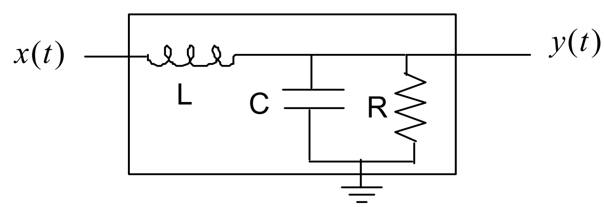
Sample Midterm Test 2 (mt2s02)
Covering Chapters 4-5 and part of Chapter 15 of *Fundamentals of Signals & Systems*

Problem 1 (25 marks)

Consider the second-order, lowpass RLC Butterworth filter depicted below. The input voltage is $x(t)$ and the output voltage is $y(t)$.



The differential equation relating the input and output voltages of this RLC filter is

$$LC\frac{d^2y(t)}{dt^2}+\frac{L}{R}\frac{dy(t)}{dt}+y(t)=x(t).$$

(a) [10 marks] Find the frequency response $H(j\omega)=Y(j\omega)/X(j\omega)$ of the filter. Express it in the form

$$H(j\omega)=\frac{A\omega_n^2}{(j\omega)^2+2\varsigma\omega_n(j\omega)+\omega_n^2}.$$

Give expressions for the damping ratio ς and the undamped natural frequency ω_n.

Answer:
The frequency response of the filter is

$$H(j\omega):=\frac{Y(s)}{X(s)}=\frac{1}{LC(j\omega)^2+\frac{L}{R}(j\omega)+1}$$

$$=\frac{\frac{1}{LC}}{(j\omega)^2+\frac{1}{RC}(j\omega)+\frac{1}{LC}}=\frac{\omega_n^2}{(j\omega)^2+2\varsigma\omega_n(j\omega)+\omega_n^2}$$

Thus, $\omega_n=\frac{1}{\sqrt{LC}}$ and $\varsigma=\frac{1}{2\frac{1}{\sqrt{LC}}RC}=\frac{\sqrt{L}}{2R\sqrt{C}}$.

(b) [15 marks] The filter has a low-pass Butterworth response with cutoff frequency $\omega_c = 1$ radian/s for $R = \frac{1}{\sqrt{2}}\Omega$, $L = 1\mathrm{H}$, and $C = 1\mathrm{F}$. Give the numerical values of the damping ratio ς and the undamped natural frequency ω_n. Compute and sketch the step response of the system to a 10-Volt step in input voltage, i.e., $x(t) = 10u(t)V$.

Answer:

$$\varsigma = \frac{1}{\sqrt{2}}, \quad \omega_n = 1$$

The frequency response of the Butterworth filter is:

$$H(j\omega) = \frac{1}{(j\omega)^2 + \sqrt{2}(j\omega) + 1} = \frac{1}{(j\omega + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})(j\omega + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})}$$

The unit step input voltage has the following Fourier transform:

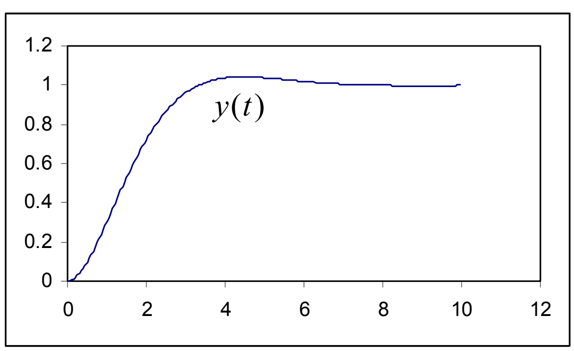
$$X(j\omega) = \frac{10}{j\omega} + 10\pi\delta(\omega)$$

