Sample Final Exam (finals00)
Covering Chapters 10-17 of *Fundamentals of Signals & Systems*

Problem 1 (10 marks)

Consider the causal FIR filter described by its impulse response

$$h[n] = -\frac{1}{2}\delta[n] + \delta[n-1] - \frac{1}{2}\delta[n-2].$$

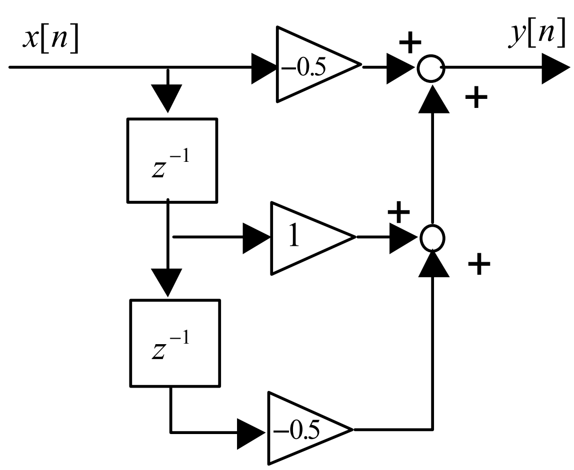
(a) [3 marks] Draw a block diagram of the direct form realization of the filter. Write the transfer function $H(z)$ of the filter and specify its ROC.

(b) [5 marks] Find the frequency response $H(e^{j\omega})$ of the filter and give its magnitude and its phase. Sketch its magnitude $\left|H(e^{j\omega})\right|$ for $\omega \in [-\pi, \pi]$. What type of filter is it? (low-pass, band-pass or high-pass?)

(c) [2 marks] Compute the -3dB cutoff frequency ω_c of the filter.

Answer:

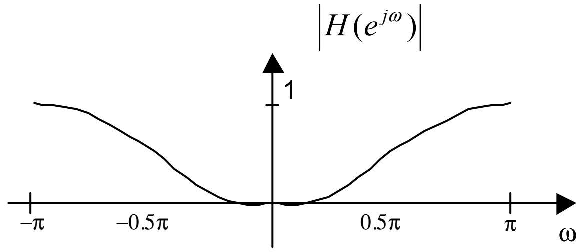
(a) [3 marks]



$$H(z) = -\frac{1}{2} + z^{-1} - \frac{1}{2}z^{-2} = \frac{-0.5z^2 + z - 0.5}{z^2}, \quad |z| > 0$$

(b) [5 marks] *Frequency response:*

$$H(e^{j\omega}) = -\frac{1}{2} + e^{-j\omega} - \frac{1}{2}e^{-j2\omega} = e^{-j\omega}\left(1 - \frac{1}{2}e^{j\omega} - \frac{1}{2}e^{-j\omega}\right)$$
$$= e^{-j\omega}(1 - \cos\omega)$$
$$\left|H(e^{j\omega})\right| = \left|1 - \cos\omega\right|$$
$$\angle H(e^{j\omega}) = -\omega$$



Highpass filter

(c) [2 marks] -3dB cutoff frequency ω_c of the filter:

$$\left|1 - \cos\omega_c\right| = \frac{1}{\sqrt{2}} \Leftarrow \omega_c = a\cos(1 - \frac{1}{\sqrt{2}}) = 1.2735$$

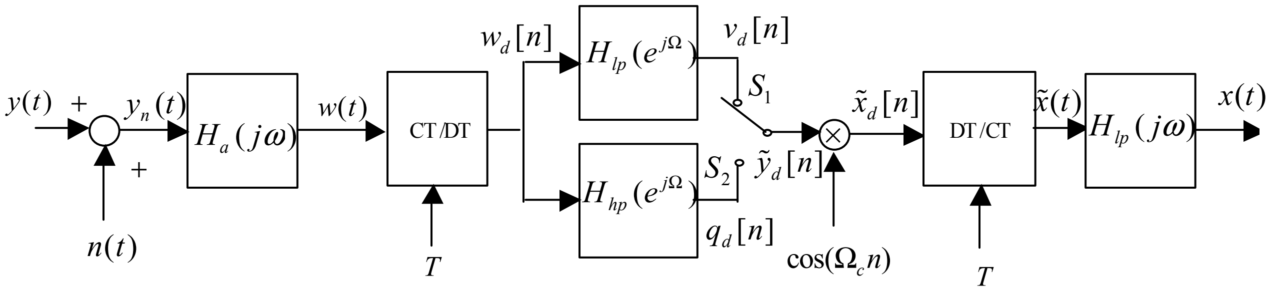
Problem 2 (20 marks)

