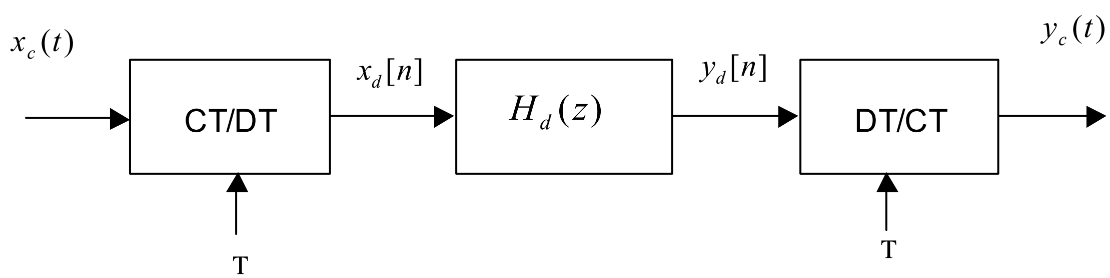
Sample Final Exam (finals02)
Covering Chapters 10-17 of *Fundamentals of Signals & Systems*

Problem 1 (30 marks)

Consider the system depicted below used for discrete-time processing of continuous-time signals. The sampling period is 100 milliseconds ($T = 0.1$ s).

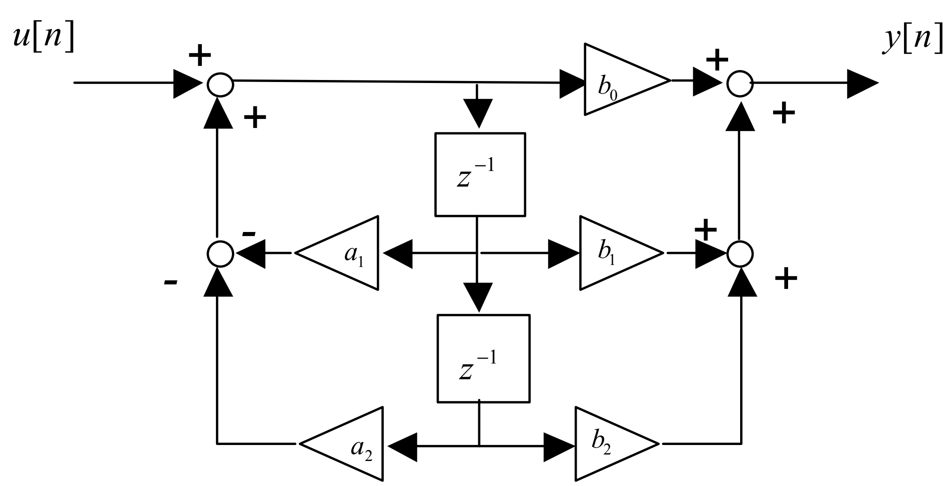


The discrete-time filter $H_d(z)$ is given by the following causal difference equation initially at rest:

$$y[n] + a_1 y[n-1] + a_2 y[n-2] = b_0 x[n] + b_1 x[n-1] + b_2 x[n-2]$$

(a) [8 marks] Find the controllable canonical state-space realization of the filter $H_d(z)$, i.e., sketch the block diagram and give the state-space equations.

Answer:



The state equation:

$$\begin{bmatrix} x_1[n+1] \\ x_2[n+1] \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix}}_{A} \begin{bmatrix} x_1[n] \\ x_2[n] \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u[n].$$

The output equation:

$$y[n] = \underbrace{\begin{bmatrix} b_2 - a_2 b_0 & b_1 - a_1 b_0 \end{bmatrix}}_{C} x[n] + \underbrace{b_0}_{D} u[n]$$

(b) [10 marks] The filter $H_d(z)$ is designed to approximate the unity-gain, second-order, continuous-time, causal LTI filter $H(s) = \frac{2}{s^2 + 3s + 2}$, $\mathrm{Re}\{s\} > -1$. Find the values of the parameters $\{a_1, a_2, b_0, b_1, b_2\}$ of $H_d(z)$ using the "c2d" transformation. Specify the ROC of $H_d(z)$.

Answer:

The state equation for $H(s)$:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u(t).$$

The output equation for $H(s)$:

$$y(t) = \underbrace{\begin{bmatrix} 2 & 0 \end{bmatrix}}_{C} x(t)$$

Diagonalize to compute matrix exponential:

$$A_1 = V^{-1} A V = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}, V = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$$

$$A_d = e^{AT} = V e^{A_1 T} V^{-1} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} e^{-T} & 0 \\ 0 & e^{-2T} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2e^{-T} & e^{-T} \\ e^{-2T} & e^{-2T} \end{bmatrix} = \begin{bmatrix} 2e^{-T} - e^{-2T} & e^{-T} - e^{-2T} \\ -2e^{-T} + 2e^{-2T} & -e^{-T} + 2e^{-2T} \end{bmatrix} = \begin{bmatrix} 0.9909 & 0.0861 \\ -0.1722 & 0.7326 \end{bmatrix}$$

$$B_d = A^{-1}\left[e^{AT} - I_2\right] B = \begin{bmatrix} -3/2 & -1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2e^{-T} - e^{-2T} - 1 & e^{-T} - e^{-2T} \\ -2e^{-T} + 2e^{-2T} & -e^{-T} + 2e^{-2T} - 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3/2 & -1/2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^{-T} - e^{-2T} \\ -e^{-T} + 2e^{-2T} - 1 \end{bmatrix} = \begin{bmatrix} -e^{-T} + \frac{1}{2} e^{-2T} + \frac{1}{2} \\ e^{-T} - e^{-2T} \end{bmatrix} = \begin{bmatrix} 0.0045 \\ 0.0861 \end{bmatrix}$$

$$C_d = C$$

Transfer function:

$$\mathcal{H}(z) = C_d(zI_n - A_d)^{-1}B_d + D_d$$

$$= \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} z-0.9909 & -0.0861 \\ 0.1722 & z-0.7326 \end{bmatrix}^{-1} \begin{bmatrix} 0.0045 \\ 0.0861 \end{bmatrix}$$

