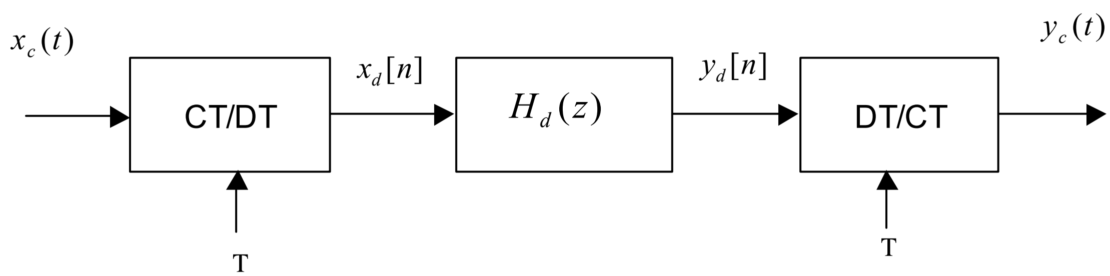
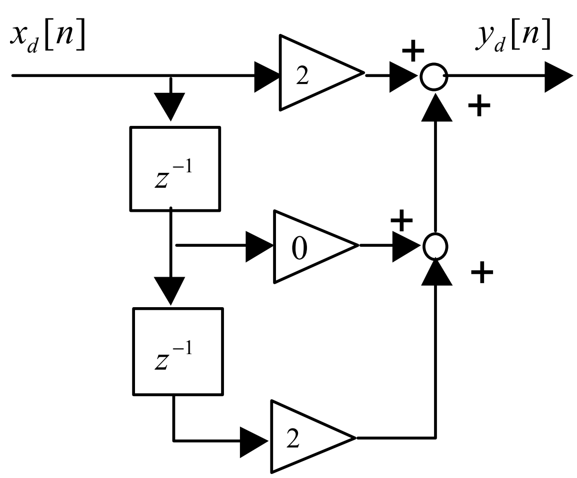
Sample Final Exam (finals01)
Covering Chapters 10-17 of *Fundamentals of Signals & Systems*

Problem 1 (15 marks)

Consider the system depicted below used for discrete-time processing of continuous-time signals. The sampling period is 1 microsecond ($T = 10^{-6}$ s).



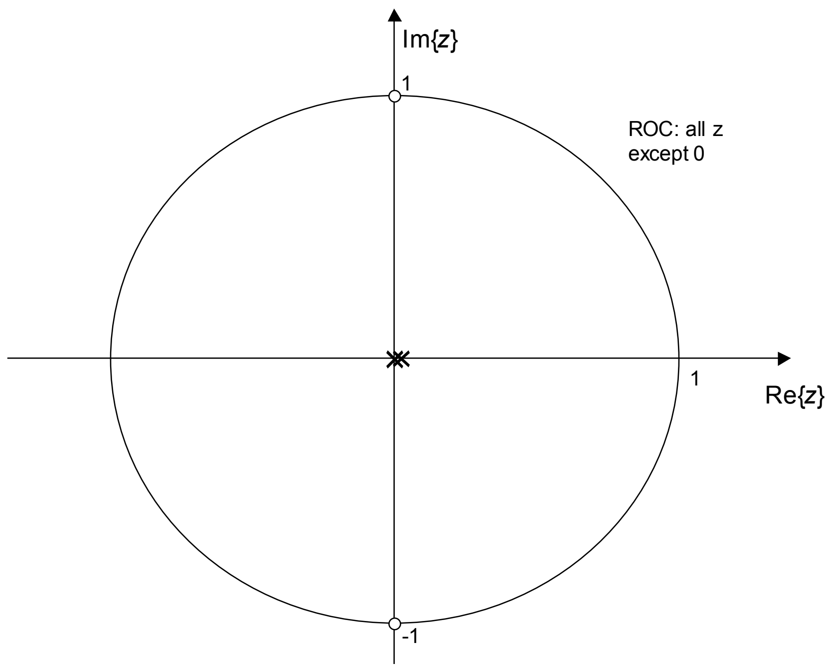
The discrete-time filter $H_d(z)$ is given by the following block diagram:



(a) [5 marks] Write the transfer function $H_d(z)$ of the filter, specify its ROC, and sketch its pole-zero plot. Compute the DC gain of $H_d(z)$.

Answer:

$$H(z) = 2 + 2z^{-2} = \frac{2z^2 + 2}{z^2}, \quad |z| > 0.$$



DC gain is $H(1) = 4$.

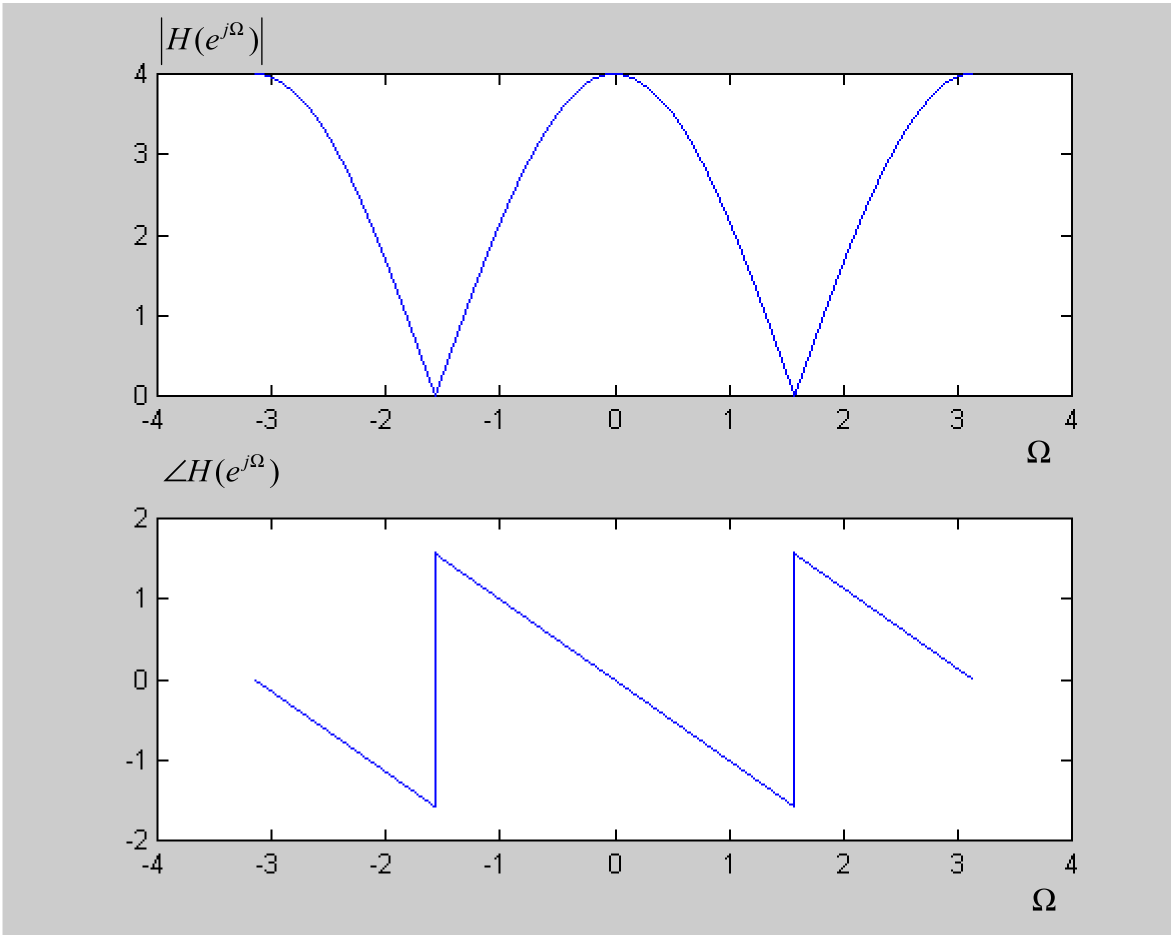
(b) [5 marks] Compute the frequency response $H(e^{j\Omega})$ of the filter and sketch its magnitude and its phase over $\Omega \in [-\pi, \pi]$.