The system shown below processes the continuous-time signal $y_n(t) = y_1(t) + y_2(t) + n(t)$ composed of the sum of:

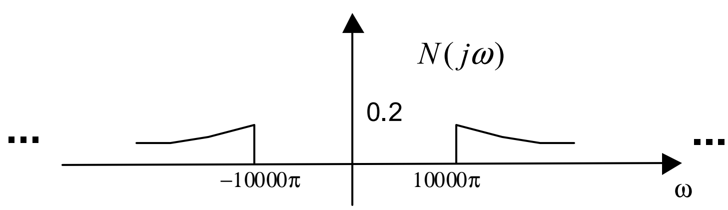
- Signal $y_1(t)$ which is a lower single-sideband amplitude modulation, suppressed carrier (lower SSB-AM/SC) of the corresponding message signal $x_1(t)$,
- Signal $y_2(t)$ which is an upper single-sideband amplitude modulation, suppressed carrier (upper SSB-AM/SC) of the corresponding message signal $x_2(t)$,
- a noise signal $n(t)$.

Both SSB-AM/SC signals share the same carrier signal $\cos(\omega_c t)$ of frequency ω_c. Also assume that the sidebands have the same magnitude as the spectra of the message signals.

The antialiasing filter $H_a(j\omega)$ is a perfect unity-gain lowpass filter with cutoff frequency ω_a. The antialiased signal $w(t)$ is first converted to a discrete-time signal $w[n]$ via the CT/DT operator. Signal $w[n]$ is processed in parallel by two discrete-time filters: a lowpass filter $H_{lp}(e^{j\Omega})$, and a highpass filter $H_{hp}(e^{j\Omega})$. Then, the output of either filter, as selected by a switch, is the input to a discrete-time synchronous demodulator with frequency Ω_c which is used to demodulate either of the two discretized message signals. Finally, the DT/CT operator followed by a continuous-time lowpass filter reconstruct the desired continuous-time message signal.



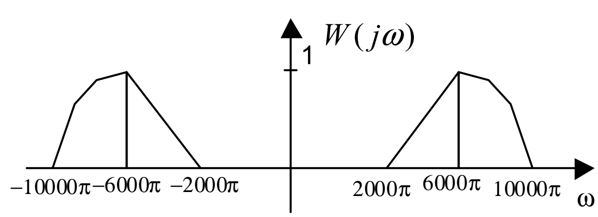
The modulating (or message signals) $x_1(t), x_2(t)$ have spectra $X_1(j\omega), X_2(j\omega)$ as shown below. The spectrum $N(j\omega)$ of the noise signal is also shown below.



(a) [5 marks] Find the maximum carrier frequency ω_c and the corresponding minimum antialiasing filter's cutoff frequency ω_a that will avoid any unrepairable distortion of the modulated signals due to the additive noise $n(t)$. Sketch the spectrum $W(j\omega)$ of the signal $w(t)$ for the frequencies ω_c and ω_a that you found. Indicate the important frequencies and magnitudes on your sketch.

Answer:

Maximum $\omega_c = 10000\pi - 4000\pi = 6000\pi$. Minimum $\omega_a = 10000\pi$.



(b) [4 marks] Find the minimum sampling frequency $\omega_s = \dfrac{2\pi}{T}$ and its corresponding sampling period T that will allow perfect reconstruction of the modulated signal.

Answer:

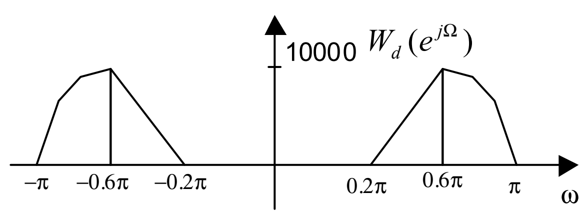
$$\omega_s = 2\omega_a = 20000\pi,\ T = \frac{2\pi}{\omega_s} = 10^{-4}$$

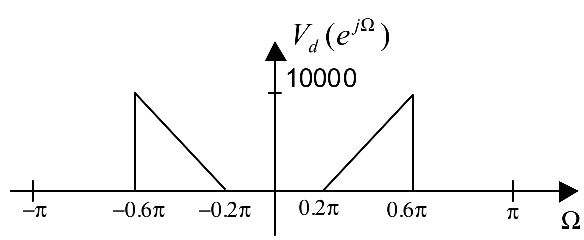
(c) [5 marks] Suppose that the frequency of the discrete-time synchronous demodulator can be set to either Ω_{c1} or Ω_{c2} corresponding to switch positions S_1 and S_2, respectively. With the frequencies that you found in (a) and (b), find the cutoff frequencies Ω_{lp} and Ω_{hp} of the ideal discrete-time lowpass and highpass filters. Also give the frequencies Ω_{c1} and Ω_{c2}. Sketch the spectra $W_d(e^{j\Omega})$, $V_d(e^{j\Omega})$ and $Q_d(e^{j\Omega})$ over the interval $[-\pi,\pi]$, indicating the important frequencies and magnitudes.