$$= \frac{1}{(z-0.9909)(z-0.7326)+0.0148} \begin{bmatrix} 2 & 0 \end{bmatrix} \begin{bmatrix} 0.0045(z-0.7326)+(0.0861)(0.0861) \\ (-0.1722)(0.0045)+(0.0861)(z-0.9909) \end{bmatrix}$$

$$= \frac{0.009(z-0.7326)+2(0.0861)(0.0861)}{(z-0.9909)(z-0.7326)+0.0148}$$

$$= \frac{0.009z+0.0082}{z^2-1.7235z+0.7407}$$

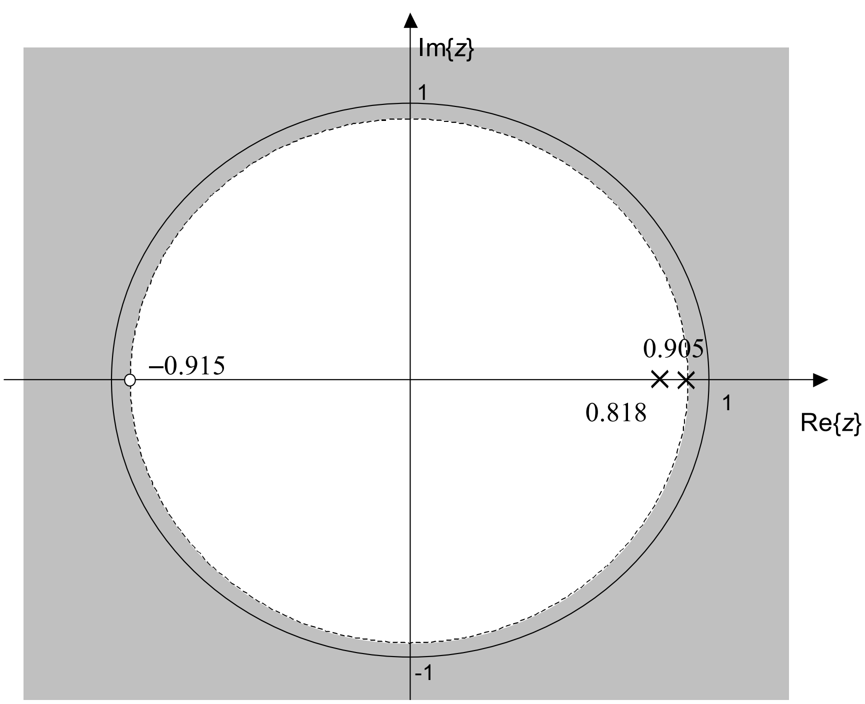
$$= \frac{0.009z^{-1}+0.0082z^{-2}}{1-1.7235z^{-1}+0.7407z^{-2}}$$

Hence, $b_0 = 0, b_1 = 0.009, b_2 = 0.0082, a_1 = -1.7235, a_2 = 0.7407$

(c) [5 marks] Sketch the pole-zero plot of $H_d(z)$. Compute the gain of $H_d(z)$ at the highest frequency.

Answer:

$$H_d(z) = \frac{0.009z+0.0082}{z^2-1.7235z+0.7407} = \frac{0.009(z+0.915)}{(z-0.8184)(z-0.9051)}, \quad |z| > 0.9051.$$



Gain at highest frequency: $\left|H_d(-1)\right| = \left|\dfrac{-0.009+0.0082}{1+1.7235+0.7407}\right| = 0.00023$

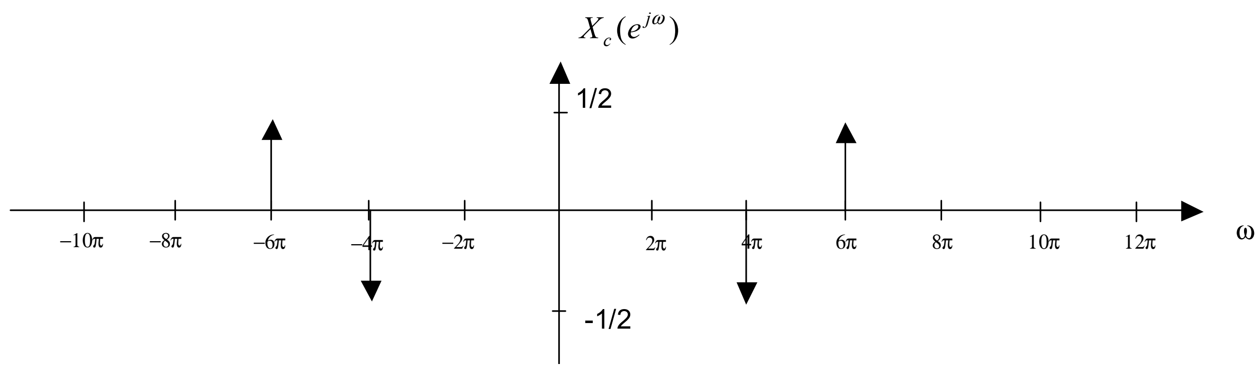
(d) [7 marks] Suppose that the continuous-time signal to be filtered is given by: $x_c(t) = \cos(6\pi t) - \cos(4\pi t)$. Sketch the spectra of the continuous-time and discrete-time versions of the input signal $X_c(j\omega)$ and $X_d(e^{j\Omega})$. Sketch the spectra $Y_d(e^{j\Omega})$ of the discrete-time output signal and $Y_c(j\omega)$ of the continuous-time output signal $y_c(t)$. Finally, give an expression for the output signal $y_c(t)$.

