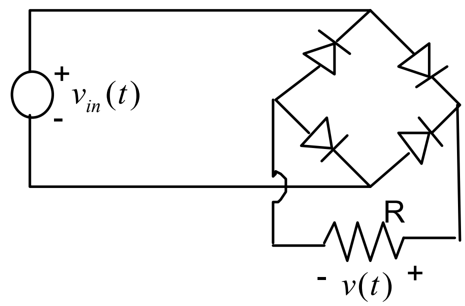
Sample Midterm Test 2 (mt2s03)
Covering Chapters 4-5 and part of Chapter 15 of *Fundamentals of Signals & Systems*

Problem 1 (30 marks)

The following nonlinear circuit is an ideal full-wave rectifier.



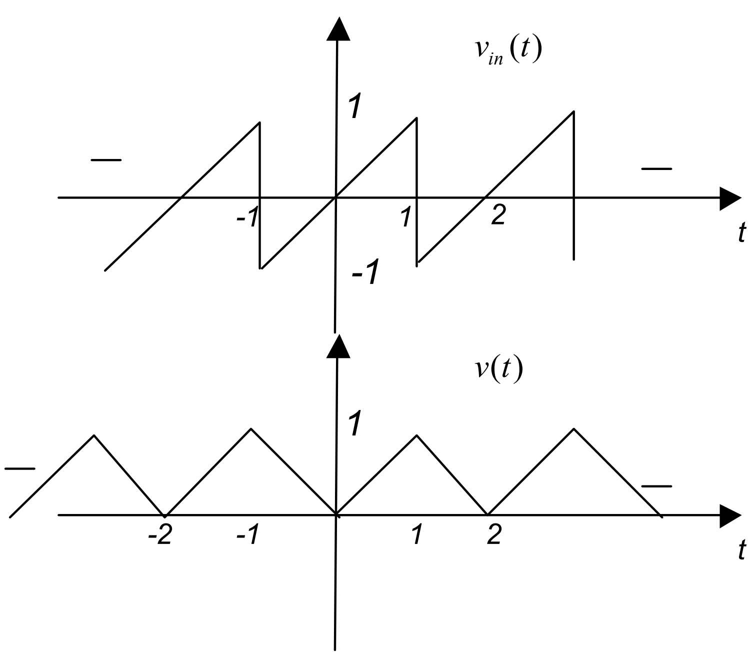
The voltages are $v_{in}(t)=\sum_{k=-\infty}^{+\infty}\delta(t-k2)*\left[t\left(u(t+1)-u(t-1)\right)\right]$, and $v(t)=\left|v_{in}(t)\right|$.

Let T_1 be the fundamental period of the rectified voltage signal $v(t)$ and let $\omega_1=2\pi/T_1$ be its fundamental frequency.

(a) [8 marks] Find the fundamental period T_1. Sketch the input and output voltages $v_{in}(t), v(t)$.

Answer:

We have $T_1=2$



(b) [12 marks] Compute the Fourier series coefficients of $v(t)$. Show that the even coefficients vanish. Write $v(t)$ as a Fourier series.

Answer:
DC component :

$$a_0 = \frac{1}{2}\int_{-1}^{1}|t|\,dt = \int_0^1 t\,dt = \frac{1}{2}t^2\Big|_0^1 = \frac{1}{2}$$

for $k \neq 0$:

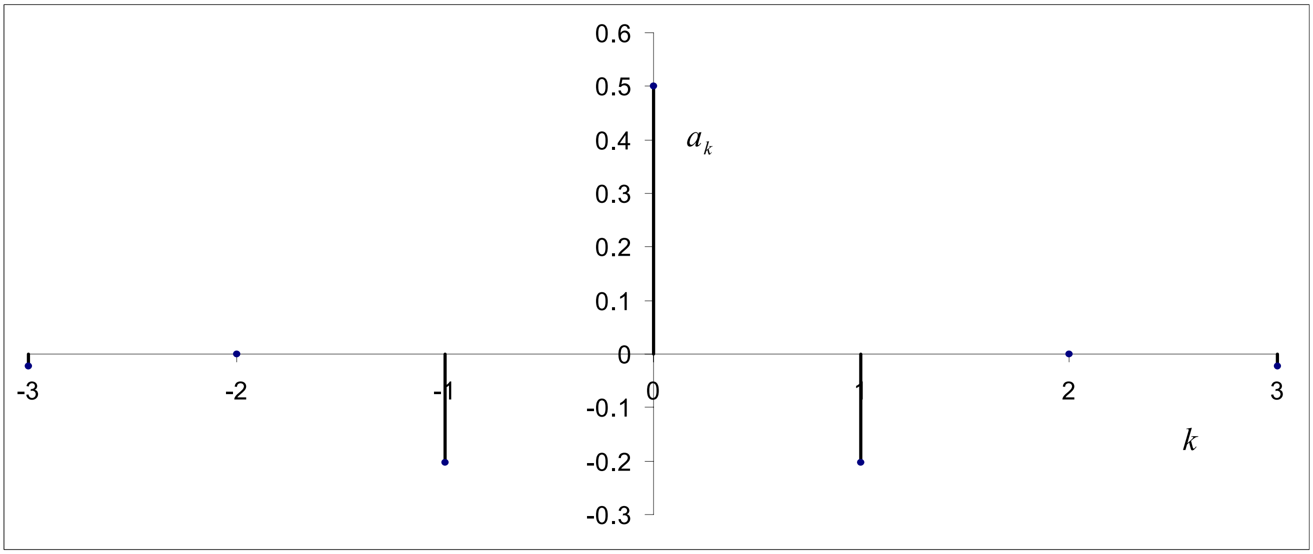
$$\begin{aligned}
a_k &= \frac{1}{2}\int_{-1}^{1}|t|e^{-jk\pi t}dt = \frac{1}{2}\left[\int_0^1 te^{-jk\pi t}dt - \int_{-1}^{0} te^{-jk\pi t}dt\right] \\
&= \frac{1}{2}\int_0^1 t(e^{-jk\pi t} + e^{jk\pi t})dt = \int_0^1 t\cos(k\pi t)dt \\
&= \frac{1}{k\pi}\int_0^1 (k\pi)t\cos(k\pi t)dt = \frac{1}{k\pi}\left[\left(t\sin(k\pi t)\right)_0^1 - \int_0^1 \sin(k\pi t)dt\right] \\
&= \frac{1}{k\pi}\left[\underbrace{\left(\sin k\pi - 0\right)}_{0} + \frac{1}{k\pi}\cos(k\pi t)_0^1\right] \\
&= \frac{1}{(k\pi)^2}\left(\cos k\pi - 1\right) \\
&= \frac{1}{(k\pi)^2}\left((-1)^k - 1\right)
\end{aligned}$$

Thus, the even coefficients vanish as $\left((-1)^k - 1\right) = 0,\ k$ even .

$$v(t) = \frac{1}{2} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{1}{(k\pi)^2}\left((-1)^k - 1\right)e^{jk\pi t}$$

is the Fourier series expansion of the full-wave rectified (triangular) voltage.

(c) [5 marks] Sketch the spectrum of $v(t)$ up to and including the third harmonic components.



(d) [5 marks] Write $v(t)$ using the real form of the Fourier series:

$$v(t) = a_0 + 2\sum_{k=1}^{+\infty}[B_k\cos(k\omega_1 t) - C_k\sin(k\omega_1 t)]$$

Answer:
Recall that the C_k coefficients are the imaginary parts of the a_k's, so they are equal to 0. Hence

$B_k = a_k$, and $x(t) = \frac{1}{2} + 2\sum_{k=1}^{+\infty} \frac{1}{(k\pi)^2}\left((-1)^k - 1\right)\cos(k\pi t)$

Problem 2 (10 marks)

Given periodic signals with spectra $x(t) \overset{\mathcal{FS}}{\leftrightarrow} a_k$ and $y(t) \overset{\mathcal{FS}}{\leftrightarrow} b_k$, show the property of

periodic convolution: $\int_T x(\tau)y(t-\tau)d\tau \overset{\mathcal{FS}}{\leftrightarrow} Ta_k b_k$

Answer:

$$\begin{aligned}
\int_T x(\tau)y(t-\tau)d\tau &= \int_T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0\tau} \sum_{p=-\infty}^{+\infty} b_p e^{jp\omega_0(t-\tau)} d\tau \\
&= \int_T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0\tau} \sum_{p=-\infty}^{+\infty} b_p e^{-jp\omega_0\tau} e^{jp\omega_0 t} d\tau \\
&= \int_T \sum_{k=-\infty}^{+\infty} \sum_{p=-\infty}^{+\infty} a_k b_p e^{j(k-p)\omega_0\tau} d\tau e^{jp\omega_0 t} \\
&= \sum_{k=-\infty}^{+\infty} \sum_{p=-\infty}^{+\infty} a_k b_p \int_T e^{j(k-p)\omega_0\tau} d\tau\, e^{jp\omega_0 t} \\
&= \sum_{k=-\infty}^{+\infty} a_k b_k T e^{jk\omega_0 t}
\end{aligned}$$