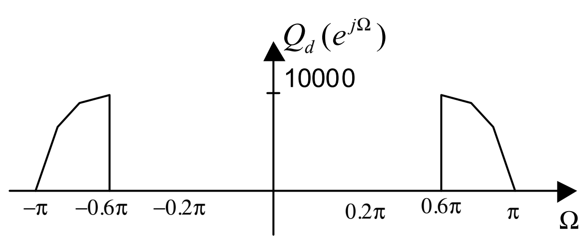
Answer:

Cutoff frequencies: $\Omega_{lp} = 6000\pi T = 0.6\pi, \quad \Omega_{hp} = 6000\pi T = 0.6\pi$

demodulator frequencies $\Omega_{c1} = 0.6\pi, \quad \Omega_{c2} = 0.6\pi$

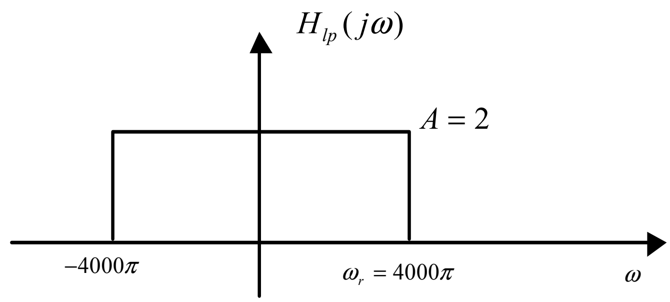




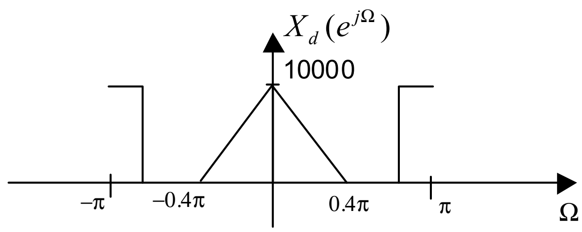


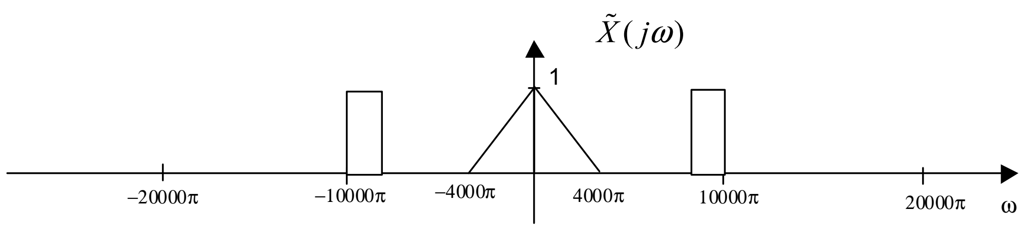
(d) [6 marks] Find the cutoff frequency ω_r of the ideal continuous-time lowpass filter $H_{lp}(j\omega)$ and its gain A so that the message signals can be recovered, i.e., $x(t) = x_1(t)$ when the switch is at position S_1, and $x(t) = x_2(t)$ when the switch is at position S_2. Sketch the spectra $X_d(e^{j\omega})$, $\tilde{X}(j\omega)$ and $X(j\omega)$ in each case (switch at S_1 and at S_2), indicating the important frequencies and magnitudes.