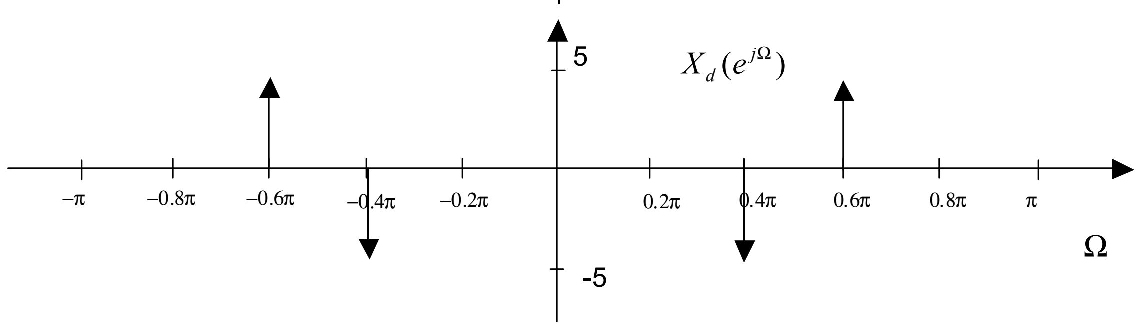
Answer:

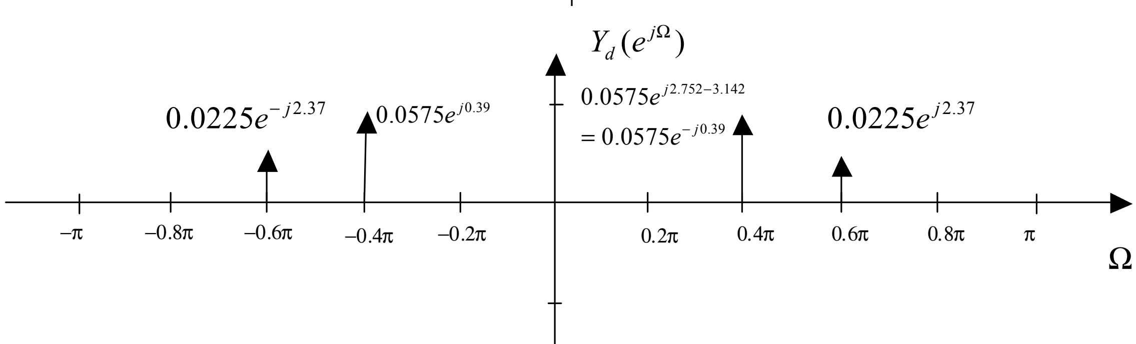
We need to compute

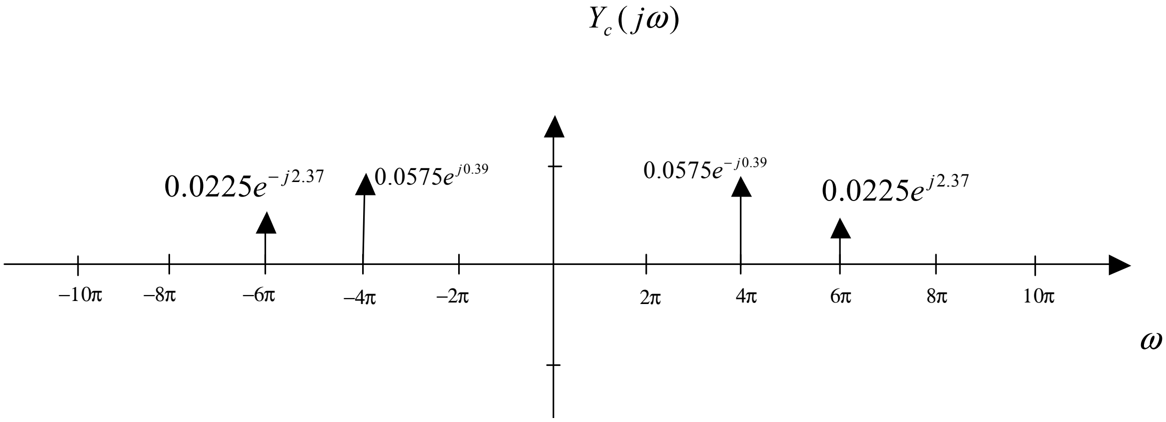
$$H_d(e^{j0.4\pi}) = \frac{0.009e^{j0.4\pi}+0.0082}{e^{j0.8\pi}-1.7235e^{j0.4\pi}+0.7407} = -0.0106 + j0.0044 = 0.0115e^{j2.7521}$$

$$H_d(e^{j0.6\pi}) = \frac{0.009e^{j0.6\pi}+0.0082}{e^{j1.2\pi}-1.7235e^{j0.6\pi}+0.7407} = -0.0032 + j0.0031 = 0.0045e^{j2.3717}.$$









$$y_c(t) = 0.0045\cos(6\pi t + 2.3717) - 0.0115\cos(4\pi t + 2.752)$$

Problem 2 (15 marks)

Suppose we want to design a causal, stable, first-order high-pass filter of the type

$$H(z) = \frac{B}{1 - az^{-1}}, |z| > |a|$$

with -3dB cutoff frequency $\omega_c = \frac{2\pi}{3}$ (i.e., frequency where the magnitude of the frequency response of the filter is $\frac{1}{\sqrt{2}}$) and a real.

(a) [2 marks] Express the real constant B in terms of the pole a to obtain unity gain at the highest frequency $\omega = \pi$.

Ans:

The gain at $\omega = \pi$ is

$$H(e^{j\pi}) = H(-1) = \frac{B}{1+a},$$

and unity gain is obtained for $B = 1 + a$.

(b) [10 marks] "Design" the filter, i.e., find the numerical values of the pole a and the constant B.