Answer:

$$\begin{aligned} H(e^{j\Omega}) &= 2 + 2e^{-j2\Omega} = 4e^{-j\Omega}\left(0.5e^{j\Omega} + 0.5e^{-j\Omega}\right) \\ &= 4\cos\Omega e^{-j\Omega} \end{aligned}$$

$$\left|H(e^{j\Omega})\right| = 4\left|\cos\Omega\right|$$

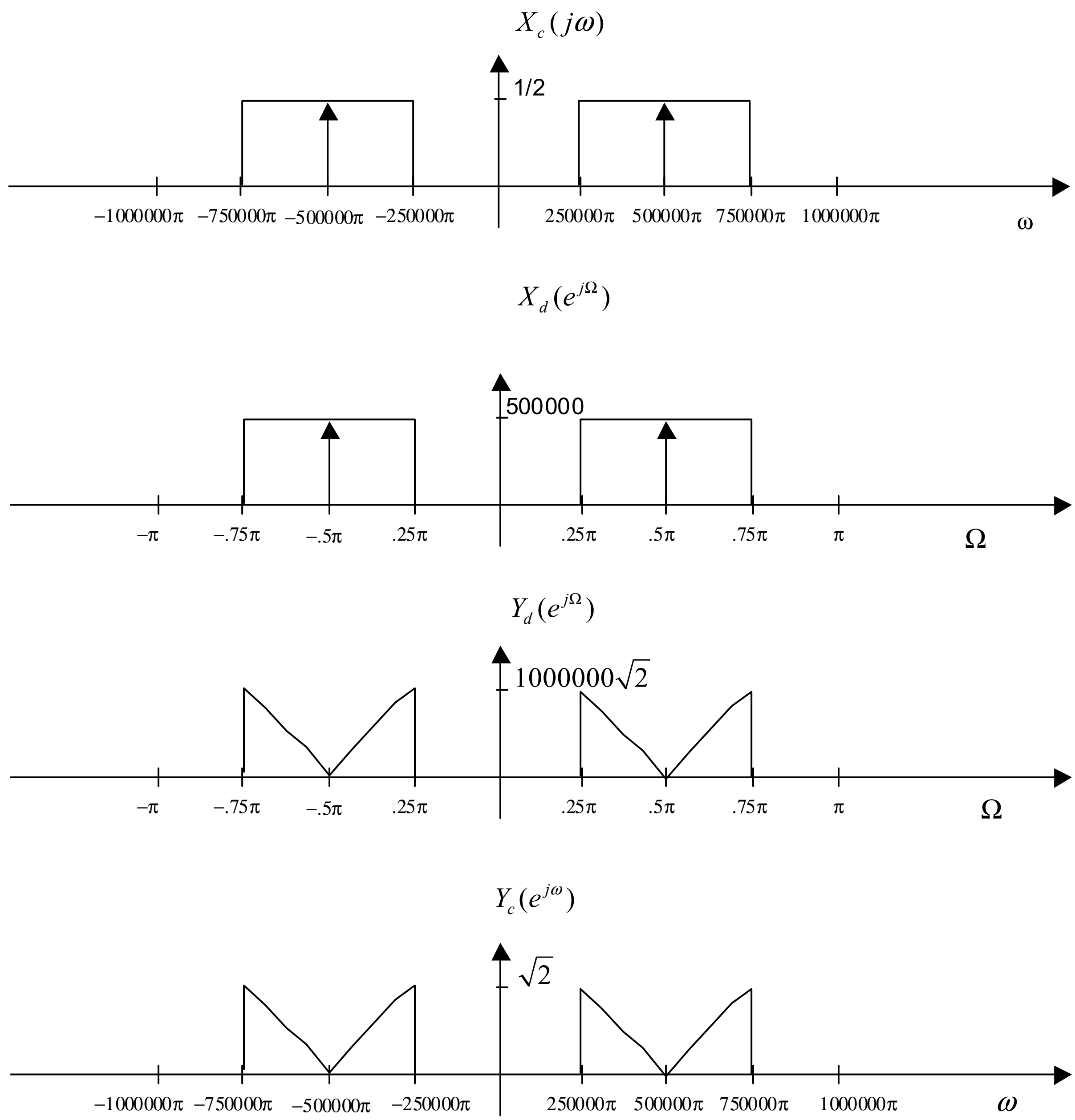
$$\angle H(e^{j\Omega}) = \begin{cases} -\Omega, & |\Omega| \le \frac{\pi}{2} \\ -\Omega + \pi, & \frac{\pi}{2} < \Omega \le \pi \\ -\Omega - \pi, & -\pi < \Omega < -\frac{\pi}{2} \end{cases}$$



(c) [5 marks] Suppose that the continuous-time signal to be filtered is given by: $x_c(t) = \cos(500000\pi t) + 250000\cos(500000\pi t)\mathrm{sinc}(250000t)$. Sketch the spectra of the continuous-time and discrete-time versions of the input signal $X_c(j\omega)$ and $X_d(e^{j\Omega})$. Sketch the spectra $Y_d(e^{j\Omega})$ of the discrete-time output signal and $Y_c(j\omega)$ of the continuous-time output signal $y_c(t)$.