Hence, the Fourier transform of the output voltage is given by:

$$\begin{aligned} H(j\omega) &= \frac{1}{(j\omega + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})(j\omega + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})}\left(\frac{10}{j\omega} + 10\pi\delta(\omega)\right) \\ &= \frac{10}{j\omega(j\omega + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})(j\omega + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})} + 10\pi\delta(\omega) \\ &= \frac{5(-1+j)}{j\omega + \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}} + \frac{5(-1-j)}{j\omega + \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}} + \frac{10}{j\omega} + 10\pi\delta(\omega) \end{aligned}$$

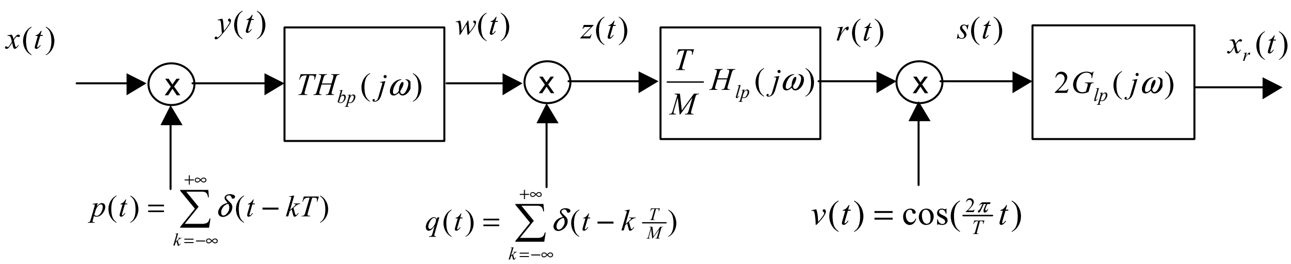
and taking the inverse transform, we get

$$\begin{aligned} y(t) &= \left[(-5 + j5)e^{(-\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})t} + (-5 - j5)e^{(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})t}\right]u(t) + 10u(t) \\ &= 5\sqrt{2}e^{-\frac{1}{\sqrt{2}}t}\left[e^{j\left[\frac{1}{\sqrt{2}}t + \frac{3\pi}{4}\right]} + e^{-j\left[\frac{1}{\sqrt{2}}t + \frac{3\pi}{4}\right]}\right]u(t) + 10u(t) = 10\sqrt{2}e^{-\frac{1}{\sqrt{2}}t}\cos(\tfrac{1}{\sqrt{2}}t + \frac{3\pi}{4})u(t) + 10u(t) \ \ V \end{aligned}$$