Answer:

$$\frac{1}{2} = \left|H(e^{j\omega_c})\right|^2 = \frac{(1+a)^2}{(1 - a\cos\omega_c)^2 + a^2\sin^2\omega_c}$$

$$= \frac{(1+a)^2}{(1+0.5a)^2 + 0.75a^2}$$

This yields the quadratic equation

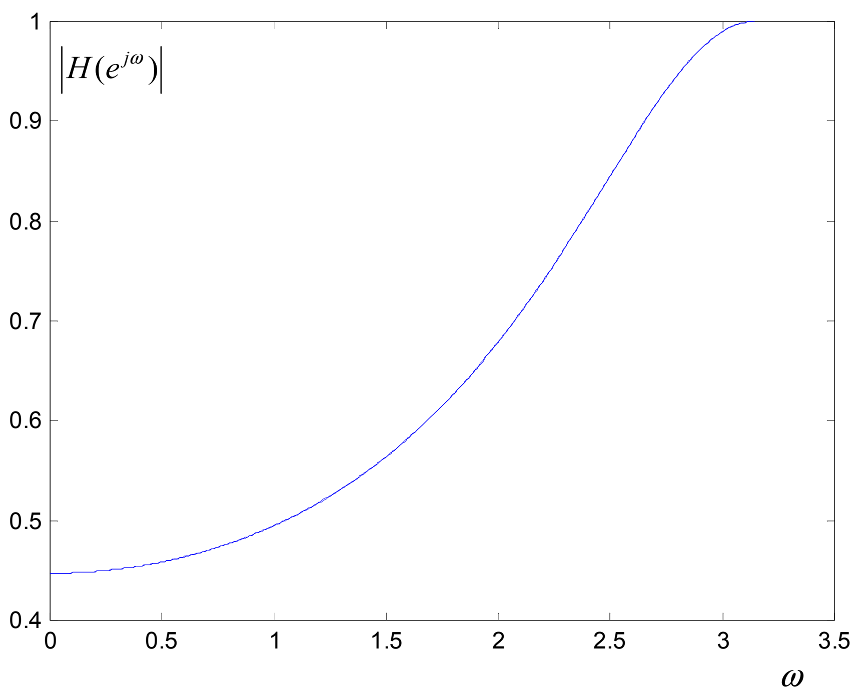
$$2(1+a)^2 = (1+0.5a)^2 + 0.75a^2$$
$$\Leftrightarrow$$
$$a^2 + 3a + 1 = 0$$

whose solutions are $a_{1,2} = \frac{-3 \pm \sqrt{5}}{2}$. But for the filter to be stable, we select the pole inside the unit circle: $a = \frac{-3+\sqrt{5}}{2} = -0.382$. Finally $B = 0.618$, and the high-pass filter is

$$H(z) = \frac{\frac{-1+\sqrt{5}}{2}}{1+\frac{3-\sqrt{5}}{2}z^{-1}} = \frac{0.618}{1+0.382z^{-1}}, |z| > 0.382$$

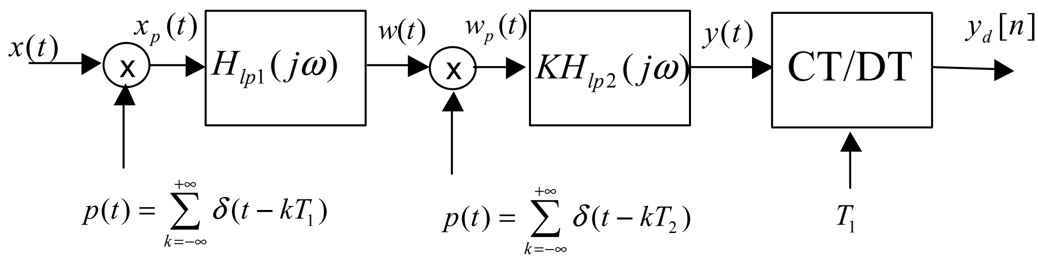
(c) [3 marks] Sketch the magnitude of the filter's frequency response.

Answer:

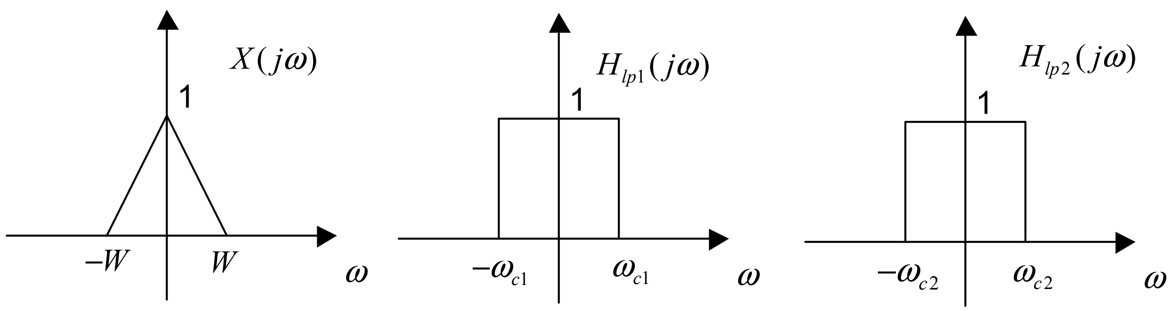


Problem 3 (20 marks)

Consider the following sampling system where the sampling frequencies are $\omega_{s1} = \frac{2\pi}{T_1}$, $\omega_{s2} = \frac{2\pi}{T_2}$.



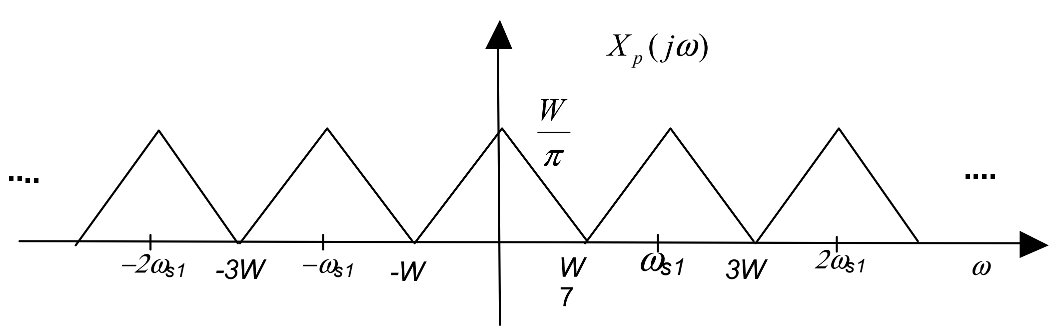
The spectrum $X(j\omega)$ of the input signal $x(t)$, and the frequency responses of the two ideal lowpass filters, are shown below. The gain of the second lowpass filter is $K > 0$.