Answer:

Recall that $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$ and the perfect unit-magnitude lowpass filter of bandwidth W has impulse response: $\frac{W}{\pi}\mathrm{sinc}(\frac{W}{\pi}t) = \frac{\sin Wt}{\pi t}$.

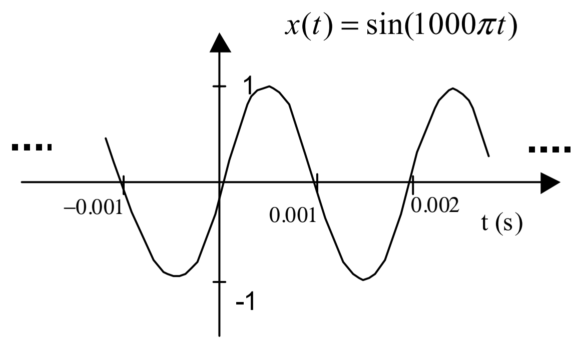


Problem 2 (15 marks)

Design an envelope detector to demodulate the AM signal:

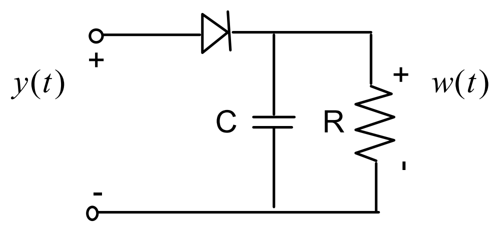
$$y(t) = [1 + 0.3x(t)]\cos(2\pi 10^6 t)$$

where $x(t)$ is the periodic modulating signal shown below. That is, draw a circuit diagram of the envelope detector and compute the values of the circuit components. Justify all of your approximations and assumptions. Provide rough sketches of the carrier signal, the modulated signal and the signal at the output of the detector. What is the modulation index m of the AM signal?



Answer:

An envelope detector can be implemented with the following simple RC circuit with a diode.



The output voltage of the detector, when it goes from one peak at voltage v_1 to the next when it intersects the modulated carrier at voltage v_2 after approximately one period $T = 1\mu s$ of the carrier, is given by:

$$v_2 \cong v_1 e^{-T/RC}$$

Since the time constant $\tau = RC$ of the detector should be large with respect to $T = 1\mu s$, we can use a first-order approximation of the exponential such that

$$v_2 \cong v_1(1 - T/RC).$$

This is a line of negative slope $-\frac{v_1}{RC}$ between the initial voltage v_1 and the final voltage v_2 so that

$$\frac{v_2 - v_1}{T} \cong -v_1/RC.$$

This slope must be more negative than the maximum negative slope of the envelope of $y(t)$, which is maximized as follows:

$$\min_t \frac{d(0.3x(t))}{dt} = \min_t \frac{d(0.3\sin(1000\pi t))}{dt}$$
$$= \min_t 300\pi\cos(1000\pi t) = -300\pi$$

Taking the worst-case $v_1 = 0.7$ that will make $-v_1/RC$ the largest, we must have

$$-\frac{0.7}{RC} < -300\pi$$
$$\Leftrightarrow$$
$$RC < \frac{0.7}{300\pi} = 0.00074$$

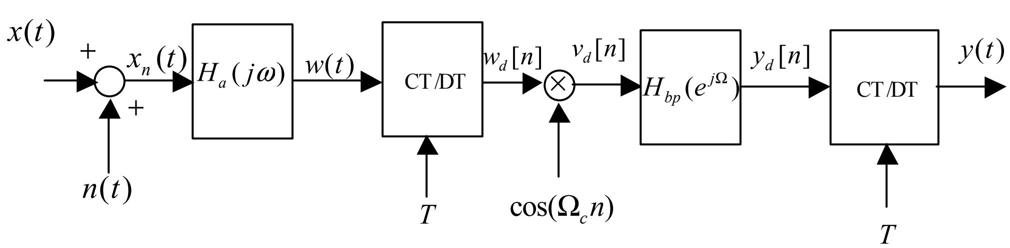
We could take $R = 7k\Omega,\ C = 0.1\mu F$ to get $RC = 0.0007$.

Let K be the maximum amplitude of $0.3x(t)$, i.e., $|0.3x(t)| < 0.3 = K$ and here $A = 1$. The *modulation index* m is the ratio $m = K/A = 0.3$.

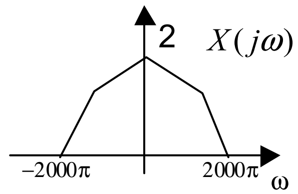
Problem 3 (15 marks)

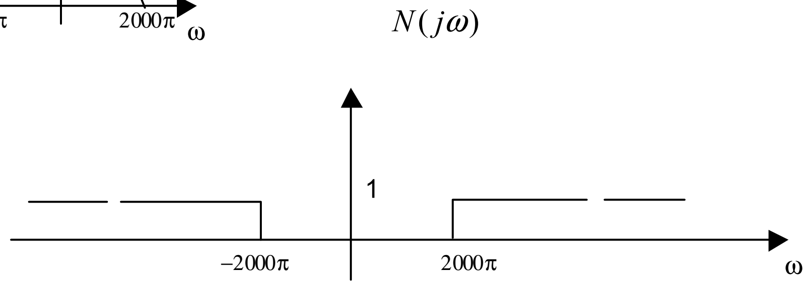
The system shown below is a lower single-sideband, suppressed-carrier AM modulator implemented in discrete time. The message signal $x(t)$ is corrupted by an additive noise $n(t)$: $x_n(t) = x(t) + n(t)$ before sampling. We want the modulator to operate at a carrier frequency $\omega_c = 2\pi \times 10^6$ rd/s.

The antialiasing filter $H_a(j\omega)$ is a perfect unity-gain lowpass filter with cutoff frequency ω_a. The antialiased signal $w(t)$ is first converted to a discrete-time signal $w_d[n]$ via the CT/DT operator. Signal $w_d[n]$ is modulated with frequency Ω_c and bandpass filtered by $H_{bp}(e^{j\Omega})$ to create the lower sidebands. Finally, the DT/CT operator produces the continuous-time lower SSB/SC AM signal $y(t)$.



