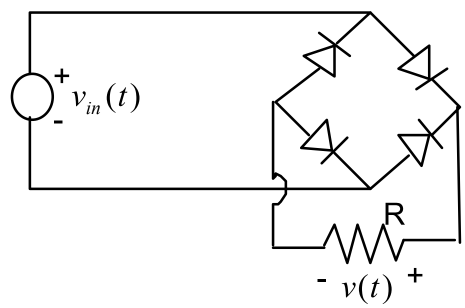
Sample Midterm Test 2 (mt2s04)
Covering Chapters 4-5 and part of Chapter 15 of *Fundamentals of Signals & Systems*

Problem 1 (35 marks)

The following nonlinear circuit is an ideal full-wave rectifier.



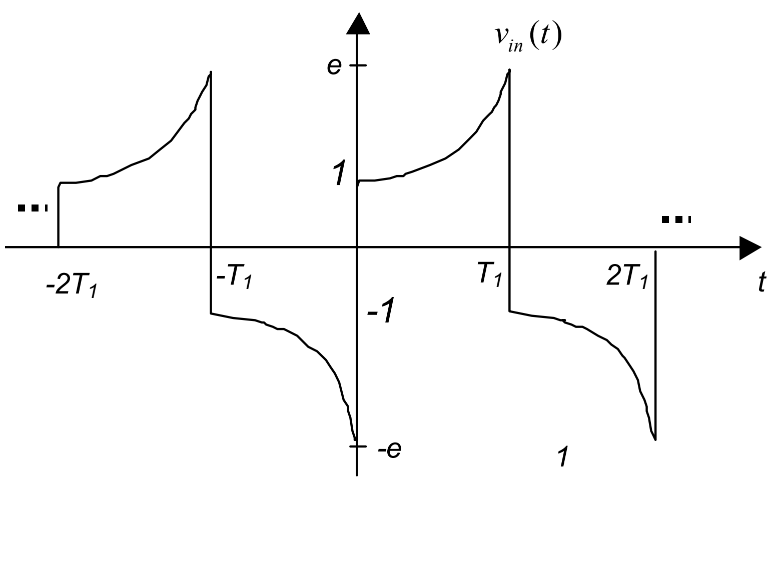
The voltages are $v_{in}(t) = e^{\alpha t}\left[u(t) - u(t-T_1)\right] * \sum_{k=-\infty}^{+\infty}\left[\delta(t-2kT_1) - \delta(t-(2k-1)T_1)\right]$ where $\alpha \in \mathbb{R}, \alpha > 0$,

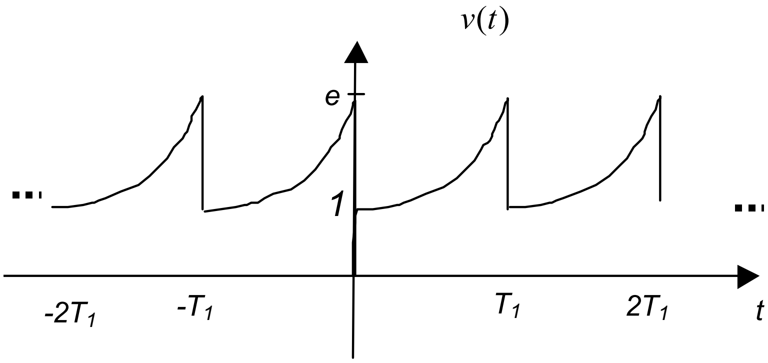
and $v(t) = \left|v_{in}(t)\right|$.

(a) [5 marks] Find the fundamental period T of the input voltage. Sketch the input and output voltages $v_{in}(t), v(t)$ for $\alpha = 1/T_1$.

Answer:

We have $T = 2T_1$





(b) [12 marks] Compute the Fourier series coefficients a_k of the input voltage $v_{in}(t)$ for any positive values of α and T_1. Write $v_{in}(t)$ as a Fourier series.