Therefore,

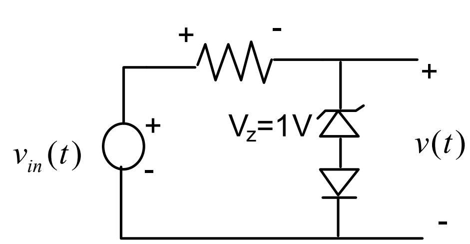
$$\int_T x(\tau)y(t-\tau)d\tau \overset{\mathcal{FS}}{\leftrightarrow} Ta_k b_k$$

Problem 3 (25 marks)

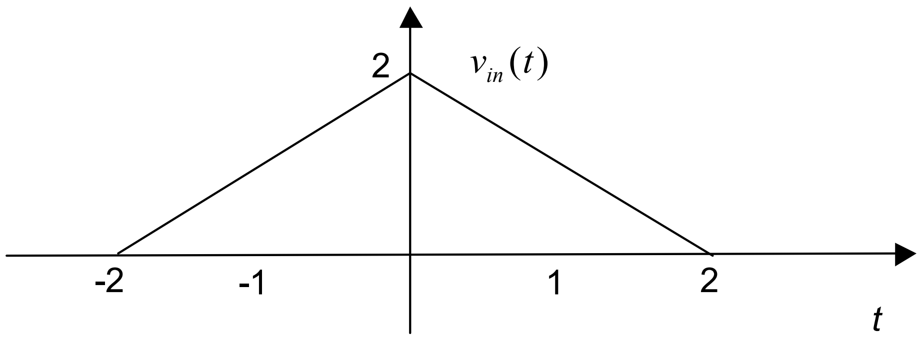
(a) [14 marks] The following circuit with a Zener diode is an ideal clamping circuit. Sketch the input and output voltages $v_{in}(t), v(t)$. Compute the Fourier transform of the output voltage $v(t)$.

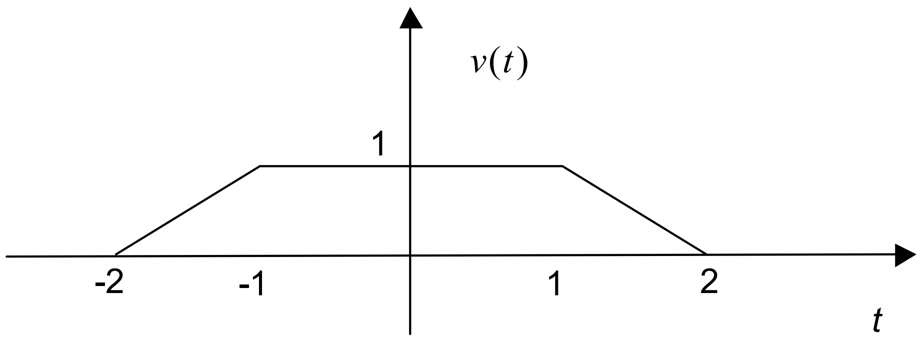
Answer:
The input voltage is
$v_{in}(t) = (t+2)u(t+2) - 2tu(t) + (t-2)u(t-2)$ Volts,

and the output voltage is $v(t) = \begin{cases} v_{in}(t), & v_{in}(t) < 1 \\ 1, & v_{in}(t) \geq 1 \end{cases}$.



Input and output voltage:





$$
\begin{aligned}
V(j\omega) &= \int_{-\infty}^{\infty} v(t)e^{-j\omega t}dt \\
&= \int_{-2}^{-1}(t+2)e^{-j\omega t}dt + \int_{-1}^{1} e^{-j\omega t}dt - \int_{1}^{2}(t-2)e^{-j\omega t}dt \\
&= \frac{-1}{j\omega}\left[(t+2)e^{-j\omega t}\right]_{-2}^{-1} + \frac{1}{j\omega}\int_{-2}^{-1} e^{-j\omega t}dt - \frac{1}{j\omega}\left[e^{-j\omega t}\right]_{-1}^{1} + \frac{1}{j\omega}\left[(t-2)e^{-j\omega t}\right]_{1}^{2} - \frac{1}{j\omega}\int_{1}^{2} e^{-j\omega t}dt \\
&= \frac{-1}{j\omega}e^{j\omega} - \frac{1}{(j\omega)^2}\left[e^{j\omega} - e^{j2\omega}\right] + \frac{1}{j\omega}\left[e^{j\omega} - e^{-j\omega}\right] + \frac{1}{j\omega}e^{-j\omega} + \frac{1}{(j\omega)^2}\left[e^{-j2\omega} - e^{-j\omega}\right] \\
&= \frac{1}{(j\omega)^2}\left[e^{j2\omega} + e^{-j2\omega}\right] - \frac{1}{(j\omega)^2}\left[e^{j\omega} + e^{-j\omega}\right] \\
&= \frac{2}{\omega^2}\left[\cos\omega - \cos 2\omega\right]
\end{aligned}
$$