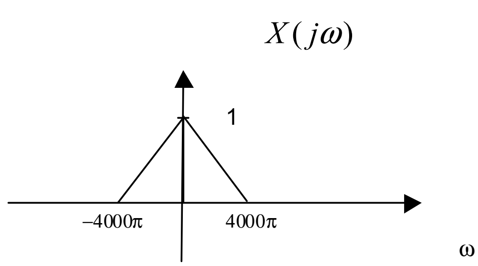
Answer:



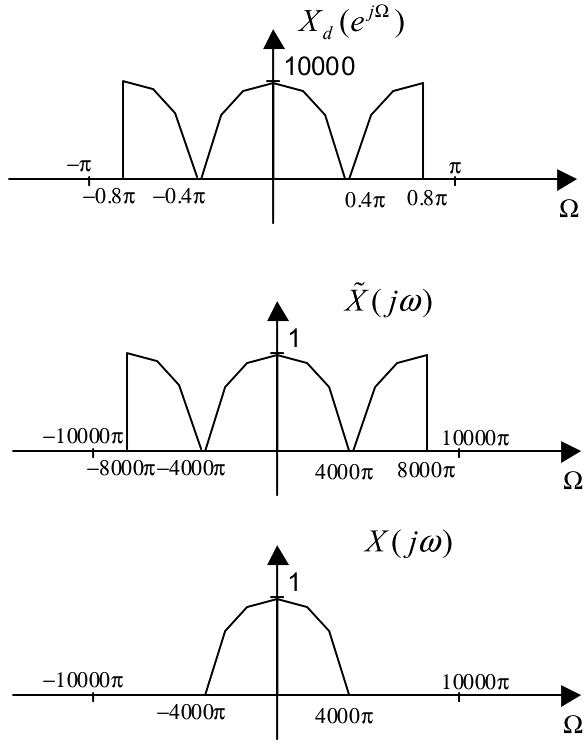
Switch at S1:





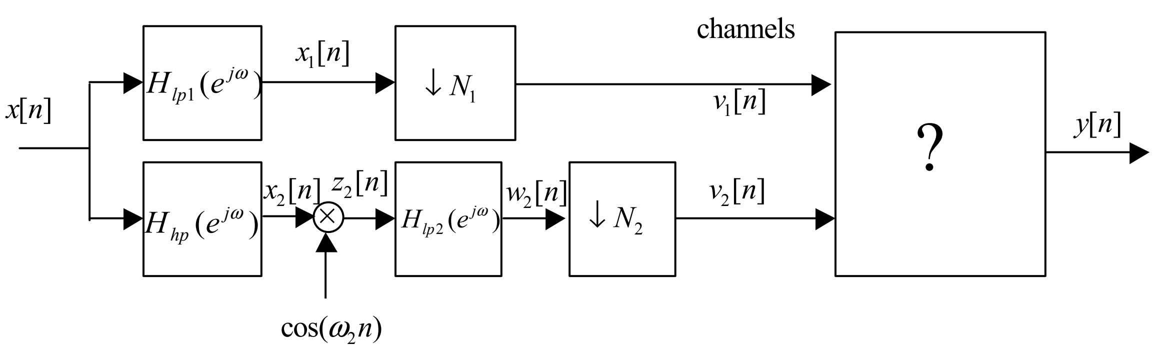


Switch at S2:



Problem 3 (20 marks)

Consider the discrete-time system shown below, where $\downarrow N$ represents decimation by N. This system transmits a signal $x[n]$ over two low-rate channels.



Numerical values:

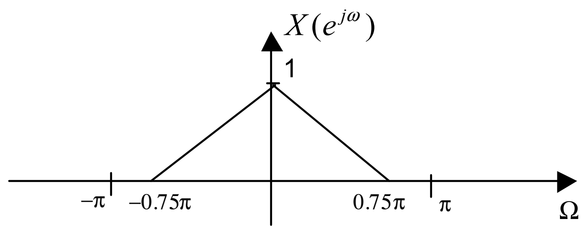
lowpass filters' cutoff frequencies $\omega_{clp1} = \omega_{clp2} = \dfrac{\pi}{2}$,

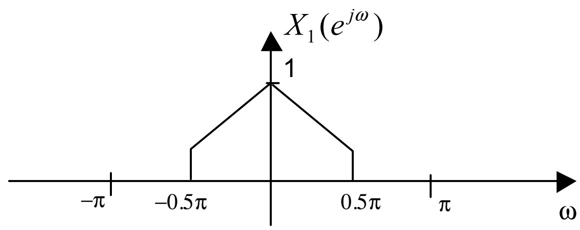
highpass filter's cutoff frequency $\omega_{chp} = \dfrac{\pi}{2}$, modulation carrier frequency: $\omega_2 = \dfrac{\pi}{2}$,

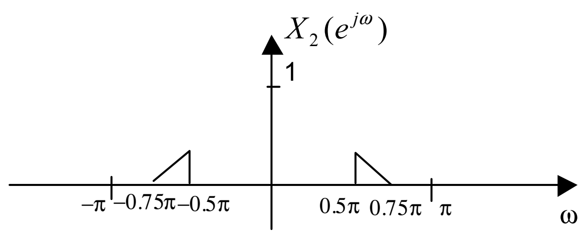
signal's spectrum over $[-\pi,\pi]$: $X(e^{j\omega}) = \begin{cases} 1 - \dfrac{4}{3\pi}|\omega|, & |\omega| \le \dfrac{3\pi}{4} \\ 0, & \dfrac{3\pi}{4} < |\omega| < \pi \end{cases}$.