Answer:

DC component :

$$a_0 = \frac{1}{2T_1}\int_{-T_1}^{T_1} x(t)dt = \frac{1}{2T_1}\int_0^{T_1} e^{\alpha t}dt - \frac{1}{2T_1}\int_{-T_1}^{0} e^{\alpha(t+T_1)}dt$$

$$= \frac{1}{2T_1}\int_0^{T_1} e^{\alpha t}dt - \frac{1}{2T_1}\int_0^{T_1} e^{\alpha\tau}d\tau = 0$$

for $k \neq 0$:

$$a_k = \frac{1}{2T_1}\int_{-T_1}^{T_1} x(t)e^{-jk\frac{2\pi}{2T_1}t}dt$$

$$= \frac{1}{2T_1}\int_0^{T_1} e^{\alpha t}e^{-jk\frac{\pi}{T_1}t}dt - \frac{1}{2T_1}\int_{-T_1}^{0} e^{\alpha(t+T_1)}e^{-jk\frac{\pi}{T_1}t}dt$$

$$= \frac{1}{2T_1}\int_0^{T_1} e^{(\alpha-jk\frac{\pi}{T_1})t}dt - \frac{1}{2T_1}e^{\alpha T_1}\int_{-T_1}^{0} e^{(\alpha-jk\frac{\pi}{T_1})t}dt$$

$$= \frac{1}{2T_1(\alpha - jk\frac{\pi}{T_1})}\left(e^{\alpha T_1}e^{-jk\pi} - 1\right) - \frac{e^{\alpha T_1}}{2T_1(\alpha - jk\frac{\pi}{T_1})}\left(1 - e^{-\alpha T_1}e^{jk\pi}\right)$$

$$= \frac{1}{2T_1(\alpha - jk\frac{\pi}{T_1})}\left(e^{\alpha T_1}e^{-jk\pi} - 1\right) - \frac{e^{\alpha T_1}}{2T_1(\alpha - jk\frac{\pi}{T_1})}\left(1 - e^{-\alpha T_1}e^{jk\pi}\right)$$

$$= \frac{e^{\alpha T_1}\left((-1)^k - 1\right) + (-1)^k - 1}{2T_1\alpha - j2k\pi}$$

$$= \frac{(e^{\alpha T_1} + 1)\left((-1)^k - 1\right)}{2T_1\alpha - j2k\pi}$$

Fourier series:

$$v_{in}(t) = \sum_{k=-\infty}^{+\infty} \frac{(e^{\alpha T_1}+1)\left((-1)^k - 1\right)}{2T_1\alpha - j2k\pi} e^{jk\frac{\pi}{T_1}t}$$

(c) [10 marks] Compute the Fourier series coefficients b_k of $v(t)$ again for any positive values of α and T_1. Here the fundamental period is $T = T_1$.

Answer:

DC component :

$$b_0 = \frac{1}{T_1}\int_0^{T_1} x(t)dt = \frac{1}{T_1}\int_0^{T_1} e^{\alpha t} dt$$

$$= \frac{1}{\alpha T_1}\left[e^{\alpha t}\right]_0^{T_1} = \frac{1}{\alpha T_1}\left[e^{\alpha T_1} - 1\right]$$

for $k \neq 0$:

$$b_k = \frac{1}{T_1}\int_0^{T_1} x(t)e^{-jk\frac{2\pi}{T_1}t} dt = \frac{1}{T_1}\int_0^{T_1} e^{(\alpha - jk\frac{2\pi}{T_1})t} dt$$

$$= \frac{1}{T_1(\alpha - jk\frac{2\pi}{T_1})}\left(e^{\alpha T_1} - 1\right) = \frac{e^{\alpha T_1} - 1}{\alpha T_1 - j2k\pi}$$

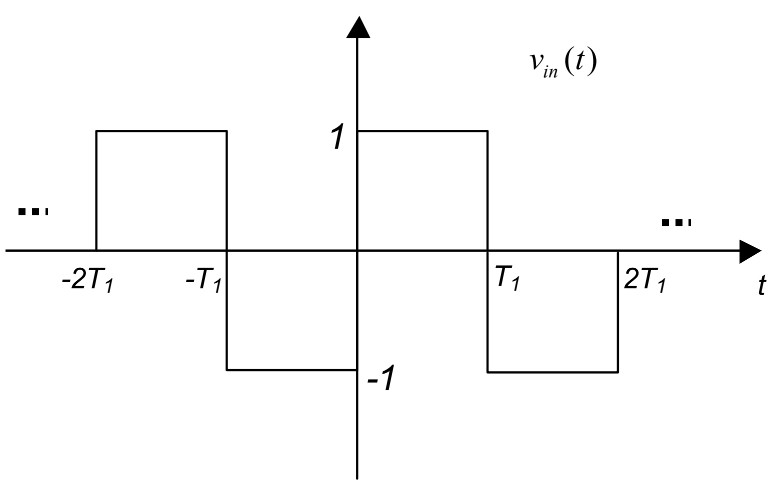
(d) [8 marks] Compute the Fourier series coefficients of the voltage signals $v_{in}(t), v(t)$ for the case $\alpha \to 0$ with T_1 held constant. What time-domain signals $v_{in}(t), v(t)$ do you obtain in this case? Sketch them.