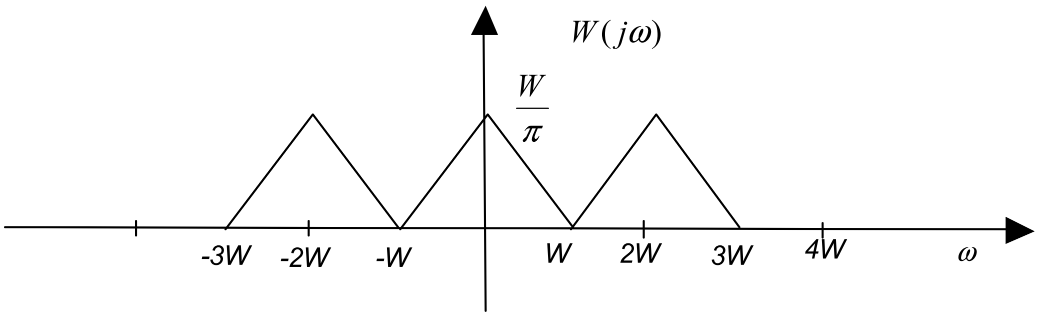


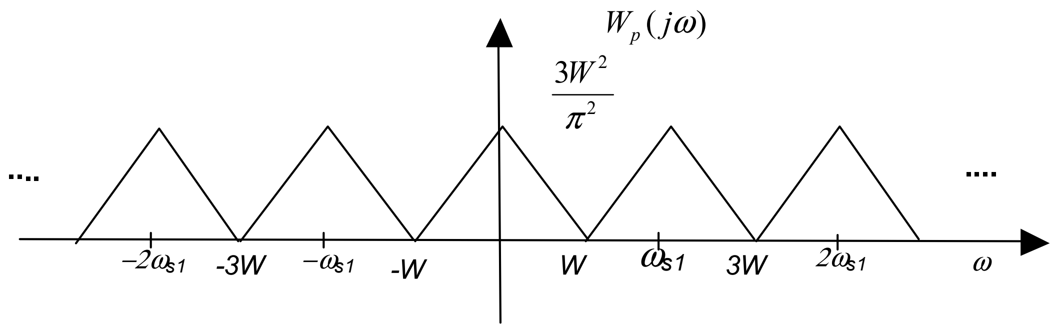
(a) [10 marks] For what range of sampling frequencies ω_{s1} is the sampling theorem satisfied for the *first* sampler (from $x(t)$ to $x_p(t)$)? Suppose that the cutoff frequencies of the lowpass filters are given by $\omega_{c1} = 3W$, $\omega_{c2} = W$. For what range of sampling frequencies ω_{s2} is the sampling theorem satisfied for the *second* sampler (from $w(t)$ to $w_p(t)$)? Choosing the lowest sampling frequencies in the ranges that you found for the two samplers, sketch the spectra $X_p(j\omega)$, $W(j\omega)$, $W_p(j\omega)$, and $Y(j\omega)$. Find the gain K of the second filter that leads to $y(t) = x(t)$.

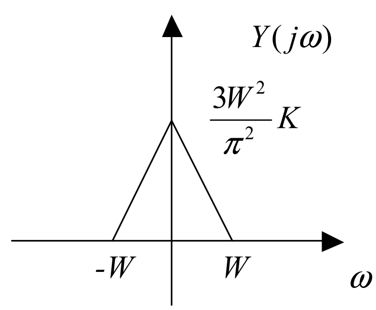
Answer:

The sampling theorem is satisfied for $\omega_{s1} > 2W$ for the first sampler, and for $\omega_{s2} > 6W$. For the spectra, we set $\omega_{s1} = 2W$ so that $T_1 = \frac{\pi}{W}$ and $\omega_{s2} = 6W$ so that $T_1 = \frac{\pi}{3W}$:







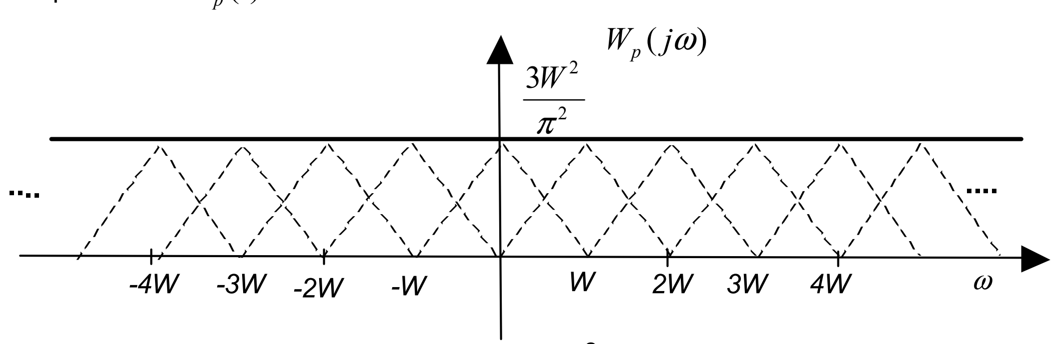


Finally $K = \dfrac{\pi^2}{3W^2}$.

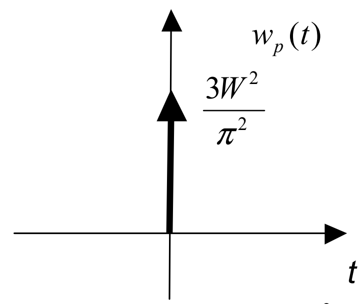
(b) [10 marks] Assume that the sampling frequencies are set to $\omega_{s1} = 2W$ and $\omega_{s2} = 3W$ and that the cutoff frequencies of the lowpass filters are given by $\omega_{c1} = 3W, \omega_{c2} = W$. Find the signals $w_p(t)$ and $y(t)$ and sketch them. Use the value for the gain K that you found in (a). Find and sketch the discrete-time signal $y_d[n]$.

