.

Answer:

$$G(s) = \frac{s}{(s+1)(s+2)} = \frac{s}{s^2 + 3s + 2}$$

The observable canonical state-space realization is given by:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u ,$$

and the output equation is

$$y = \underbrace{\begin{bmatrix} 0 & 1 \end{bmatrix}}_{C} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

and, $D = 0$.

(b) [10 marks] Compute the discrete-time state-space system $(A_{c2d}, B_{c2d}, C_{c2d}, D_{c2d})$ for $G(s)$ and its associated transfer function $G_{c2d}(z)$, specifying its ROC. Sketch the pole zero plot of $G_{c2d}(z)$.

Answer:

The A matrix is first diagonalized:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}}_{W} \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}}_{\Lambda} \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}^{-1}}_{W}$$

Discretized state-space system using "c2d":

$$A_d = e^{AT} = We^{\Lambda T}W^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} e^{-0.1} & 0 \\ 0 & e^{-0.2} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 2e^{-0.1} & e^{-0.1} \\ -e^{-0.2} & -e^{-0.2} \end{bmatrix} = \begin{bmatrix} 2e^{-0.1} - e^{-0.2} & e^{-0.1} - e^{-0.2} \\ -2e^{-0.1} + 2e^{-0.2} & -e^{-0.1} + 2e^{-0.2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.990944 & 0.086107 \\ -0.17221 & 0.73262 \end{bmatrix}$$

$$B_d := A^{-1}\left[e^{AT} - I_n\right]B = \begin{bmatrix} -3/2 & -1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2e^{-0.1} - e^{-0.2} - 1 & e^{-0.1} - e^{-0.2} \\ -2e^{-0.1} + 2e^{-0.2} & -e^{-0.1} + 2e^{-0.2} - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/2 & -1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-0.1} - e^{-0.2} \\ -e^{-0.1} + 2e^{-0.2} - 1 \end{bmatrix} = \begin{bmatrix} -e^{-0.1} + 1/2e^{-0.2} + 1/2 \\ e^{-0.1} - e^{-0.2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.0045280 \\ 0.086107 \end{bmatrix}$$

$$C_d := C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$D_d := D = 1$$

transfer function:

$$G_{c2d}(z) = C(zI_n - A)^{-1}B + D$$
$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \text{z-0.990944} & \text{-0.086107} \\ 0.17221 & \text{z-0.73262} \end{bmatrix}^{-1} \begin{bmatrix} 0.0045280 \\ 0.086107 \end{bmatrix}$$
$$= \frac{1}{(\text{z-0.990944})(\text{z-0.73262}) + 0.17221(0.086107)} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \text{z-0.73262} & 0.086107 \\ \text{-0.17221} & \text{z-0.990944} \end{bmatrix} \begin{bmatrix} 0.0045280 \\ 0.086107 \end{bmatrix}$$
$$= \frac{0.086107z - 0.086107}{z^2\text{-1.72356}z + 0.74081} = \frac{0.086107(z-1)}{(z-0.90484)(z-0.81872)}, \quad |z| > 0.90484$$

(c) [6 marks] Simulate the unit step response of $G(s)$ by computing recursively the first 6 values (for $n = 0,\ldots,5$) of the unit step response of the difference equation corresponding to $G_{c2d}(z)$. Also find the settling value of the step response by applying the final value theorem to $G_{c2d}(z)$.

Answer:

$$G_{c2d}(z) = \frac{0.086107z - 0.086107}{z^2\text{-1.72356}z + 0.74081}$$
$$= \frac{0.086107z^{-1} - 0.086107z^{-2}}{1\text{-1.72356}z^{-1} + 0.74081z^{-2}}$$

Corresponding difference equation:

$$y[n] = 1.72356y[n-1] - 0.74081y[n-2] + 0.086107x[n-1] - 0.086107x[n-2]$$

Step response:

$$y[0] = 1.72356(0) - 0.74081(0) + 0.086107(0) - 0.086107(0)$$
$$= 0$$
$$y[1] = 1.72356(0) - 0.74081(0) + 0.086107(1) - 0.086107(0)$$
$$= 0.086107$$
$$y[2] = 1.72356(0.086107) - 0.74081(0) + 0.086107(1) - 0.086107(1)$$
$$= 0.1484$$
$$y[3] = 1.72356(0.1484) - 0.74081(0.086107) + 0.086107(1) - 0.086107(1)$$
$$= 0.1920$$
$$y[4] = 1.72356(0.1920) - 0.74081(0.1484) + 0.086107(1) - 0.086107(1)$$
$$= 0.2210$$
$$y[5] = 1.72356(0.2210) - 0.74081(0.1920) + 0.086107(1) - 0.086107(1)$$
$$= 0.2386$$

Final value: $y[+\infty] = \lim_{z\to 1}(1 - z^{-1})\frac{G_{c2d}(z)}{(1-z^{-1})} = G_{c2d}(1) = 0$

END OF EXAMINATION