Answer:

When $\alpha \to 0$ we get a rectangular wave for $v_{in}(t)$ and a constant signal for $v(t)$.

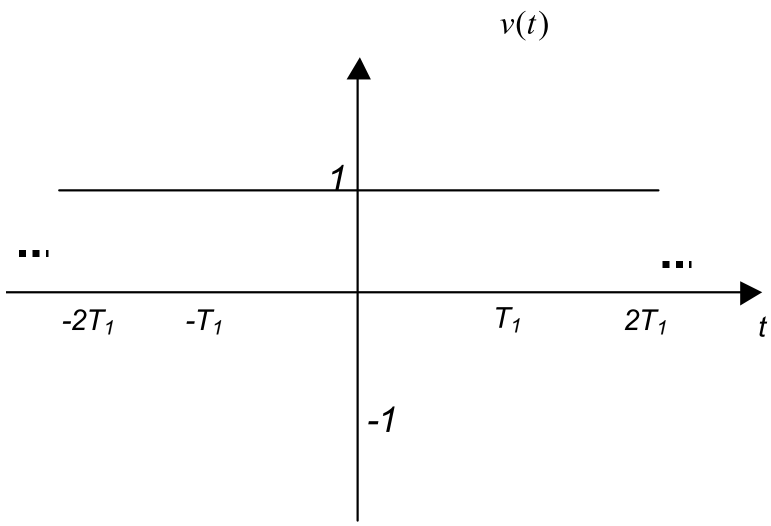
$$\lim_{\alpha \to 0} a_0 = 0,$$

$$\lim_{\alpha \to 0} a_k = \lim_{\alpha \to 0} \frac{(e^{\alpha T_1}+1)\left((-1)^k - 1\right)}{2T_1\alpha - j2k\pi} = \frac{\left(1-(-1)^k\right)}{jk\pi}, \quad k \neq 0$$



$$\lim_{\alpha \to 0} b_0 = \lim_{\alpha \to 0} \frac{1}{\alpha T_1}\left[e^{\alpha T_1} - 1 \right] = 1$$

$$\lim_{\alpha \to 0} b_k = \lim_{\alpha \to 0} \frac{e^{\alpha T_1} - 1}{\alpha T_1 - j2k\pi} = \frac{1-1}{-j2k\pi} = 0$$



Problem 2 (30 marks)

Consider the causal LTI differential system initially at rest:

$$\frac{d^2 y(t)}{dt^2} + 3\sqrt{3}\frac{dy(t)}{dt} + 9y(t) = 0.5\frac{dx(t)}{dt} + x(t)$$

(a) [5 marks] Compute the frequency response $H(j\omega)$ of the system.

Answer:

By inspection: $H(j\omega) = \dfrac{0.5 j\omega + 1}{(j\omega)^2 + 3\sqrt{3}(j\omega) + 9}$.

(b) [20 marks] Compute the response of the system to the input $x(t) = te^{-2t}u(t) + u(t)$ using the Fourier transform.