Answer:

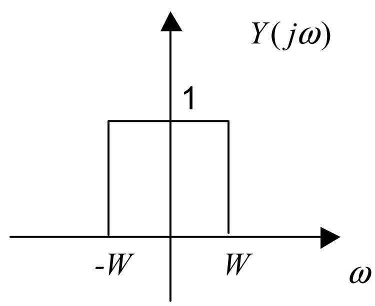
The spectrum of $w_p(t)$ is constant:



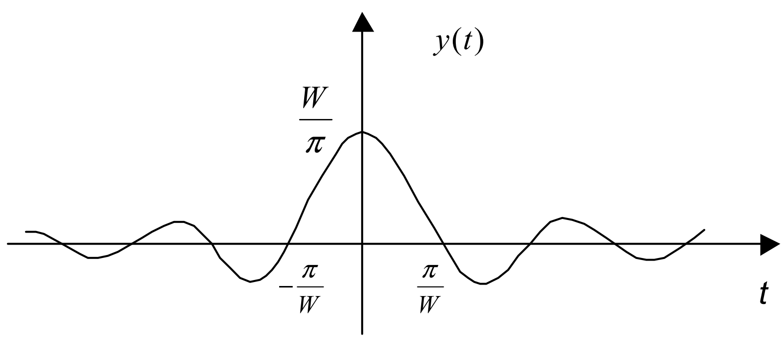
Therefore $w_p(t) = \frac{3W^2}{\pi^2}\delta(t)$.



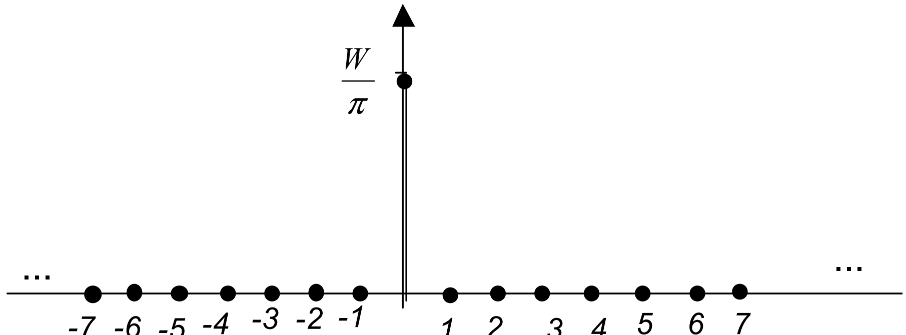
After filtering with the second lowpass with gain $K = \frac{\pi^2}{3W^2}$, we get



Which has the inverse FT: $y(t) = \frac{W}{\pi}\operatorname{sinc}\left(\frac{W}{\pi}t\right)$.

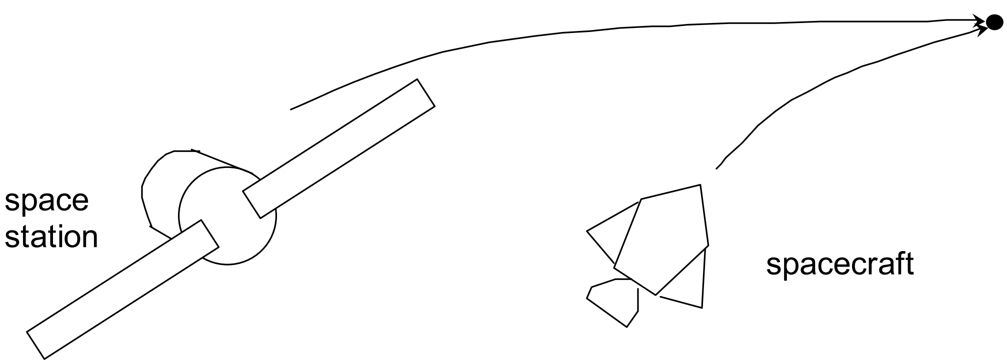


After the CT/DT operation, we obtain: $y_d[n] = y(nT_1) = y(n\frac{\pi}{W}) = \frac{W}{\pi}\operatorname{sinc}(n) = \frac{W}{\pi}\delta[n]$



Problem 4 (20 marks)

Space rendez-vous
Consider the spacecraft shown below which has to maneuver in order to dock on a space station.



For simplicity, we consider the one-dimensional case where the state of each vehicle consists of its position and velocity along axis z.

Assume that the space station moves autonomously according to the state-space system

$$\dot{x}_s(t) = A_s x_s(t)$$

where $x_s = \begin{bmatrix} z_s \\ \dot{z}_s \end{bmatrix}$, and $A_s = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$,

and the spacecraft's equation of motion is:

$$\dot{x}_c(t) = A_c x_c(t) + B_c u_c(t)$$

where $x_c = \begin{bmatrix} z_c \\ \dot{z}_c \end{bmatrix}$, $u_c(t)$ is the thrust, $A_c = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B_c = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$.

(a) [5 marks] Write down the state-space system of the state error $e := x_c - x_s$ which describes the evolution of the difference in position and velocity between the spacecraft and the space station. The output is the difference in position.

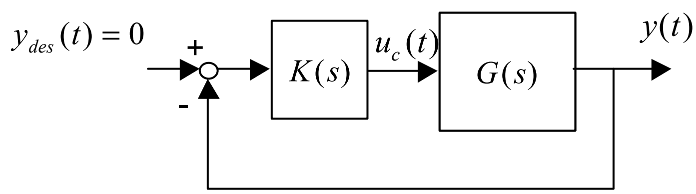
Answer:

Let $e := x_c - x_s = \begin{bmatrix} z_c \\ \dot{z}_c \end{bmatrix} - \begin{bmatrix} z_s \\ \dot{z}_s \end{bmatrix}$, so that

$$\begin{aligned}\dot{e}(t) &= A_c(x_c(t) - x_s(t)) + B_c u_c(t)\\ &= A_c e(t) + B_c u_c(t)\\ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} e(t) + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} u_c(t)\\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} e(t)\end{aligned}$$

(b) [5 marks] A controller is implemented in a unity feedback control system to drive the position difference to zero for docking. The controller is given by:

$$K(s) = \frac{100(s+1)}{0.01s+1}, \ \ \mathrm{Re}\{s\} > -100$$



Find $G(s)$ and assess the stability of this feedback control system (hint: one of the closed-loop poles is at -10.)