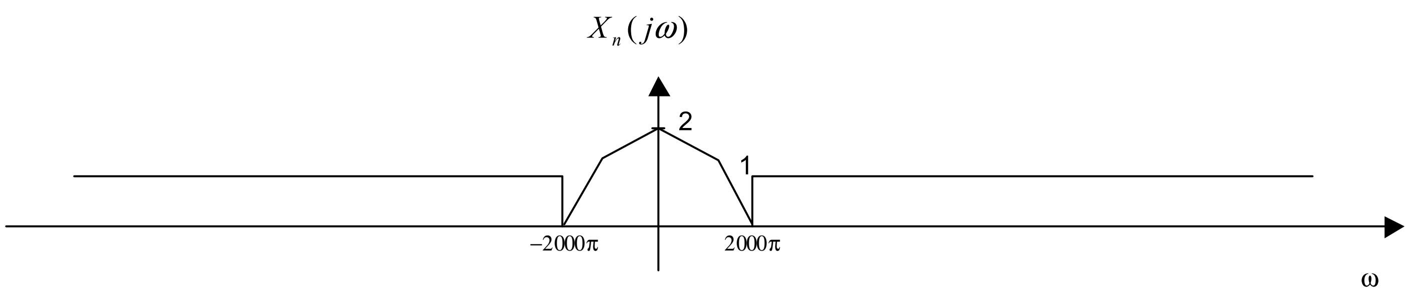
The modulating (or message) signal $x(t)$ has spectrum $X(j\omega)$ as shown below. The spectrum $N(j\omega)$ of the noise signal is also shown.

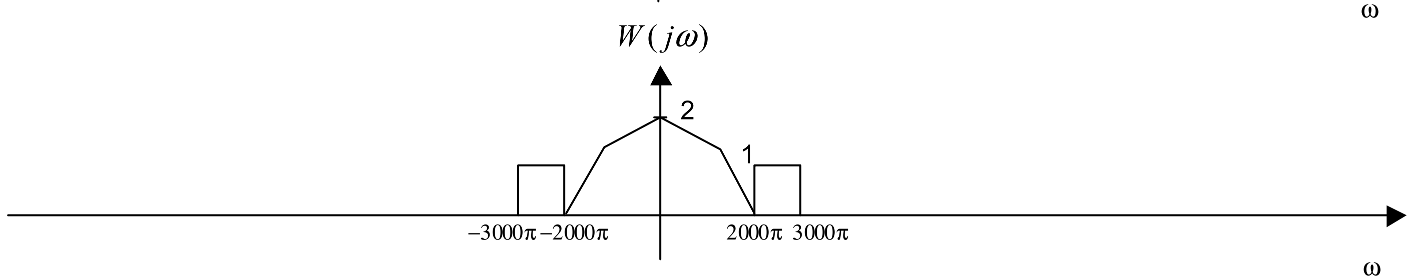




(a) [5 marks] The antialiasing filter's cutoff frequency is given as $\omega_a = 3000\pi$. Sketch the spectra $X_n(j\omega)$ and $W(j\omega)$ of signals $x_n(t)$ and $w(t)$. Indicate the important frequencies and magnitudes on your sketch.

Answer:





(b) [10 marks] Find the minimum sampling frequency ω_s and corresponding sampling period T that will produce the required modulated signal. Find the discrete-time modulation frequency Ω_c that will result in a continuous-time carrier frequency $\omega_c = 2\pi \times 10^6$ rd/s. Give the cutoff frequencies $\Omega_1 < \Omega_2$ of the bandpass filter to obtain a lower SSB/SC AM signal. Using these frequencies, sketch the spectra $W_d(e^{j\Omega}), V_d(e^{j\Omega}), Y_d(e^{j\Omega})$ and $Y(j\omega)$.

Answer:

In order to find the minimum sampling rate, we need to look at the desired lower SSB signal at the output. Its highest frequency component has to be at $\omega_c = 2\pi \times 10^6 \text{ rd/s}$, and therefore the sampling frequency has to be greater than $2\omega_c = 4\pi \times 10^6 \text{ rd/s}$, which is the minimum.

$$\omega_s = 2(\omega_c) = 4\pi 10^6 \text{ rd/s}, \quad T = \frac{2\pi}{\omega_s} = \frac{2\pi}{4\pi 10^6} = \frac{10^{-6}}{2} = 5\times 10^{-7}$$

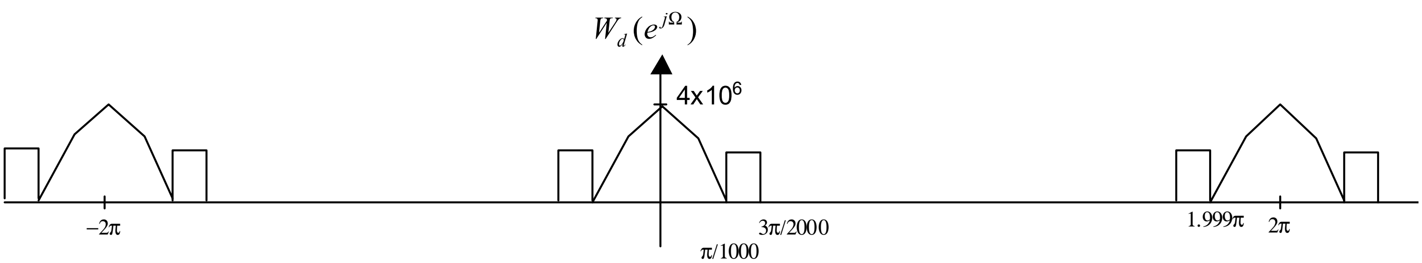
The carrier frequency: $\Omega_c = \dfrac{2\pi}{\omega_s}\omega_c = \pi$

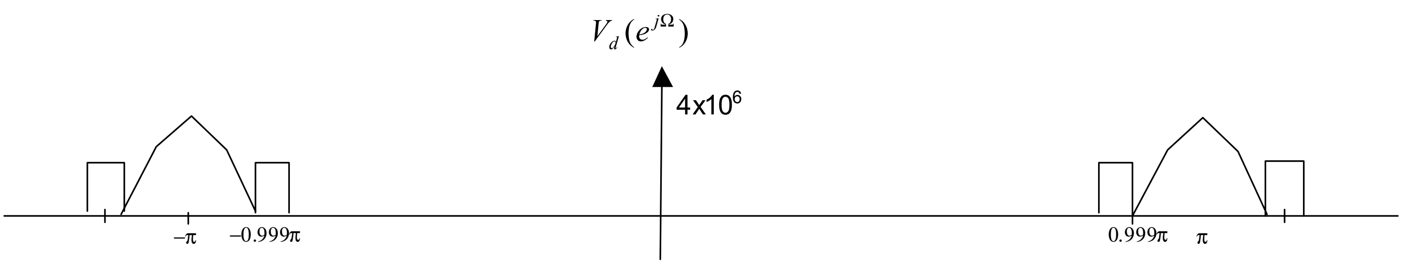
Bandpass filter's frequencies:

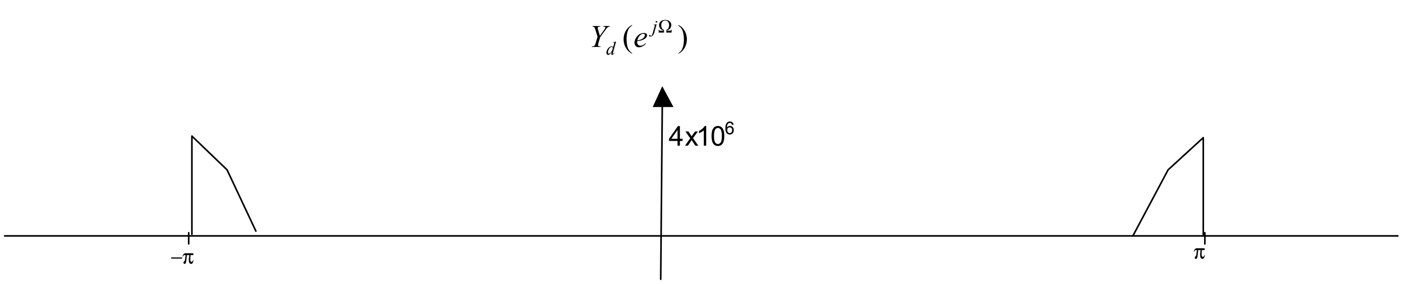
$$\Omega_1 = \pi - \left(\frac{2\pi}{\omega_s}\right)2\pi 10^3 = \pi - \frac{4\pi^2 10^3}{4\pi 10^6} = \pi - \pi 10^{-3} = 0.999\pi$$

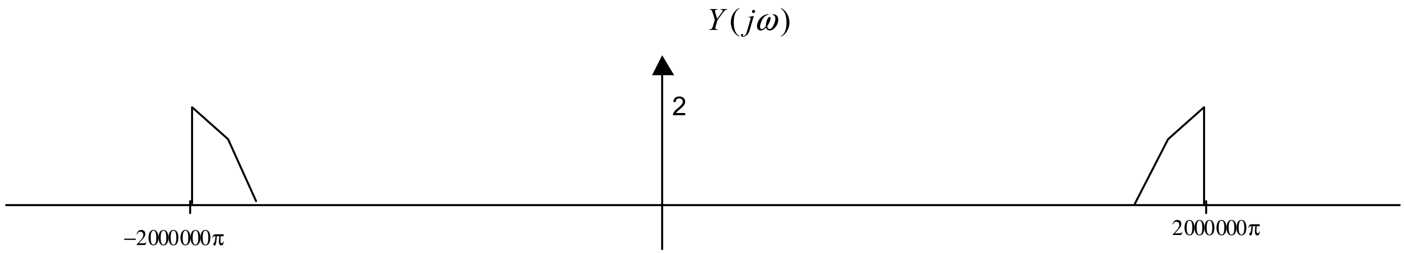
$$\Omega_2 = \pi$$

(assume unity gain)



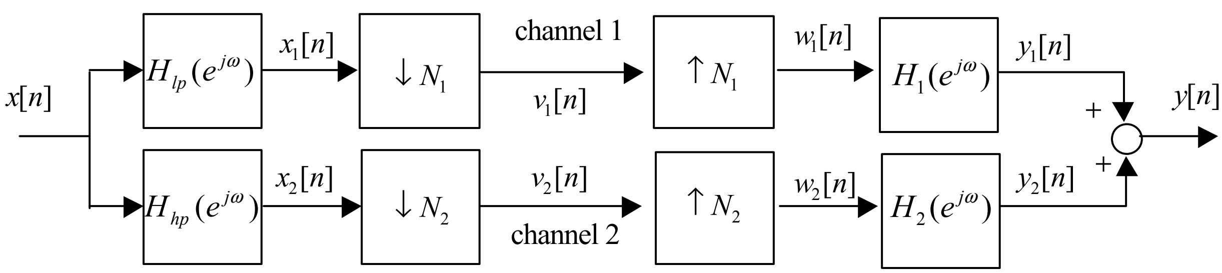






Problem 4 (25 marks)

Consider the *maximally decimated multirate filter bank* shown below, used for voice data compression. This system transmits a signal $x[n]$ over two low-bit-rate channels. The sampled voice signal is bandlimited to $\omega_M = \frac{2\pi}{5}$.



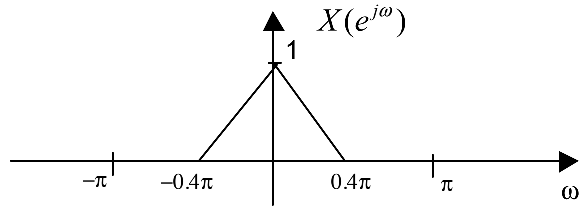
Numerical values:

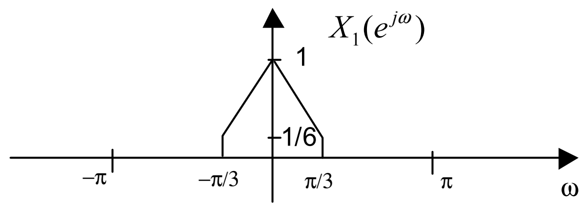
Input lowpass filters' cutoff frequency $\omega_{clp} = \frac{\pi}{3}$,

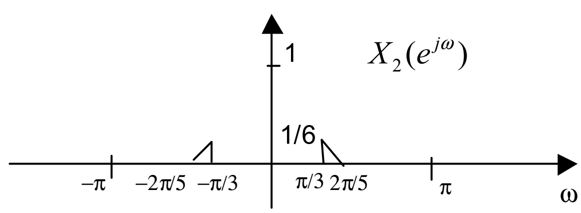
Input highpass filter's cutoff frequency $\omega_{chp} = \frac{\pi}{3}$,

signal's spectrum over $[-\pi, \pi]$: $X(e^{j\omega}) = \begin{cases} 1 - \frac{5}{2\pi}|\omega|, & |\omega| \le \frac{2\pi}{5} \\ 0, & \frac{2\pi}{5} < |\omega| < \pi \end{cases}$.