Answer:

$$x(t) = te^{-2t}u(t) + u(t) \quad \overset{FT}{\longleftrightarrow} \quad X(j\omega) = \frac{1}{(2 + j\omega)^2} + \frac{1}{j\omega} + \pi\delta(\omega)$$

$$Y(j\omega)=H(j\omega)X(j\omega)=\frac{0.5j\omega+1}{(j\omega)^2+3\sqrt{3}(j\omega)+9}\left[\frac{1}{(2+j\omega)^2}+\frac{1}{j\omega}+\pi\delta(\omega)\right]$$

$$=\frac{0.5(j\omega+2)}{(j\omega)^2+3\sqrt{3}(j\omega)+9}\frac{1}{(2+j\omega)^2}+\frac{0.5j\omega+1}{(j\omega)^2+3\sqrt{3}(j\omega)+9}\frac{1}{j\omega}+\frac{\pi}{9}\delta(\omega)$$

$$=\underbrace{\frac{0.5}{(j\omega)^2+3\sqrt{3}(j\omega)+9}\frac{1}{(2+j\omega)}}_{Y_1(j\omega)}+\underbrace{\frac{0.5j\omega+1}{(j\omega)^2+3\sqrt{3}(j\omega)+9}\frac{1}{j\omega}+\frac{\pi}{9}\delta(\omega)}_{Y_2(j\omega)}$$

Partial fraction expansion of $Y_1(j\omega)$:

$$\frac{0.5}{(j\omega)^2+3\sqrt{3}(j\omega)+9}\frac{1}{(2+j\omega)}=\frac{A}{\left(j\omega+\frac{3\sqrt{3}}{2}-j\frac{3}{2}\right)}+\frac{B}{\left(j\omega+\frac{3\sqrt{3}}{2}+j\frac{3}{2}\right)}+\frac{C}{(2+j\omega)}$$

$$=\frac{-0.09587+j0.03822}{\left(j\omega+\frac{3\sqrt{3}}{2}-j\frac{3}{2}\right)}+\frac{-0.09587-j0.03822}{\left(j\omega+\frac{3\sqrt{3}}{2}+j\frac{3}{2}\right)}+\frac{0.1917}{(2+j\omega)}$$

Time domain:

$$y_1(t)=(-0.09587+j0.03822)e^{(-\frac{3\sqrt{3}}{2}+j\frac{3}{2})t}u(t)+(-0.09587-j0.03822)e^{(-\frac{3\sqrt{3}}{2}-j\frac{3}{2})t}u(t)+0.1917e^{-2t}u(t)$$

$$=2\,\mathrm{Re}\left\{(-0.09587+j0.03822)e^{(-\frac{3\sqrt{3}}{2}+j\frac{3}{2})t}\right\}u(t)+0.1917e^{-2t}u(t)$$

$$=2e^{-\frac{3\sqrt{3}}{2}t}\left\{-0.09587\cos(\frac{3}{2}t)-0.03822\sin(\frac{3}{2}t)\right\}u(t)+0.1917e^{-2t}u(t)$$

Partial fraction expansion of $Y_2(j\omega)$:

$$\frac{0.5j\omega+1}{(j\omega)^2+3\sqrt{3}(j\omega)+9}\frac{1}{(j\omega)}+\frac{\pi}{9}\delta(\omega)=\frac{D}{\left(j\omega+\frac{3\sqrt{3}}{2}-j\frac{3}{2}\right)}+\frac{E}{\left(j\omega+\frac{3\sqrt{3}}{2}+j\frac{3}{2}\right)}+\frac{F}{j\omega}+\frac{\pi}{9}\delta(\omega)$$

$$=\frac{-0.0556-j0.0704}{\left(j\omega+\frac{3\sqrt{3}}{2}-j\frac{3}{2}\right)}+\frac{-0.0556+j0.0704}{\left(j\omega+\frac{3\sqrt{3}}{2}+j\frac{3}{2}\right)}+\frac{1/9}{j\omega}+\frac{\pi}{9}\delta(\omega)$$

Time domain:

$$y_2(t)=(-0.0556-j0.0704)e^{(-\frac{3\sqrt{3}}{2}+j\frac{3}{2})t}u(t)+(-0.0556+j0.0704)e^{(-\frac{3\sqrt{3}}{2}-j\frac{3}{2})t}u(t)+\frac{1}{9}u(t)$$

$$=2\,\mathrm{Re}\left\{(-0.0556-j0.0704)e^{(-\frac{3\sqrt{3}}{2}+j\frac{3}{2})t}\right\}u(t)+\frac{1}{9}u(t)$$

$$=2e^{-\frac{3\sqrt{3}}{2}t}\left\{-0.0556\cos(\frac{3}{2}t)+0.0704\sin(\frac{3}{2}t)\right\}u(t)+\frac{1}{9}u(t)$$