Answer:

$$\begin{aligned}G(s) &= C_e(sI - A_e)^{-1}B_e + D_e\\ &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}\\ &= \frac{0.1}{s^2}, \ \mathrm{Re}\{s\} > 0\end{aligned}$$

Closed-loop characteristic polynomial with coprime numerators and denominators:

$$\begin{aligned}p(s) &= n_G n_K + d_G d_K = 0.1(100s+100) + s^2(0.01s+1)\\ &= 0.01s^3 + s^2 + 10s + 10 = 0.01(s^3 + 100s^2 + 1000s + 1000)\end{aligned}$$

Thus,

$$\begin{aligned}s^3 + 100s^2 + 1000s + 1000 &= (s+a)(s+b)(s+10)\\ &= s^3 + (10+a+b)s^2 + (10a+10b+ab)s + 10ab\end{aligned}$$

By identifying coefficients, we find three equations in two unknowns (one is redundant):

$$10 + a + b = 100$$

$$10a + 10b + ab = 1000$$

$$10ab = 1000$$

We combine the first and third equations to find a quadratic equation:

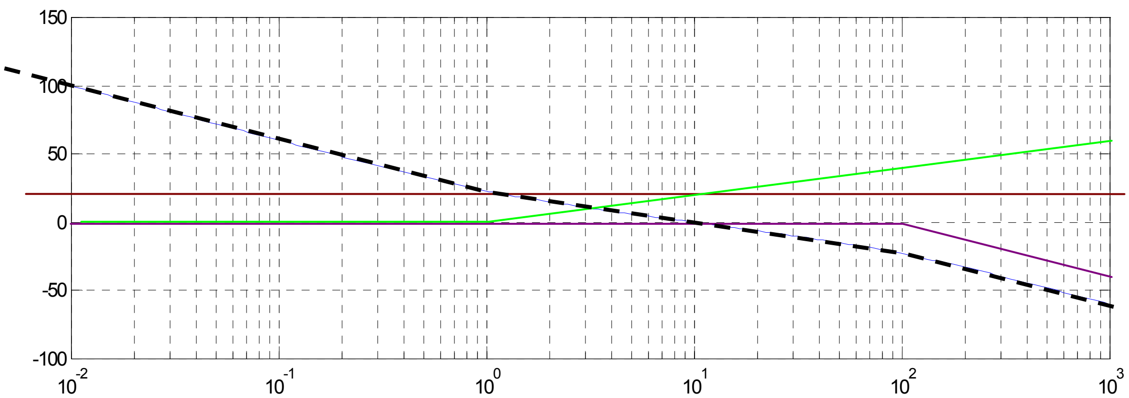
$$a^2 - 90a + 100 = 0$$
$$\Rightarrow a_1 = 45 + 10\sqrt{4.5^2 - 1} = 88.8748$$
$$a_2 = 45 - 10\sqrt{4.5^2 - 1} = 1.1252$$

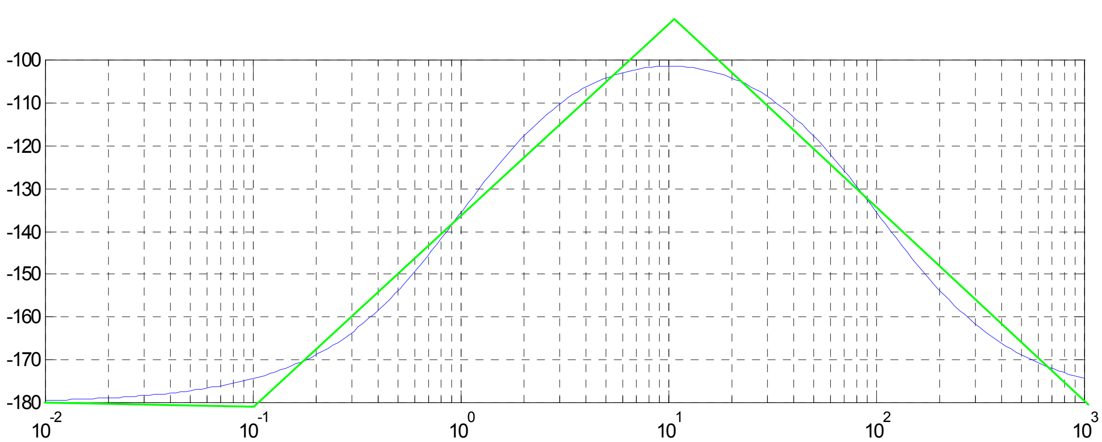
It turns out that for the choice $a = 88.8748 \Rightarrow b = 1.1252$ and for the choice $a = 1.1252 \Rightarrow b = 88.8748$. Therefore the three closed-loop poles are at $-10, -1.1252, -88.8748$, and the closed-loop system is stable.

(c) [7 marks] Find the loop gain, sketch its Bode plot, and compute the phase margin of the closed-loop system. Assuming for the moment that the controller would be implemented on earth, what would be the longest communication delay that would not destabilize the automatic docking system?

Answer:

Loop gain: $L(s) = \dfrac{10(s+1)}{s^2(0.01s+1)}$





Phase margin:
The crossover frequency is approximately $\omega_{co} = 10\text{rd/s}$. The phase margin is given by:

$\angle L(j\omega_{co}) \cong -102^\circ \Rightarrow \phi_m = 78^\circ$. From the broken line approximation, we find a phase margin of $\angle L(j\omega_{co}) \cong -90^\circ \Rightarrow \phi_m = 90^\circ$. Using the latter, we compute the maximum time-delay that can be tolerated:

$$\omega_{co}\tau = \frac{\pi}{180}\phi_m$$

$$\Rightarrow \tau = \frac{\pi}{180\omega_{co}}\phi_m = \frac{90\pi}{180 \cdot 10} = 0.1571\text{s}$$

(d) [3 marks] Compute the sensitivity function of the system and give the steady-state error to a unit step disturbance on the output.