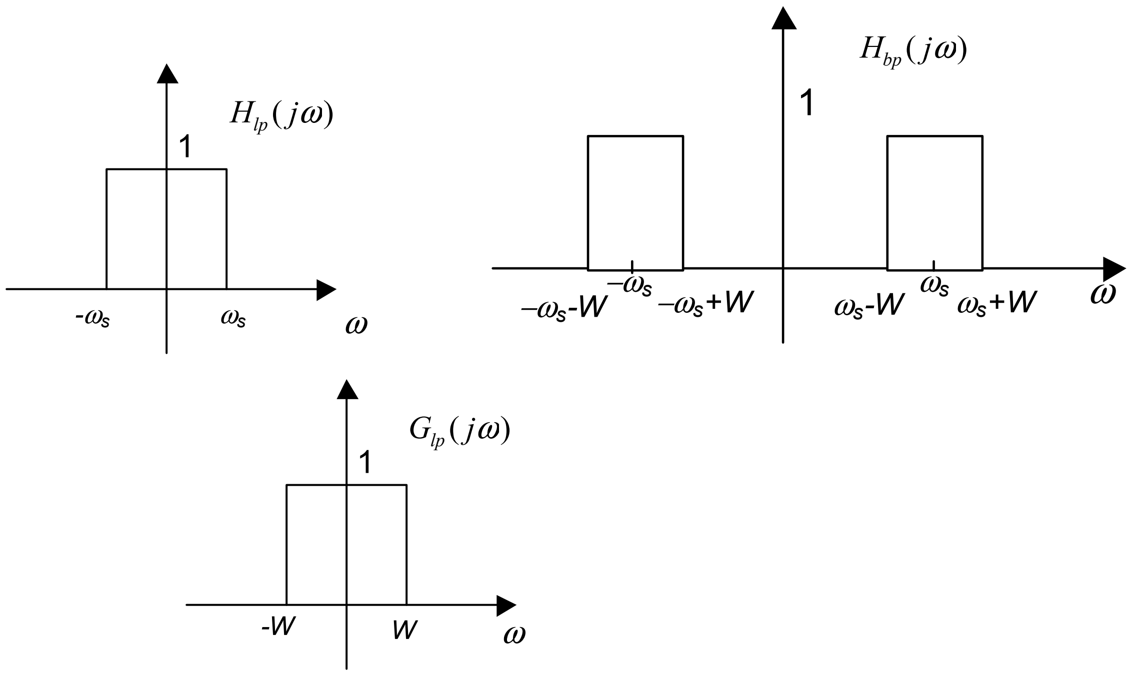


Problem 2 (25 marks)

Consider the following sampling/modulation system.

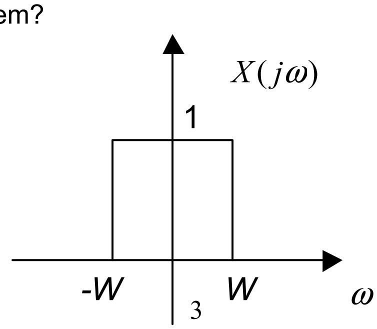


Where the input signal is the convolution $x(t)=\frac{W^2}{\pi^2}\text{sinc}\left(\frac{W}{\pi}t\right)*\text{sinc}\left(\frac{W}{\pi}t\right)$ and the spectra of the ideal bandpass filter and the ideal lowpass filters are shown below.



(a) [7 marks] Find and sketch $X(j\omega)$, the Fourier transform of the input signal $x(t)$. For what range of sampling frequencies $\omega_s=\frac{2\pi}{T}$ is the sampling theorem satisfied for the first sampler? What is the smallest positive integer M that will ensure that the second sampler satisfies the sampling theorem?

Answer:



The sampling theorem is satisfied for $\omega_s > 2W$ for the first sampler. The second sampler satisfies the sampling theorem if

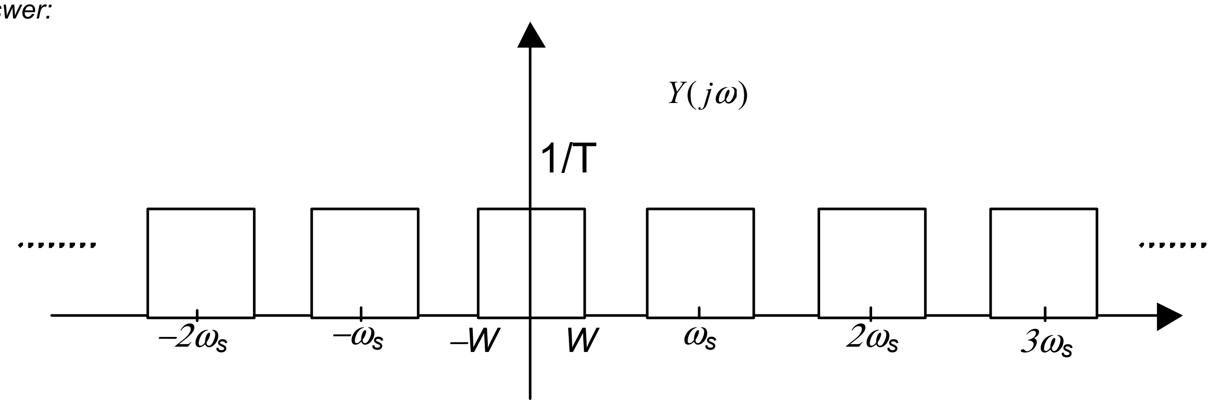
$$\omega_{s_2} = \frac{2\pi M}{T} = M\omega_s > 2(\omega_s + W)$$