Overall response:

$$
\begin{aligned}
y(t) &= y_1(t) + y_2(t) \\
&= 2e^{-\frac{3\sqrt{3}}{2}t}\left\{(-0.0959-0.0556)\cos(\frac{3}{2}t) + (-0.0382+0.0704)\sin(\frac{3}{2}t)\right\}u(t) + 0.1917e^{-2t}u(t) + \frac{1}{9}u(t) \\
&= 2e^{-\frac{3\sqrt{3}}{2}t}\left\{-0.1515\cos(\frac{3}{2}t) + 0.0322\sin(\frac{3}{2}t)\right\}u(t) + 0.1917e^{-2t}u(t) + \frac{1}{9}u(t) \\
&= e^{-\frac{3\sqrt{3}}{2}t}\left\{-0.3030\cos(\frac{3}{2}t) + 0.0644\sin(\frac{3}{2}t)\right\}u(t) + 0.1917e^{-2t}u(t) + \frac{1}{9}u(t)
\end{aligned}
$$

(c) [5 marks] Compute the response of the system to the input $x(t) = 5\sin(10t)$.

Answer:

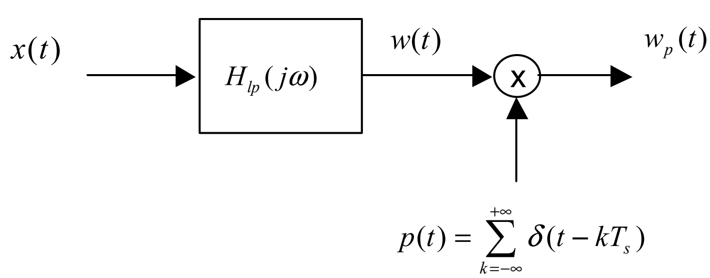
$$
\begin{aligned}
H(j10) &= \frac{0.5j(10)+1}{(j(10))^2 + 3\sqrt{3}(j(10)) + 9} \\
&= \frac{j5+1}{j(30\sqrt{3})-91} = 0.0154 - j0.0462 = 0.0487e^{-j1.2494}
\end{aligned}
$$

The response is:

$$
\begin{aligned}
y(t) &= 5(0.0487)\sin(10t - 1.2494) \\
&= 0.2435\sin(10t - 1.2494)
\end{aligned}
$$

Problem 3 (20 marks)

Consider the sampling system with an ideal unit-gain, low-pass filter $H_{lp}(j\omega)$ (called an *antialiasing filter*) with cutoff frequency ω_c, and where the sampling frequency is $\omega_s = \frac{2\pi}{T_s}$.



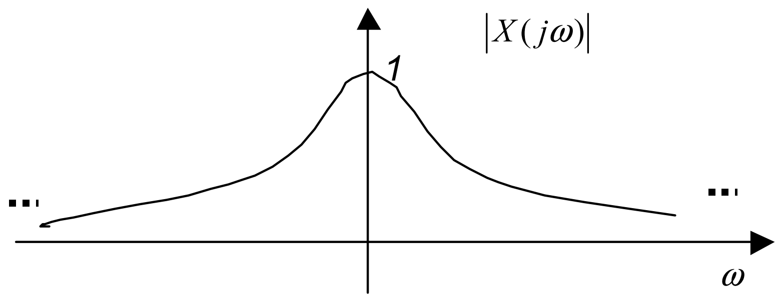
The input signal is $x(t) = e^{-t}u(t)$, and the sampling period is set at $T_s = \frac{\pi}{\sqrt{3}}$.

(a) [10 marks] Give a mathematical expression for $X(j\omega)$, the Fourier transform of the input signal, and sketch it (magnitude and phase.) Design the antialiasing filter (i.e., find its cutoff frequency) so that its bandwidth is maximized while avoiding aliasing of its output $w(t)$ in the sampling operation. With this value of ω_c, sketch the magnitudes of the Fourier transforms $W(j\omega)$ and $W_p(j\omega)$.