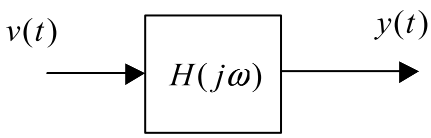
(b) [4 marks] Give an expression (no need to simplify) for the total energy contained in signal $v(t)$ between 1kHz and 10kHz (including the part at negative frequencies.)
Answer:

$$E = \frac{1}{2\pi}\int_{-20000\pi}^{-2000\pi} \left|V(j\omega)\right|^2 d\omega + \frac{1}{2\pi}\int_{2000\pi}^{20000\pi} \left|V(j\omega)\right|^2 d\omega$$

$$= \frac{2}{\pi}\int_{-20000\pi}^{-2000\pi} \frac{1}{\omega^4}\left|\left[\cos\omega - \cos 2\omega\right]\right|^2 d\omega + \frac{2}{\pi}\int_{2000\pi}^{20000\pi} \frac{1}{\omega^4}\left|\left[\cos\omega - \cos 2\omega\right]\right|^2 d\omega$$

$$= \frac{4}{\pi}\int_{2000\pi}^{20000\pi} \frac{1}{\omega^4}\left|\left[\cos\omega - \cos 2\omega\right]\right|^2 d\omega$$

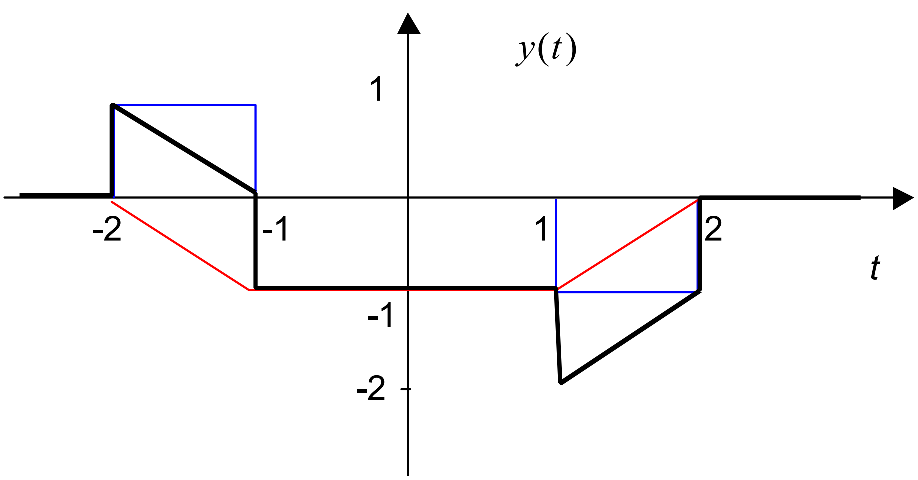
(c) [7 marks] The clamped output voltage is passed through an LTI filter whose frequency response is $H(j\omega) = j\omega - 1$. Find and sketch the output signal $y(t)$.



Answer:

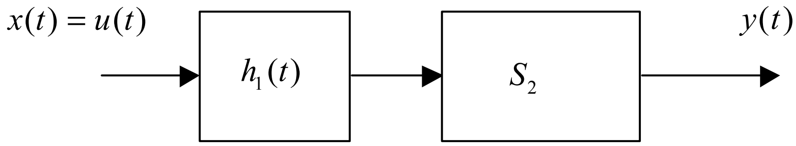
The frequency response of the filter indicates that it is a differentiator plus the negative of the signal. Then, all we have to do is to differentiate the signal $v(t)$ and subtract $v(t)$ to obtain $y(t)$:

$$\begin{aligned} y(t) &= \frac{d}{dt}v(t) - v(t) \\ &= [u(t+2) - u(t+1) - u(t-1) + u(t-2)] \\ &\quad -[(t+2)u(t+2) - (t+1)u(t+1) - (t-1)u(t-1) + (t-2)u(t-2)] \\ &= -(t+1)u(t+2) + tu(t+1) + (t-2)u(t-1) - (t-3)u(t-2) \end{aligned}$$



Problem 4 (20 marks)

Compute the unit step response $y(t)$ of the overall LTI system shown below using the Fourier transform.