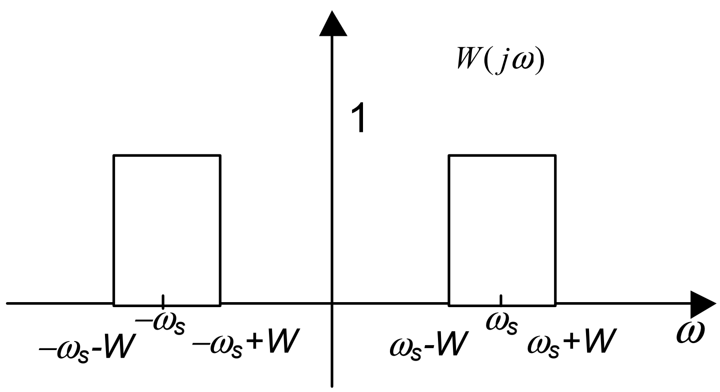
$$\Rightarrow \quad M > 2 + \frac{W}{\omega_s}$$

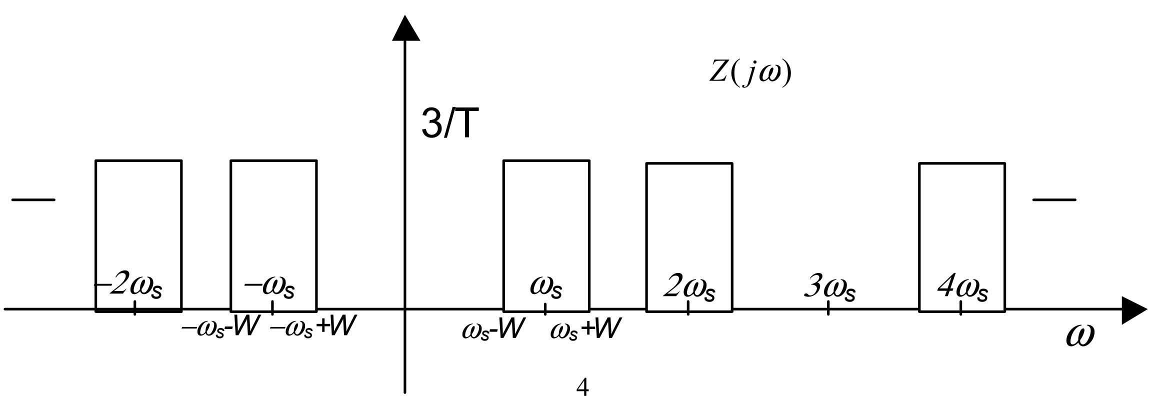
and the smallest integer M satisfying this inequality is M=3 such that $\omega_{s_2} = 3\omega_s$.

(b) [8 marks] Assume that the sampling theorem is satisfied for the first sampler and pick M to be the smallest integer that you obtained in (a) in the remaining questions. Sketch the Fourier transforms $Y(j\omega)$, $W(j\omega)$ and $Z(j\omega)$.

Answer:

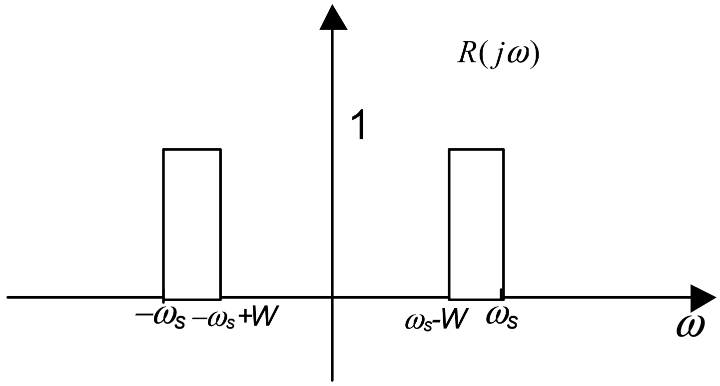




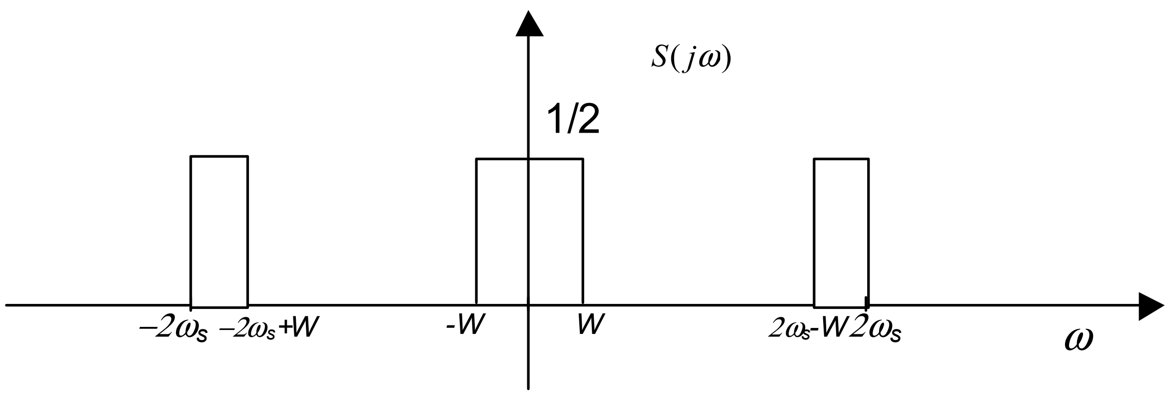


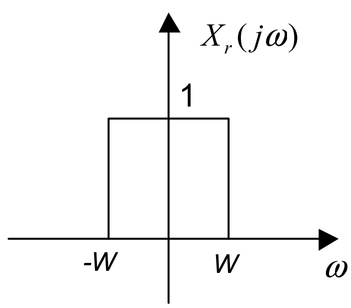
(c) [5 marks] Sketch the Fourier transforms $R(j\omega)$, $S(j\omega)$ and $X_r(j\omega)$.

Answer:



The modulation with a cosine signal makes two copies of $R(j\omega)$, one around ω_s and on around $-\omega_s$, which reconstructs the spectrum around $\omega = 0$:





(d) [5 marks] Using Parseval's relation (see appended Table), find the total energy of the error signal $e(t) := x(t) - x_r(t)$ defined as the difference between the input signal $x(t)$ and the "reconstructed" output signal $x_r(t)$.

Answer:

Total energy of the error signal is $Energy = \frac{1}{2\pi}\int_{-\infty}^{\infty} |E(j\omega)|^2 \, d\omega = 0$

where the Fourier transform of the error signal is $E(j\omega) = X(j\omega) - X_r(j\omega) = 0$

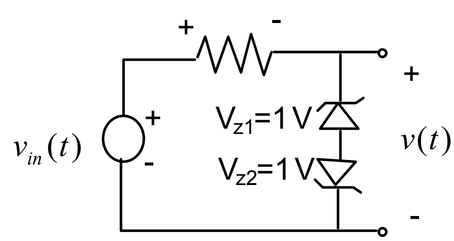
Problem 3 (20 marks)

The following circuit with two Zener diodes is an ideal clamping circuit.