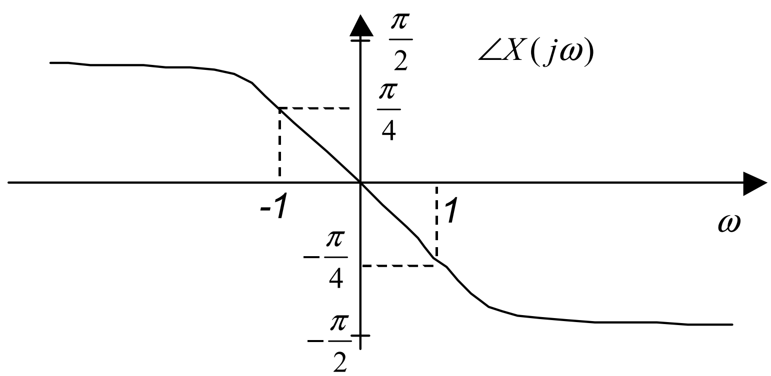
Answer:

The Fourier transform of the input signal $x(t)=e^{-t}u(t)$ is $X(j\omega)=\dfrac{1}{j\omega+1}$.

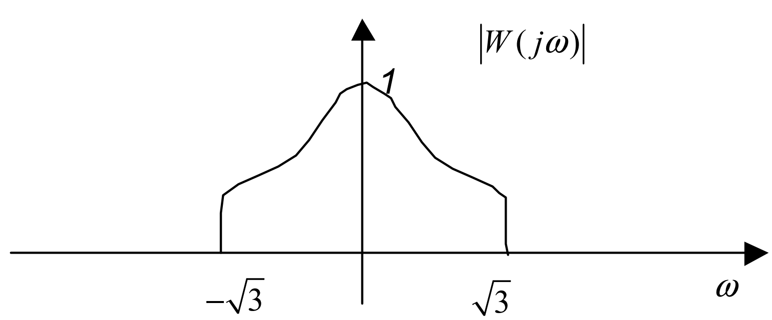
Magnitude:

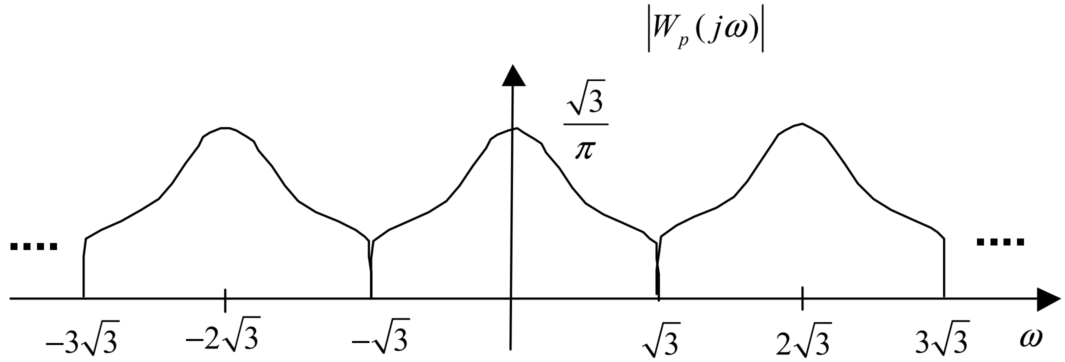


Phase:



The cutoff frequency of the antialiasing filter must be set equal to half of the sampling frequency to satisfy the sampling theorem. Therefore, $\omega_c=\dfrac{1}{2}\omega_s=\dfrac{1}{2}\dfrac{2\pi}{T_s}=\sqrt{3}$. With this cutoff frequency, the magnitudes of $W(j\omega)$ and $W_p(j\omega)$ are shown below.





(b) [10 marks] Compute the ratios of total energies $r = 100\frac{E_{\infty w}}{E_{\infty x}}$ in percent, where $E_{\infty w}$ is the total energy in signal $w(t)$ and $E_{\infty x}$ is the total energy in signal $x(t)$. This ratio gives us an idea of how similar $w(t)$ is to $x(t)$ before sampling. How similar is it? (HINT: $\int \frac{du}{\alpha^2 + u^2} = \frac{1}{a}\arctan\left(\frac{u}{a}\right) + C$)

Answer:

The total energy in the input signal is easier to compute in the time domain:

$$E_{\infty x} = \int_0^\infty \left|e^{-t}\right|^2 dt = \int_0^\infty e^{-2t} dt = -\frac{1}{2}\left[e^{-2t}\right]_0^\infty = \frac{1}{2}$$