(a) [5 marks] Sketch the spectra $X(e^{j\omega})$, $X_1(e^{j\omega})$, $X_2(e^{j\omega})$, indicating the important frequencies and magnitudes.

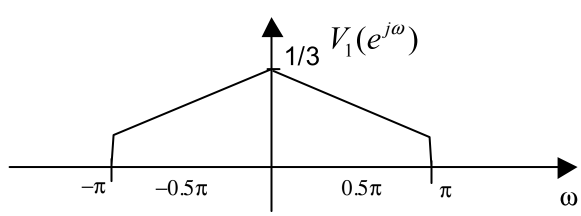


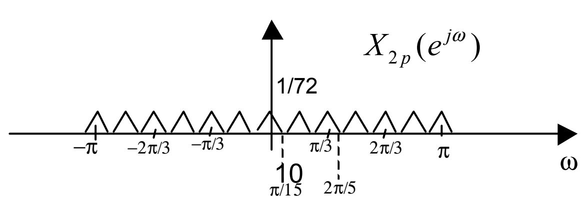


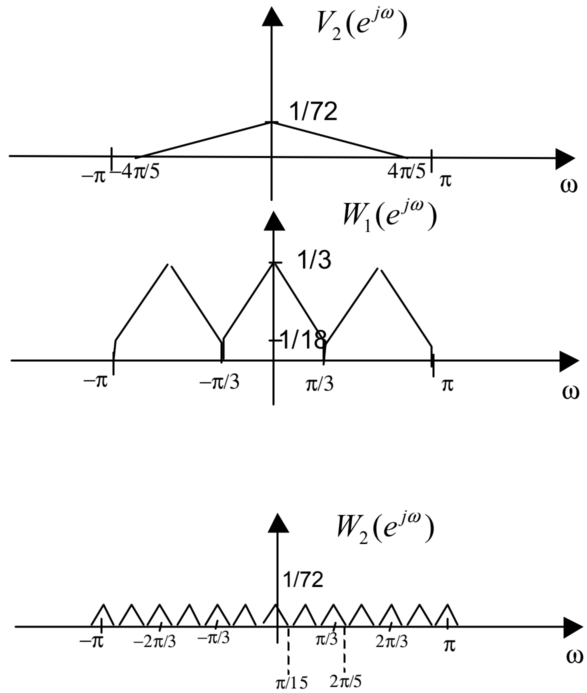


(b) [10 marks] Find the maximum decimation factors N_1 and N_2 avoiding aliasing and sketch the corresponding spectra $V_1(e^{j\omega})$, $V_2(e^{j\omega})$, $W_1(e^{j\omega})$, $W_2(e^{j\omega})$, indicating the important frequencies and magnitudes. Specify what the ideal output filters $H_1(e^{j\omega})$ and $H_2(e^{j\omega})$ should be for perfect signal reconstruction. Sketch their frequency responses, indicating the important frequencies and magnitudes.

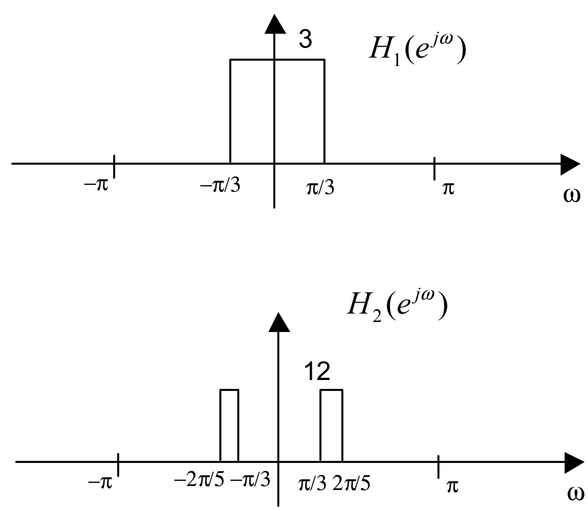
Maximum decimation factors: $N_1 = 3$ *and* $N_2 = 12$







Filter $H_1(e^{j\omega})$ should be lowpass and filter $H_2(e^{j\omega})$ should be bandpass:



(c) [6 marks] Compute the percentage ratio of energies at the output between the high-frequency subband $y_2[n]$ and the output signal $y[n]$, i.e., compute $r := 100\dfrac{E\{y_2\}}{E\{y\}}$, where the energy is given by $E\{y\} := \sum_{n=-\infty}^{+\infty} \left|y[n]\right|^2$.

Answer:
Using Parseval's relationship:

$$E\{y\} := \sum_{n=-\infty}^{+\infty} \left|y[n]\right|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|Y(e^{j\omega})\right|^2 d\omega$$

$$= \frac{1}{\pi}\int_{0}^{2\pi/5} \left|1-\frac{5}{2\pi}\omega\right|^2 d\omega = \frac{1}{\pi}\int_{0}^{2\pi/5}\left(1-\frac{5}{\pi}\omega+\frac{25}{4\pi^2}\omega^2\right)d\omega$$

$$= \frac{1}{\pi}\left(\omega-\frac{5}{2\pi}\omega^2+\frac{25}{12\pi^2}\omega^3\right)_0^{2\pi/5} = \frac{1}{\pi}\left(\frac{2\pi}{5}-\frac{2\pi}{5}+\frac{25(8\pi^3)}{12\pi^2(125)}\right)_0^{2\pi/5}$$

$$= \frac{1}{\pi}\left(\frac{2\pi}{15}\right) = \frac{2}{15} = 0.1333$$

$$E\{y_2\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|Y_2(e^{j\omega})\right|^2 d\omega$$

$$= \frac{1}{\pi}\int_{\pi/3}^{2\pi/5} \left|1-\frac{5}{2\pi}\omega\right|^2 d\omega$$

$$= \frac{1}{\pi}\left(\omega-\frac{5}{2\pi}\omega^2+\frac{25}{12\pi^2}\omega^3\right)_{\pi/3}^{2\pi/5}$$

$$= \frac{1}{\pi}\left(\frac{2\pi}{5}-\frac{2\pi}{5}+\frac{25(8\pi^3)}{12\pi^2(125)}-\frac{\pi}{3}+\frac{5\pi}{18}-\frac{25(\pi^3)}{12\pi^2(27)}\right)$$

$$= \frac{1}{\pi}\left(\frac{2\pi}{15}-\frac{\pi}{18}-\frac{25\pi}{12(27)}\right) = \frac{2}{15}-\frac{1}{18}-\frac{25}{324} = 0.000617$$

$$r = 100\frac{E\{y_2\}}{E\{y\}} = 100\frac{0.000617}{0.1333} = 0.46\%$$

(d) [4 marks] Now suppose each of the two signal subbands $v_1[n]$ and $v_2[n]$ are quantized with a different number of bits to achieve good data compression, without losing intelligibility of the voice message. According to your result in (c), it would seem to make sense to use less bits to quantize the high-frequency subband. Suppose that the filter bank operates at an input/output sample rate of 10kHz, and you decide to use 12 bits to quantize $v_1[n]$ and 4 bits to quantize $v_2[n]$. Compute the bit rates of channel 1 and channel 2, and the overall bit rate of the system.