The input voltage is

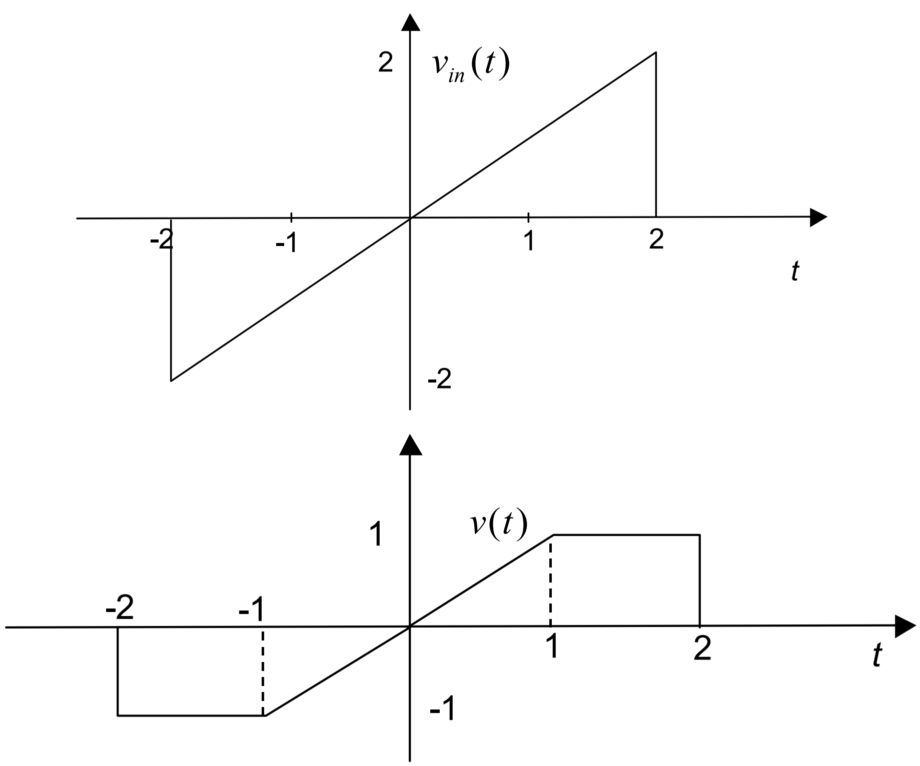
$$v_{in}(t) = tu(t+2) - tu(t-2) \text{ Volts,}$$

and the output voltage is $v(t) = \begin{cases} v_{in}(t), & -1 < v_{in}(t) < 1 \\ 1, & v_{in}(t) \geq 1 \\ -1, & v_{in}(t) \leq -1 \end{cases}$.



(a) [8 marks] Sketch the input voltage $v_{in}(t)$ and the output voltage $v(t)$.

Answer:
Input and output voltage:



(b) [12 marks] Compute the Fourier transform $V(j\omega)$ of the output voltage $v(t)$. Show that it is purely imaginary and odd.

Answer:

$$\begin{aligned}
V(j\omega) &= \int_{-\infty}^{\infty} v(t)e^{-j\omega t}\,dt = \int_{-2}^{-1} -e^{-j\omega t}\,dt + \int_{1}^{2} e^{-j\omega t}\,dt + \int_{-1}^{1} te^{-j\omega t}\,dt \\
&= \int_{1}^{2} (-e^{j\omega t} + e^{-j\omega t})\,dt + \frac{1}{-j\omega}\left[te^{-j\omega t}\right]_{-1}^{1} - \frac{1}{-j\omega}\int_{-1}^{1} e^{-j\omega t}\,dt \\
&= -2j\int_{1}^{2} \sin(\omega t)\,dt + \frac{e^{-j\omega} + e^{j\omega}}{-j\omega} - \frac{1}{(-j\omega)^2}\left[e^{-j\omega t}\right]_{-1}^{1} \\
&= \frac{2j}{\omega}\left[\cos(\omega t)\right]_{1}^{2} + \frac{2j\cos\omega}{\omega} + \frac{e^{-j\omega} - e^{j\omega}}{\omega^2} \\
&= \frac{2j}{\omega}\left[\cos(2\omega) - \cos(\omega)\right] + \frac{2j\cos\omega}{\omega} - 2j\frac{\sin(\omega)}{\omega^2} \\
&= \frac{2j\cos(2\omega)}{\omega} - 2j\frac{\sin\omega}{\omega^2}
\end{aligned}$$