The total energy in the filtered signal is easier to compute in the frequency domain:

$$\begin{aligned} E_{\infty w} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |W(j\omega)|^2 d\omega = \frac{1}{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}} \left|\frac{1}{1+j\omega}\right|^2 d\omega \\ &= \frac{1}{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}} \left|\frac{1}{1+j\omega}\right|^2 d\omega = \frac{1}{2\pi}\int_{-\sqrt{3}}^{\sqrt{3}} \frac{1}{1+\omega^2} d\omega \\ &= \frac{1}{2\pi}\left[\arctan(\omega)\right]_{-\sqrt{3}}^{\sqrt{3}} = \frac{1}{2\pi}\left[\arctan(\sqrt{3}) - \arctan(-\sqrt{3})\right] \\ &= \frac{1}{2\pi}\left[\frac{\pi}{3} - (-\frac{\pi}{3})\right] = \frac{1}{3} \end{aligned}$$

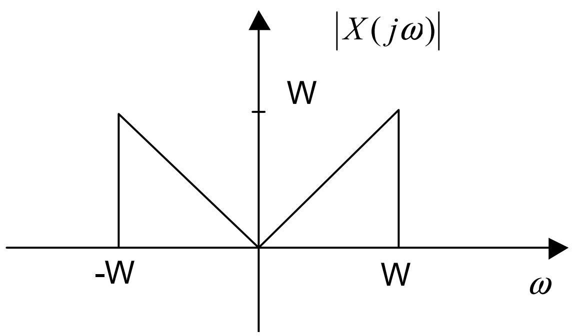
ratios of total energies $r = 100\frac{E_{\infty w}}{E_{\infty x}}\% = 100\frac{1/3}{1/2}\% = 67\%$.

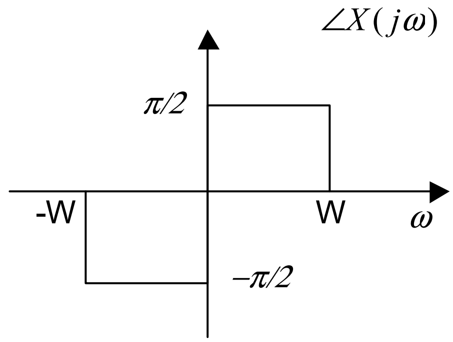
This ratio indicates that quite a bit of energy is lost in the antialiasing filter and therefore $w(t)$ is not very similar to $x(t)$.

To do a better job, one would have to increase both the cutoff frequency and the sampling frequency.

Problem 4 (15 marks)

(a) [10 marks] Find the inverse Fourier transform $x(t)$ of $X(j\omega)$ whose magnitude and phase are shown below.





Answer:

Let $Y(j\omega)$ be the Fourier transform of a rectangular window of unit magnitude and zero phase (i.e., it is real) from -W to W. Then the Fourier transform of $x(t)$ is

$$X(j\omega) = j\omega Y(j\omega).$$

Using the differentiation property, the signal $x(t)$ is given by

$$\begin{aligned} x(t) &= \frac{dy(t)}{dt} \\ &= \frac{d\frac{W}{\pi}\operatorname{sinc}(\frac{W}{\pi}t)}{dt} \\ &= \frac{W}{\pi}\frac{d}{dt}\frac{\sin(Wt)}{Wt} \\ &= \frac{W}{\pi}\left[\frac{W^2 t\cos(Wt) - W\sin(Wt)}{W^2 t^2}\right] \end{aligned}$$

(b) [5 marks] Find the DC gain of the LTI filter characterized by its impulse response $h(t) = \begin{cases} e^{-|t|}, & |t| < 2 \\ 0, & \text{otherwise} \end{cases}$.

Answer:

$$\begin{aligned} H(j\omega)\big|_{\omega=0} &= \int_{-2}^{2} h(t)e^{-j\omega t}dt\bigg|_{\omega=0} \\ &= \int_{-2}^{2} h(t)dt = \int_{-2}^{0} e^{t}dt + \int_{0}^{2} e^{-t}dt = (1-e^{-2}) - (e^{-2}-1) = 2 - 2e^{-2} = 1.73 \end{aligned}$$