Answer:
After decimation by 3, channel 1 operates at 3.333 kilosamples/s which results in a bit rate of $3333.33333 * 12 = 40\,000 \text{ bits/s}$. And after decimation by 2, channel 2 operates at 5 kilosamples/s which results in a bit rate of $5000 * 12 = 60\,000 \text{ bits/s}$.
Therefore, the overall bit rate of the filter bank is 100 000 bits/s.

Problem 5 (10 marks)

(a) [5 marks] A causal recursive DLTI system described by its zero-state impulse response given below has an initial condition $y[-1] = 5$. Calculate the zero-input response $y_{zi}[n]$ of the system for $n \geq 0$ using the unilateral z-transform.

$$h[n] = (0.2)^n u[n] + \delta[n]$$

Answer:
Taking the unilateral z-transform, we get:

$$\mathcal{H}(z) = \frac{1}{1-0.2z^{-1}} + 1 = \frac{2-0.2z^{-1}}{1-0.2z^{-1}}, \ |z| > 0.2$$

which yields the difference equation:

$$y[n] - 0.2y[n-1] = 2x[n] - 0.2x[n-1].$$

Taking the unilateral z-transform, we can include the initial condition:

$$\mathcal{Y}(z) - 0.2z^{-1}\mathcal{Y}(z) - 0.2y[-1] = 2\mathcal{X}(z) - 0.2z^{-1}\mathcal{X}(z)$$

$$\mathcal{Y}(z) = \frac{2-0.2z^{-1}}{1-0.2z^{-1}}\mathcal{X}(z) + \frac{0.2}{1-0.2z^{-1}}y[-1], \quad |z| > 0.2$$

$$y_{zi}[n] = \mathcal{Z}^{-1}\left\{\frac{1}{1-0.2z^{-1}}\right\} = (0.2)^n u[n]$$

(b) [2 marks] Determine if the following signal is periodic. If it is, find its fundamental period N and fundamental frequency Ω_0. If it is not periodic, explain why.

$$x[n] = e^{-j2\pi n} \sin\left(\frac{0.13\pi}{3} n + 0.13\pi\right).$$

Answer:

$$x[n] = \sin\left(\frac{0.13\pi}{3}(n+3)\right)$$

$$\omega_0 = \frac{0.13\pi}{3} = \frac{13\pi}{300} = \frac{13(2\pi)}{600}$$

Yes the signal is periodic.

The fundamental period is $N = 600$, fundamental frequency is $\Omega_0 = \frac{2\pi}{600} = \frac{\pi}{300}$

(c) [3 marks] Compute the zero-state response of the system in (a) for the input signal in (b).

Answer:
Using the TF found in (a), we have the frequency response:

$$H(e^{j\omega}) = \frac{2e^{j\omega} - 0.2}{e^{j\omega} - 0.2}$$

So

$$y[n] = \left|H(e^{j\omega_0})\right| \sin\left[\frac{0.13\pi}{3}(n+3) + \angle H(e^{j\omega_0})\right]$$

$$\omega_0 = \frac{0.13\pi}{3}$$

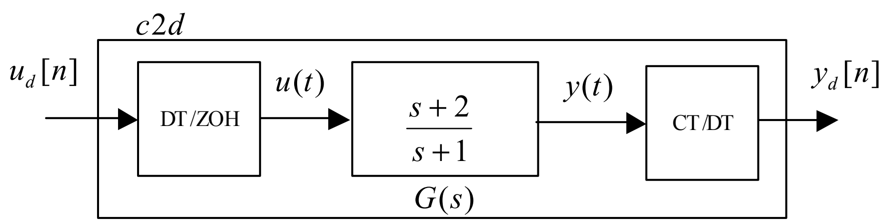
$$H(e^{j\omega_0}) = \frac{2e^{j\frac{0.13\pi}{3}} - 0.2}{e^{j\frac{0.13\pi}{3}} - 0.2} = 2.2457 - 0.0422j = 2.246e^{-j0.01877}$$

$\Rightarrow$

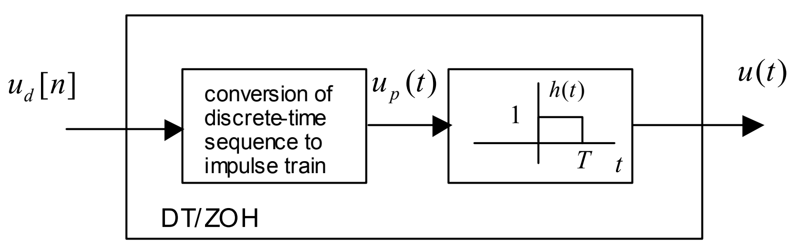
$$y[n] = 2.246\sin\left[\frac{0.13\pi}{3}(n+3) - 0.01877\right]$$

Problem 6 (20 marks)

The causal continuous-time LTI system given by its transfer function $G(s) = \frac{s+2}{s+1}$ is discretized with sampling period $T = 0.1$ s for simulation purposes. The "c2d" transformation as shown below is used first, and then it is compared to the bilinear transformation.



Recall that the DT/ZOH operator is defined as a mapping from a discrete-time signal to an impulse train (consisting of forming a train of impulses occurring every T seconds, with the impulse at time nT having an area $u_d[n]$), followed by a "hold" function that holds each value for a period T.



(a) [5 marks] Find a state-space realization of $G(s)=\dfrac{s+2}{s+1},\ \operatorname{Re}\{s\}>-1$.

Answer:

The transfer function corresponding to the system is

$$\frac{s+2}{s+1},\ \operatorname{Re}\{s\}>-1.$$

Partial fraction expansion: $\dfrac{s+2}{s+1}=1+\dfrac{1}{s+1},\ \operatorname{Re}\{s\}>-1$.