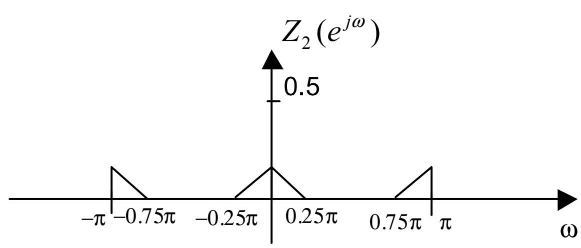
(a) [6 marks] Sketch the spectra $X(e^{j\omega})$, $X_1(e^{j\omega})$, $X_2(e^{j\omega})$, $Z_2(e^{j\omega})$ and $W_2(e^{j\omega})$, indicating the important frequencies and magnitudes.

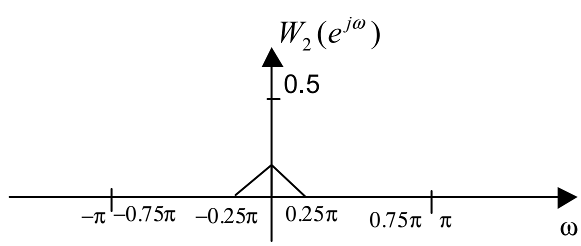
Answer:







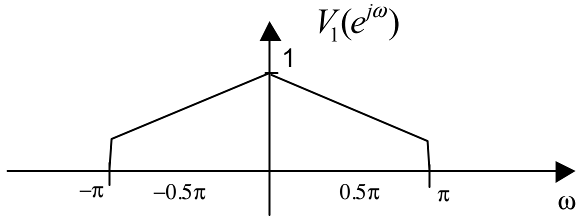


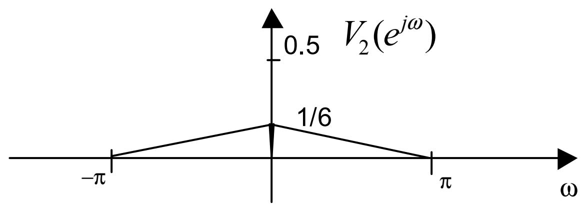


(b) [6 marks] Find the maximum decimation factors N_1 and N_2 avoiding aliasing and sketch the corresponding spectra $V_1(e^{j\omega})$, $V_2(e^{j\omega})$, indicating the important frequencies and magnitudes.

Answer:

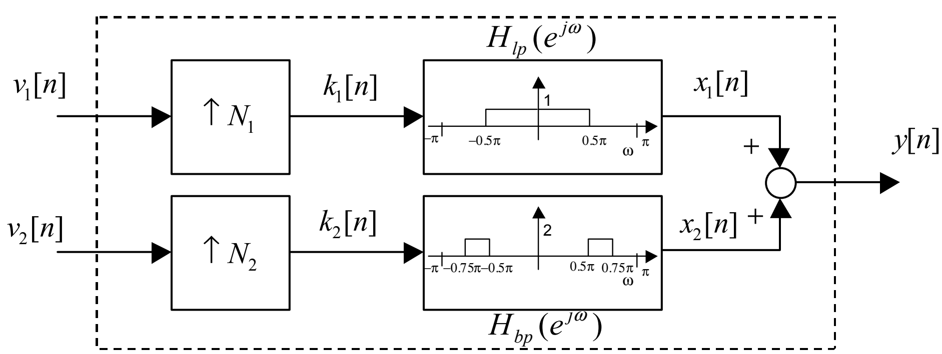
Maximum decimation factors: $N_1 = 2$ *and* $N_2 = 4$.





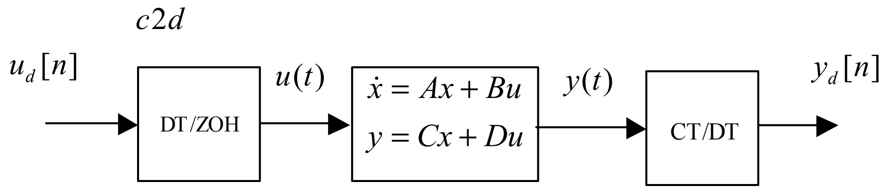
(c) [8 marks] Design the receiver system (draw its block diagram) such that $y[n] = x[n]$. You can use upsamplers (with symbol $\uparrow N$), synchronous demodulators, ideal filters and summing junctions.