Answer:

$$S(s) = \frac{1}{1+L(s)} = \frac{1}{1+\dfrac{10(s+1)}{s^2(0.01s+1)}} = \frac{s^2(0.01s+1)}{0.01s^3+s^2+10s+10}$$

The Laplace transform of the error signal is

$\hat{e}(s) = \frac{1}{s}S(s)$, and from the final value theorem, we have $\lim_{t\to+\infty} e(t) = S(0) = \frac{0}{10} = 0$

Problem 5 (15 marks)

Consider a fourth-order ($M = 4$) causal moving average filter $H(z)$.

(a) [5 marks] Compute $H(z)$ and give its pole-zero plot including the ROC.

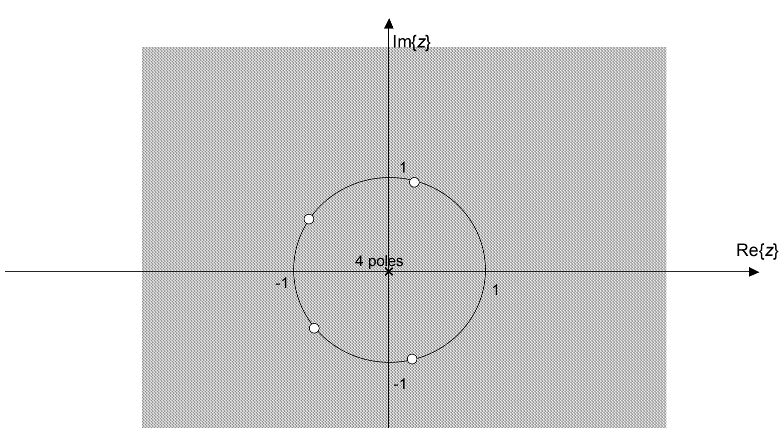
Answer:

The *z*-transform of this filter is given by

$$H(z) = \frac{1}{5} + \frac{1}{5}z^{-1} + \frac{1}{5}z^{-2} + \frac{1}{5}z^{-3} + \frac{1}{5}z^{-4} = \frac{1}{5}\frac{z^4+z^3+z^2+z+1}{z^4}$$

with ROC $\{z \in \mathbb{C}\ ,\ z \neq 0\}$, i.e., the whole complex plane excluding 0.

Four poles at 0, and zeros are on the unit circle at $\frac{2\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{8\pi}{5}$.



(b) [5 marks] Compute the filter's frequency response $H(e^{j\omega})$ and give its magnitude and phase.

Answer:

$$\begin{aligned} H(e^{j\omega}) &= \frac{1}{5}+\frac{1}{5}e^{-j\omega}+\frac{1}{5}e^{-j2\omega}+\frac{1}{5}e^{-j3\omega}+\frac{1}{5}e^{-j4\omega} \\ &= \frac{1}{5}e^{-j2\omega}\left(e^{j2\omega}+e^{j\omega}+1+e^{-j\omega}+e^{-j2\omega}\right) \\ &= e^{-j2\omega}\left(\frac{1}{5}+\frac{2}{5}\cos(\omega)+\frac{2}{5}\cos(2\omega)\right) \end{aligned}$$

$$\left|H(e^{j\omega})\right| = \left|\frac{1}{5}+\frac{2}{5}\cos(\omega)+\frac{2}{5}\cos(2\omega)\right|$$

Phase in the passband is sufficient here: $\angle H(e^{j\omega}) = -2\omega, \quad \omega \in \left(-\frac{2\pi}{5},\frac{2\pi}{5}\right)$, but the complete answer is:

$$\angle H(e^{j\omega}) = \begin{cases} -2\omega, & \omega \in \left(-\frac{2\pi}{5},\frac{2\pi}{5}\right) \\ -\pi-2\omega, & \omega \in \left(\frac{2\pi}{5},\frac{4\pi}{5}\right) \\ -2\omega, & \omega \in \left(\frac{4\pi}{5},\pi\right) \\ -2\omega, & \omega \in \left(-\pi,-\frac{4\pi}{5}\right) \\ \pi-2\omega, & \omega \in \left(-\frac{4\pi}{5},-\frac{2\pi}{5}\right) \end{cases}$$

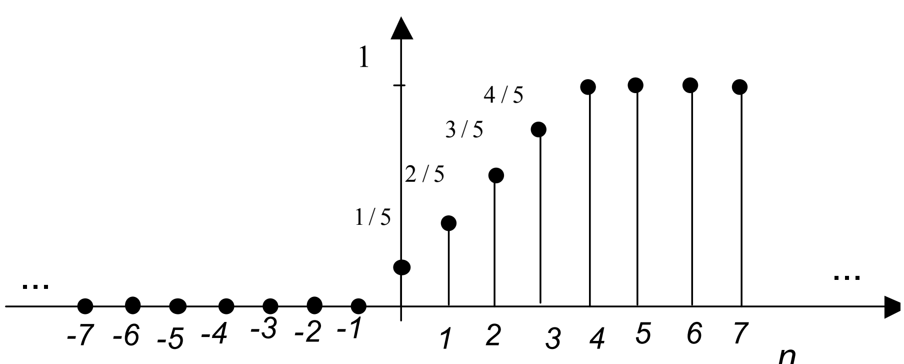
(c) [5 marks] Compute and sketch the unit step response of the filter.

Answer:

The step response is the running sum of the impulse response:

$$h[n] = \frac{1}{5}\delta[n]+\frac{1}{5}\delta[n-1]+\frac{1}{5}\delta[n-2]+\frac{1}{5}\delta[n-3]+\frac{1}{5}\delta[n-4]$$

$$y[n] = \frac{1}{5}(n+1)u[n]-\frac{1}{5}(n-5+1)u[n-5]$$



END OF EXAMINATION