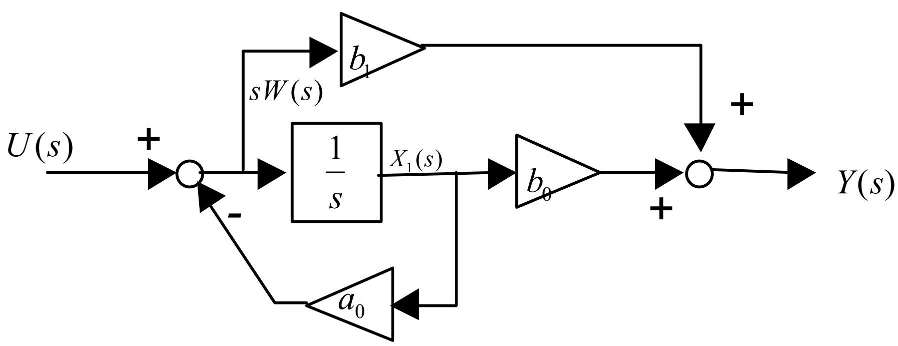
Corresponding state-space system:

$$\dot{x}=-x+u$$
$$y=x+u$$

Thus, $A=-1,\ B=1,\ C=1,\ D=1$.

Alternate solution using the controllable canonical state-space realization:

$A=-1,\ B=1,\ C=1,\ D=1$.



Where $a_0=1, b_0=2, b_1=1$. State-space system:

$$\dot{x}=-a_0x+u=\underbrace{-1}_{A}x+\underbrace{1}_{B}u$$
$$y=(b_0-a_0b_1)x+b_1u=\underbrace{1}_{C}x+\underbrace{1}_{D}u$$

where $A=-1,\ B=1,\ C=1,\ D=1$.

(b) [8 marks] Write the mathematical relationships mapping the continuous-time state-space matrices (A,B,C,D) into the discrete-time state-space matrices (A_d,B_d,C_d,D_d) in the "c2d" discretization technique. Compute the discrete-time state-space system (A_d,B_d,C_d,D_d) for $G(s)$ and its associated transfer function $G_{c2d}(z)$, specifying its ROC.

Answer:

Discretized state-space system using "c2d":

$$
\begin{aligned}
A_d &= e^{AT} = e^{-0.1} = 0.9048 \\
B_d &:= A^{-1}\left[e^{AT} - I_n\right]B = -(e^{-0.1} - 1)(1) = 0.0952 \\
C_d &:= C = 1 \\
D_d &:= D = 1
\end{aligned}
$$

transfer function:

$$
\begin{aligned}
G_{c2d}(z) &= C_d(zI_n - A_d)^{-1}B_d + D_d \\
&= 1(z-0.9048)^{-1}(0.0952) + 1 = \frac{0.0952}{z-0.9048} + 1 \\
&= \frac{z-0.8096}{z-0.9048}, \ |z| > 0.9048
\end{aligned}
$$

(c) [7 marks] Find the bilinear transformation $G_{bilin}(z)$ of $G(s)$, specifying its ROC. Compute the difference between the two frequency responses obtained with *c2d* and the bilinear transformation, i.e., compute $E(e^{j\omega}) := G_{bilin}(e^{j\omega}) - G_{c2d}(e^{j\omega})$. Evaluate this difference at dc and at the highest discrete-time frequency.

Answer:

The bilinear transformation is $s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}} = 20\frac{1-z^{-1}}{1+z^{-1}}$. Thus

$$
\begin{aligned}
G_{bilin}(z) &= \left.\frac{s+2}{s+1}\right|_{s=20\frac{1-z^{-1}}{1+z^{-1}}} = \frac{20\frac{1-z^{-1}}{1+z^{-1}} + 2}{20\frac{1-z^{-1}}{1+z^{-1}} + 1} = \frac{\frac{20(1-z^{-1}) + 2 + 2z^{-1}}{1+z^{-1}}}{\frac{20(1-z^{-1}) + 1 + z^{-1}}{1+z^{-1}}} \\
&= \frac{20(1-z^{-1}) + 2 + 2z^{-1}}{20(1-z^{-1}) + 1 + z^{-1}} = \frac{-18z^{-1} + 22}{-19z^{-1} + 21} \\
&= \frac{22z - 18}{21z - 19}, \ |z| > \frac{19}{21} \\
&= 1.048\frac{z-0.8182}{z-0.9048}, \ |z| > \frac{19}{21}
\end{aligned}
$$

Alternate solution:

Discretized state-space system using "bilinear transf.":

$$A_d := \left(1 - \frac{T}{2}A\right)^{-1}\left(1 + \frac{T}{2}A\right) = \left(1 + \frac{0.1}{2}\right)^{-1}\left(1 - \frac{0.1}{2}\right) = \frac{0.95}{1.05} = 0.9048$$

$$B_d := T\left(1 - \frac{T}{2}A\right)^{-2} B = 0.1\left(1 + \frac{0.1}{2}\right)^{-2} = 0.0907$$

$$C_d := C = 1$$

$$D_d := D + \frac{T}{2}C\left(1 - \frac{T}{2}A\right)^{-1} B = 1 + 0.05(1)\left(1 + \frac{0.1}{2}\right)^{-1} = 1.0476$$

transfer function:

$$\begin{aligned}G_{bilin}(z) &= C_d(zI_n - A_d)^{-1}B_d + D_d \\ &= 1(z - 0.9048)^{-1}(0.0907) + 1.0476 = \frac{0.0907}{z - 0.9048} + 1.0476 \\ &= \frac{1.0476z - 0.8572}{z - 0.9048} = \frac{1.0476(z - 0.8182)}{z - 0.9048}, \ |z| > 0.9048\end{aligned}$$

$$\begin{aligned}E(z) &:= G_{bilin}(z) - G_{c2d}(z) = 1.048\frac{z - 0.8182}{z - 0.9048} - \frac{z - 0.8096}{z - 0.9048} \\ &= \frac{0.048z - 0.8575 + 0.8096}{z - 0.9048} = \frac{0.048z - 0.048}{z - 0.9048} \qquad E(e^{j\omega}) = \frac{0.048(e^{j\omega} - 1)}{e^{j\omega} - 0.9048}, \\ &= \frac{0.048(z - 1)}{z - 0.9048},\end{aligned}$$

$$E(e^{j0}) = 0$$

$$E(e^{j\pi}) = \frac{0.048(-2)}{-1.9048} = 0.05$$

END OF EXAMINATION