Answer:

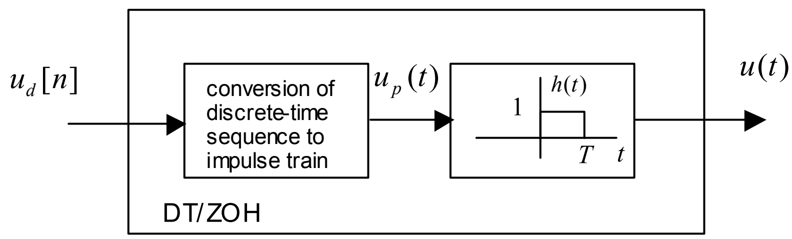


Problem 4 (20 marks)

One way to discretize a continuous-time state-space system with sampling period T for simulation purposes is to use the step-invariant transformation called "c2d".



The DT/ZOH operator is defined as a mapping from a discrete-time signal to an impulse train (consisting of forming a train of impulses occurring every T seconds, with the impulse at time nT having an area $u_d[n]$), followed by a "hold" function that holds each value for a period T.



(a) [10 marks] Derive the mathematical relationships mapping the continuous-time state-space matrices (A, B, C, D) to the discrete-time state-space matrices (A_d, B_d, C_d, D_d). (Hint: start with the general state response given by $x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A\tau}Bu(t-\tau)d\tau$)

Answer:

The state response of the state-space system (A, B, C, D) to be discretized is given by the combination of the zero-input state response and the zero-state state response.

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A\tau}Bu(t-\tau)d\tau$$

$$= e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Notice that the input $u(t)$ is a "staircase" function as it is held constant over every sampling period. Thus, if we take $t_0 = kT$ and $t = (k+1)T$, we can write

$$
\begin{aligned}
x((n+1)T) &= e^{AT}x(nT) + \int_{nT}^{(n+1)T} e^{A((n+1)T-\tau)}d\tau\, Bu(nT) \\
&= e^{AT}x(nT) + e^{A(n+1)T}\int_{nT}^{(n+1)T} e^{-A\tau}d\tau\, Bu(nT) \\
&= e^{AT}x(nT) - e^{A(n+1)T}A^{-1}\left[e^{-A\tau}\right]_{nT}^{(n+1)T} Bu(nT) \\
&= e^{AT}x(nT) - A^{-1}e^{A(n+1)T}\left[e^{-A(n+1)T} - e^{-AnT}\right]Bu(nT) \\
&= e^{AT}x(nT) - A^{-1}\left[I_n - e^{AT}\right]Bu(nT) \\
&= \underbrace{e^{AT}}_{A_d}x(nT) + \underbrace{A^{-1}\left[e^{AT} - I_n\right]B}_{B_d}u(nT)
\end{aligned}
$$

where we used the fact that A^{-1} commutes with $e^{A(n+1)T}$. If we define the discrete-time state $x_d[n] = x(nT)$, the last equation can be rewritten as a DT state equation as follows, while the output equation doesn't change.

$$
\begin{aligned}
x_d[n+1] &= A_d x_d[n] + B_d u_d[n] \\
y_d[n] &= C_d x_d[n] + D_d u_d[n]
\end{aligned},
$$

where

$$
\begin{aligned}
A_d &:= e^{AT} \\
B_d &:= A^{-1}\left[e^{AT} - I_n\right]B \\
C_d &:= C \\
D_d &:= D
\end{aligned}
$$

(b) [10 marks] We want to simulate the causal LTI differential system initially at rest:

$$2\frac{dy}{dt} + y = \frac{du}{dt}$$

with input $u(t) = q(t)$ (unit step) by discretizing it using "c2d". Find a continuous state-space representation (A,B,C,D) of this system. Then, discretize it with $T = 1$ to get the system

$$
\begin{aligned}
x_d[n+1] &= A_d x_d[n] + B_d u_d[n] \\
y_d[n] &= C_d x_d[n] + D_d u_d[n]
\end{aligned}
$$

using your formulas obtained (a). Finally, compute and plot the first 5 values of the step response $y_d[n]$. Give the steady-state value $\lim_{n\to+\infty} y_d[n]$.