Here $h_1(t) = te^{-t}u(t)$ and S_2 is the causal LTI differential system $\frac{dy}{dt} + 2y = \frac{dx}{dt}$. Give the DC gain of the overall system.

Answer:

$$H_1(j\omega) = \frac{1}{(j\omega+1)^2}$$

$$H_2(j\omega) = \frac{j\omega}{j\omega+2}$$

$$H(j\omega) = \frac{j\omega}{(j\omega+2)(j\omega+1)^2}$$

$$Y(j\omega) = H(j\omega)X(j\omega) = \frac{j\omega}{(j\omega+2)(j\omega+1)^2}\left(\frac{1}{j\omega} + \pi\delta(\omega)\right)$$

$$= \frac{1}{(j\omega+2)(j\omega+1)^2} + 0 = \frac{1}{(j\omega+2)(j\omega+1)^2}$$

Partial fraction expansion for inverse FT:

$$Y(j\omega) = \frac{1}{(j\omega+2)(j\omega+1)^2} = \frac{A}{(j\omega+2)} + \frac{B}{(j\omega+1)} + \frac{C}{(j\omega+1)^2}$$

$$= \frac{1}{(j\omega+2)} + \frac{-1}{(j\omega+1)} + \frac{1}{(j\omega+1)^2}$$

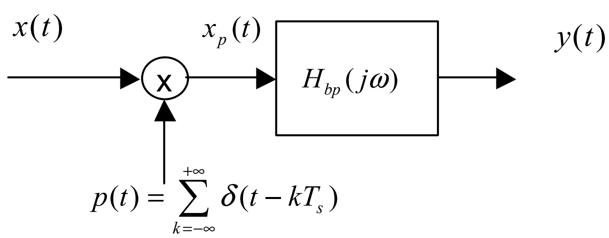
which yields: $y(t) = \left(e^{-2t} + te^{-t} - e^{-t}\right)u(t)$

DC gain:

$$H(j0) = \frac{j0}{(j0+2)(j0+1)^2} = 0$$

Problem 5 (15 marks)

Consider the following system with an ideal bandpass filter and where the sampling frequency is $\omega_s = \dfrac{2\pi}{T_s}$.



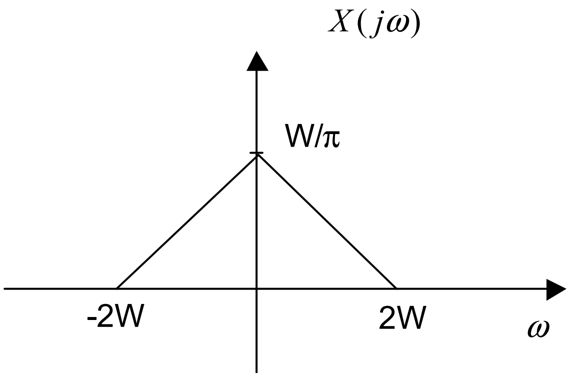
The input signal is $x(t) = \left[\dfrac{W}{\pi} \mathrm{sinc}\left(\dfrac{W}{\pi} t \right) \right]^2$, and $\omega_s = 4W$.

Give a mathematical expression for $X(j\omega)$, the Fourier transform of the input signal, and sketch it. Assuming that $\omega_{bp1} < \omega_{bp2}$ are the cutoff frequencies of the unit-magnitude ideal bandpass filter $H_{bp}(j\omega)$ with $\omega_{bp1} = 2W$, find the value of ω_{bp2} such that $\omega_{bp2} - \omega_{bp1}$ is minimized while the spectrum of $y(t)$ contains one complete copy of the spectrum $X(j\omega)$ at positive (and negative) frequencies. Sketch the Fourier transform $Y(j\omega)$ of $y(t)$.

Answer:

The input signal $x(t) = \left[\dfrac{W}{\pi} \mathrm{sinc}\left(\dfrac{W}{\pi} t \right) \right]^2$ is the multiplication of two identical sinc functions whose FT is a rectangular window of unit amplitude and bandwidth W. The resulting FT is the convolution of these two rectangular windows multiplied by $\dfrac{1}{2\pi}$:

Fourier transform $X(j\omega) = \begin{cases} \dfrac{1}{2\pi}\left(2W - |\omega|\right), & |\omega| < 2W \\ 0, & |\omega| \geq 2W \end{cases}$



The cutoff frequency such that $\omega_{bp2} - \omega_{bp1}$ is minimized, and a copy of the spectrum of $x(t)$ is kept is: $\omega_{bp2} = 6W$.