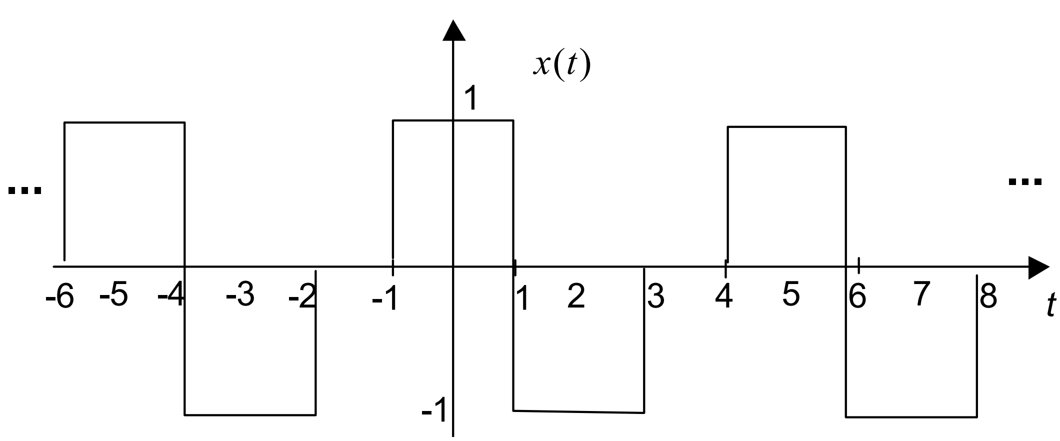
The FT is purely imaginary: $V(j\omega) = j\underbrace{\left(\frac{2\cos(2\omega)}{\omega} - 2\frac{\sin\omega}{\omega^2}\right)}_{\in\mathbb{R}}$.

It is also odd as

$$V(-j\omega) = j\left(\frac{2\cos(-2\omega)}{-\omega} - 2\frac{\sin(-\omega)}{\omega^2}\right) = j\left(\frac{2\cos(2\omega)}{-\omega} + 2\frac{\sin(\omega)}{\omega^2}\right) = -j\left(\frac{2\cos(2\omega)}{\omega} - 2\frac{\sin(\omega)}{\omega^2}\right) = -V(j\omega)$$

Problem 4 (30 marks)

(a) [15 marks] Consider the periodic signal $x(t)$ depicted below. Give a mathematical expression for $x(t)$. Find its fundamental frequency ω_0. Compute its Fourier series coefficients a_k. Express $x(t)$ as a Fourier series.



Answer:
This signal can be written as:

$$x(t) = \sum_{m=-\infty}^{+\infty} \left[u(t+1-5m) - 2u(t-1-5m) + u(t-3-5m) \right]$$

Its fundamental period and frequency are $T = 5,\ \omega_0 = \frac{2\pi}{5}$. The average value over one period is given by: $a_0 = \frac{1}{5}\int_{-1}^{3} x(t)dt = \frac{1}{5}(2-2) = 0$. The FS coefficients a_k for $k \neq 0$ are given by

$$\begin{aligned}
a_k &= \frac{1}{T}\int_T x(t) e^{-jk\frac{2\pi}{5}t} dt \\
&= \frac{1}{5}\int_{-1}^{1} e^{-jk\frac{2\pi}{5}t} dt - \frac{1}{5}\int_{1}^{3} e^{-jk\frac{2\pi}{5}t} dt \\
&= \frac{1}{5(-jk\frac{2\pi}{5})}\left[e^{-jk\frac{2\pi}{5}t} \right]_{-1}^{1} - \frac{1}{5(-jk\frac{2\pi}{5})}\left[e^{-jk\frac{2\pi}{5}t} \right]_{1}^{3}
\end{aligned}$$

$$\begin{aligned}
&= \frac{-e^{-jk\frac{2\pi}{5}} + e^{jk\frac{2\pi}{5}}}{jk2\pi} + \frac{e^{-jk\frac{2\pi}{5}3} - e^{-jk\frac{2\pi}{5}}}{jk2\pi} \\
&= \frac{e^{jk\frac{2\pi}{5}} - e^{-jk\frac{2\pi}{5}}}{jk2\pi} + \frac{e^{-jk\frac{2\pi}{5}2}\left(e^{-jk\frac{2\pi}{5}} - e^{jk\frac{2\pi}{5}} \right)}{jk2\pi} \\
&= \frac{\sin(k\frac{2\pi}{5})}{k\pi} - \frac{\sin(k\frac{2\pi}{5})}{k\pi} e^{-jk\frac{4\pi}{5}} = \left(1 - e^{-jk\frac{4\pi}{5}} \right) \frac{\sin(k\frac{2\pi}{5})}{k\pi}
\end{aligned}$$