Answer:

The transfer function corresponding to the system is

$$\frac{s}{2s+1},\ \mathrm{Re}\{s\} > -\frac{1}{2}.$$

Partial fraction expansion: $\frac{s}{2s+1} = \frac{1}{2} - \frac{1}{4}\frac{1}{2s+1},\ \mathrm{Re}\{s\} > -\frac{1}{2}$.

Corresponding state-space system:

$$\dot{x} = -\frac{1}{2}x - \frac{1}{4}u$$
$$y = x + \frac{1}{2}u$$

Thus, $A = -\frac{1}{2},\ B = -\frac{1}{4},\ C = 1,\ D = \frac{1}{2}$.

Discretized state-space system:

$$A_d = e^{AT} = e^{-0.5} = 0.6065$$
$$B_d := A^{-1}\left[e^{AT} - I_n\right]B = -2(e^{-0.5} - 1)(-\frac{1}{4}) = \frac{1}{2}(e^{-0.5} - 1) = -0.1967$$
$$C_d := 1$$
$$D_d := \frac{1}{2}$$

Recursion:

$$x_d[n+1] = 0.6065x_d[n] - 0.1967u_d[n]$$
$$y_d[n] = 1x_d[n] + 0.5u_d[n]$$

$n = 0:$

$$x_d[1] = 0.6065(0) - 0.1967(1) = -0.1967$$
$$y_d[0] = 1(0) + 0.5(1) = 0.5$$

$n = 1:$

$$x_d[2] = 0.6065(-0.1967) - 0.1967(1) = -0.3160$$
$$y_d[1] = 1(-0.1967) + 0.5(1) = 0.3033$$

$n = 2:$

$$x_d[3] = 0.6065(-0.3160) - 0.1967(1) = -0.3884$$
$$y_d[2] = 1(-0.3160) + 0.5(1) = 0.1840$$

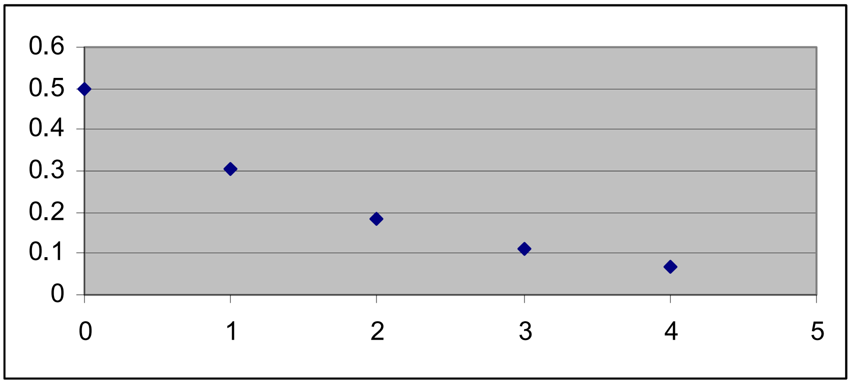
$n = 3:$

$$x_d[4] = 0.6065(-0.3884) - 0.1967(1) = -0.4323$$
$$y_d[3] = 1(-0.3884) + 0.5(1) = 0.1116$$

$n = 4:$

$$x_d[5] = 0.6065(-0.4323) - 0.1967(1) = -0.4589$$
$$y_d[4] = 1(-0.4323) + 0.5(1) = 0.0677$$

plot



Steady-state:
$$x[\infty] = \frac{-0.1967}{1-0.6065} = -0.5$$
$$y[\infty] = -0.5 + 0.5 = 0$$

Problem 5 (15 marks)

A causal recursive DLTI system is described by its (zero-state) impulse response given below:

$$h[n] = (0.8)^n \sin\left(\frac{\pi}{4}n\right)u[n] + 0.5\delta[n]$$

and has initial conditions $y[-1] = \frac{1}{\sqrt{2}}$, $y[-2] = 0$. Calculate the zero-input response $y_{zi}[n]$ of the system for $n \geq 0$ using the unilateral z-transform.

Answer:
From the table:

$$H(z) = \frac{0.8\sin(\frac{\pi}{4})z^{-1}}{1 - 1.6\cos(\frac{\pi}{4})z^{-1} + (0.8)^2 z^{-2}} + \frac{1}{2}, |z| > 0.8$$

$$= \frac{\frac{1}{2} + 0.32z^{-2}}{1 - 1.6\cos(\frac{\pi}{4})z^{-1} + (0.8)^2 z^{-2}}, |z| > 0.8$$

Difference equation: $y[n] - 1.6\cos(\frac{\pi}{4})y[n-1] + (0.8)^2 y[n-2] = \frac{1}{2} + 0.32x[n-2]$

Using the unilateral z-transform with zero input:

$$\mathcal{Y}(z) - 1.6\cos(\frac{\pi}{4})\left(z^{-1}\mathcal{Y}(z) + y[-1]\right) + (0.8)^2\left(z^{-2}\mathcal{Y}(z) + z^{-1}y[-1] + y[-2]\right) = 0$$

$$\mathcal{Y}(z)-1.6\cos(\frac{\pi}{4})z^{-1}\mathcal{Y}(z)+(0.8)^2 z^{-2}\mathcal{Y}(z)=1.6\cos(\frac{\pi}{4})y[-1]-(0.8)^2 y[-1]z^{-1}-(0.8)^2 y[-2]z^{-2}$$

$$\left[1-1.6\cos(\frac{\pi}{4})z^{-1}+(0.8)^2 z^{-2}\right]\mathcal{Y}(z)=1.6\cos(\frac{\pi}{4})^2-(0.8)^2\cos(\frac{\pi}{4})z^{-1}=0.8-(0.8)^2\cos(\frac{\pi}{4})z^{-1}$$

$$\mathcal{Y}(z)=\frac{0.8\left(1-0.8\cos(\frac{\pi}{4})z^{-1}\right)}{1-1.6\cos(\frac{\pi}{4})z^{-1}+(0.8)^2 z^{-2}},\quad |z|>0.8$$

and from the table, we get the zero-input response:

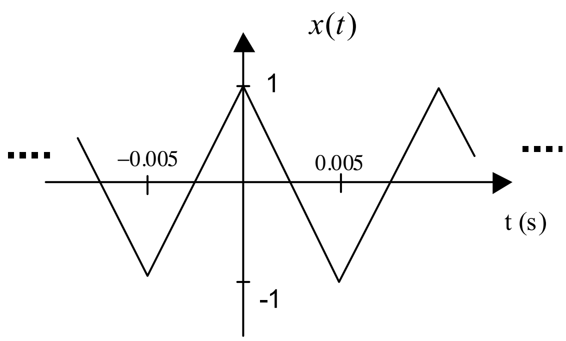
$$y_{zi}[n]=(0.8)^{n+1}\cos\left(\frac{\pi}{4}n\right)u[n]$$

Problem 6 (15 marks)

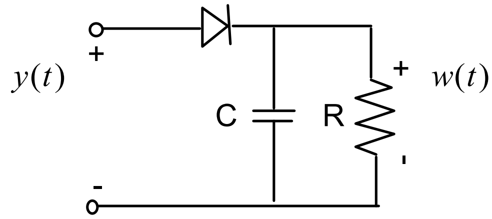
Design an envelope detector to demodulate the AM signal:

$$y(t)=[1+0.5x(t)]\cos(2\pi 10^6 t)$$

where $x(t)$ is the periodic modulating signal shown below. That is, draw a circuit diagram of the envelope detector and compute the values of the circuit components. Justify all of your approximations and assumptions. Provide rough sketches of the carrier signal, the modulated signal and the signal at the output of the detector. What is the modulation index m of the AM signal?



An envelope detector can be implemented with the following simple RC circuit with a diode.



The output voltage of the detector, when it goes from one peak at voltage v_1 to the next when it intersects the modulated carrier at voltage v_2 after approximately one period $T = 1\mu s$ of the carrier, is given by:

$$v_2 \cong v_1 e^{-T/RC}$$

Since the time constant $\tau = RC$ of the detector should be large with respect to $T = 1\mu s$, we can use a first-order approximation of the exponential such that

$$v_2 \cong v_1(1 - T/RC).$$

This is a line of negative slope $-\frac{v_1}{RC}$ between the initial voltage v_1 and the final voltage v_2 so that

$$\frac{v_2 - v_1}{T} \cong -v_1/RC.$$

This slope must be more negative than the maximum negative slope of the envelope of $y(t)$, which is maximized as follows:

$$\min_t \frac{d(0.5x(t))}{dt} = -200$$

Taking the worst-case $v_1 = 0.5$, we must have

$$-\frac{0.5}{RC} < -200$$

$$\Leftrightarrow$$

$$RC < \frac{0.5}{200} = 0.0025$$

We could take $R = 2k\Omega,\ C = 1\mu F$ to get $RC = 0.002$.

Let K be the maximum amplitude of $0.5x(t)$, i.e., $|0.5x(t)| < 0.5 = K$ and let $A = 1$. The *modulation index* m is the ratio $m = K/A = 0.5$.

END OF EXAMINATION