The Fourier series representation of $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{5}t} = \sum_{k=-\infty}^{+\infty} \left(1 - e^{-jk\frac{4\pi}{5}} \right) \frac{\sin(k\frac{2\pi}{5})}{k\pi} e^{jk\frac{2\pi}{5}t}.$$

(b) [5 marks] Find the coefficients B_k, C_k of the real form of the Fourier series of $x(t)$:

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty} [B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t)]$$

Answer:
It is easy to show that

$B_k = \mathrm{Re}\{a_k\}$, $C_k = \mathrm{Im}\{a_k\}$, $k \geq 1$, thus

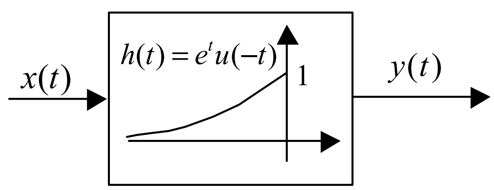
$$B_k = \mathrm{Re}\left\{\left(1 - e^{-jk\frac{4\pi}{5}}\right)\frac{\sin(k\frac{2\pi}{5})}{k\pi}\right\}$$

$$= \frac{\sin(k\frac{2\pi}{5})}{k\pi}\left(1 - \cos(k\frac{4\pi}{5})\right)$$

$$C_k = \mathrm{Im}\left\{\left(1 - e^{-jk\frac{4\pi}{5}}\right)\frac{\sin(k\frac{2\pi}{5})}{k\pi}\right\}$$

$$= \frac{\sin(k\frac{2\pi}{5})}{k\pi}\sin(k\frac{4\pi}{5})$$

(c) [10 marks] Suppose that $x(t)$ is the input to an LTI system with impulse response $h(t)$ as shown below:



Calculate the output $y(t)$ and the power in its fifth harmonic components.

Answer:
The frequency response of the system is given by:

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt = \int_{-\infty}^{0} e^{(1-j\omega)t}dt = \frac{1}{1-j\omega}\left[e^{(1-j\omega)t}\right]_{-\infty}^{0},$$

$$= \frac{1}{1-j\omega}[1-0] = \frac{1}{1-j\omega}$$

then we obtain the Fourier series coefficients of the output:

$$d_k = H(j\frac{2\pi k}{5})a_k = \frac{\sin(k\frac{2\pi}{5})}{(1-j\frac{2\pi k}{5})k\pi}\left(1 - e^{-jk\frac{4\pi}{5}}\right), \ k \neq 0.$$

$$d_0 = H(j0)a_0 = \underbrace{\left[\frac{1}{1-j\omega}\right]_{\omega=0}}_{=1}[0] = 0$$

Thus,

$$y(t)=\sum_{k=-\infty}^{+\infty} d_k e^{jk\frac{2\pi}{5}t}=\sum_{k=-\infty}^{+\infty}\left(1-e^{-jk\frac{4\pi}{5}}\right)\frac{\sin(k\frac{2\pi}{5})}{(1-jk\frac{2\pi}{5})k\pi}e^{jk\frac{2\pi}{5}t}$$

Power in fifth harmonic components:

$$P_5=\left|d_{-5}\right|^2+\left|d_5\right|^2=2\left|d_5\right|^2$$

$$=2\left|\left(1-e^{-j4\pi}\right)\frac{\sin(2\pi)}{(1-j2\pi)5\pi}\right|^2=0$$