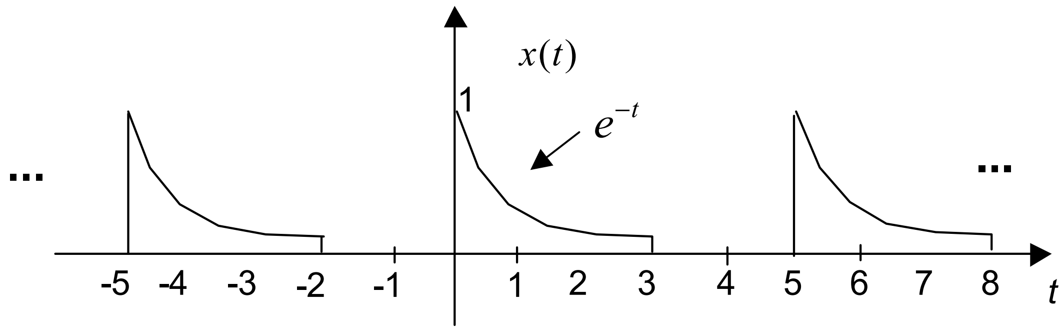
Sample Midterm Test 2 (mt2s01)
Covering Chapters 4-5 and part of Chapter 15 of *Fundamentals of Signals & Systems*

Problem 1 (30 marks)

(a) [10 marks] Consider the periodic signal $x(t)$ depicted below. Give a mathematical expression for $x(t)$. Find its fundamental frequency ω_0. Compute its Fourier series coefficients a_k. Express $x(t)$ as a Fourier series.



Answer:
This signal can be written as:

$$x(t) = \sum_{m=-\infty}^{+\infty} e^{-(t-5m)} \left[u(t-5m) - u(t-3-5m) \right]$$

Its fundamental period and frequency are $T = 5,\ \omega_0 = \frac{2\pi}{5}$.

The average value over one period is given by:

$$a_0 = \frac{1}{5}\int_0^5 x(t)dt = \frac{1}{5}\int_0^3 e^{-t}\,dt = -\frac{1}{5}\left[e^{-t} \right]_0^3 = \frac{1-e^{-3}}{5}$$

The FS coefficients a_k for $k \neq 0$ are given by

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\frac{2\pi}{5}t}dt$$
$$= \frac{1}{5}\int_0^3 e^{-t}e^{-jk\frac{2\pi}{5}t}dt = -\frac{1}{5(1+jk\frac{2\pi}{5})}\left[e^{-(1+jk\frac{2\pi}{5})t}\right]_0^3$$
$$= \frac{1-e^{-3}e^{-jk\frac{2\pi}{5}3}}{5+jk2\pi} = \frac{1-e^{-(3+jk\frac{2\pi}{5}3)}}{5+jk2\pi}$$

The Fourier series representation of $x(t)$ is

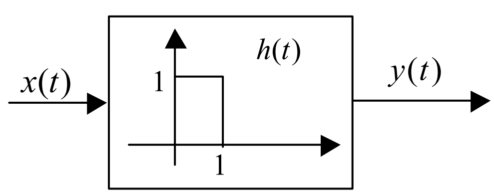
$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{5}t} = \sum_{k=-\infty}^{+\infty} \frac{1-e^{-(3+jk\frac{2\pi}{5}3)}}{5+jk2\pi} e^{jk\frac{2\pi}{5}t}.$$

(b) [5 marks] Compute the total average power of the third harmonic components of $x(t)$.

Answer:

$$P_{3tot} = |a_{-3}|^2 + |a_3|^2 = 2|a_3|^2$$
$$= 2\left|\frac{1-e^{-(3+j\frac{2\pi}{5}9)}}{5+j6\pi}\right|^2$$
$$= \frac{2}{25+36\pi^2}\left[\left(1-e^{-3}\cos\frac{18\pi}{5}\right)^2 + e^{-6}(\sin\frac{18\pi}{5})^2\right]$$
$$= \frac{2}{25+36\pi^2}\left[1-2e^{-3}\cos\frac{18\pi}{5}+e^{-6}\right]$$
$$= 0.00511$$

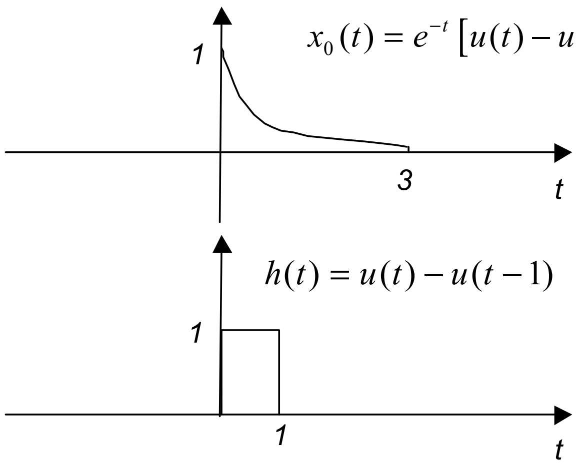
(c) [15 marks] Suppose that $x(t)$ is the input to an LTI system with impulse response $h(t)$ as shown below:



Calculate the output $y(t)$. Compute the total average power in the fundamental component of $y(t) \overset{\mathcal{FS}}{\leftrightarrow} d_k$. What is the ratio of total average power in the fundamental components P_{1tot} of the output signal to its total average power P_∞?

Answer:

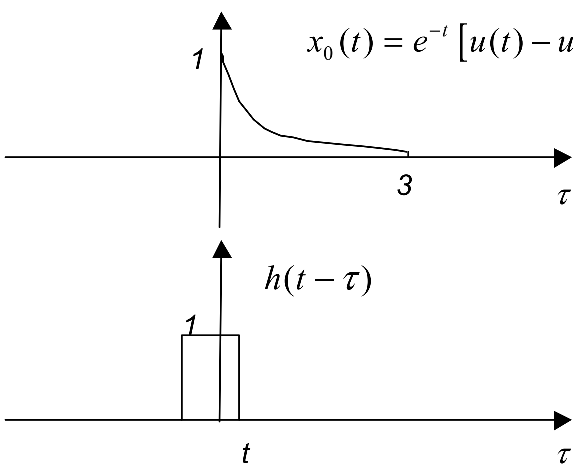
To find $y(t)$, it is best to compute the convolution in the time domain. Note that there is no overlap on either side of the nonzero part of the period starting at zero. Thus, we can concentrate on that part of the signal only which will generate one full period of the output signal.



Let's time-reverse and shift the impulse response. The intervals of interest are:

$t < 0$: no overlap, so $y(t) = 0$.

$0 \le t < 1$: overlap for $0 \le \tau < t$



Then

$$y_0(t) = \int_0^t h(t-\tau)x_0(\tau)d\tau = \int_0^t e^{-\tau}d\tau = -e^{-\tau}\Big|_0^t$$
$$= (1 - e^{-t})$$

$1 \le t < 3$: overlap for $t-1 < \tau < t$

Then

$$y_0(t) = \int_{t-1}^{t} h(t-\tau)x_0(\tau)d\tau = \int_{t-1}^{t} e^{-\tau}d\tau = -e^{-\tau}\Big|_{t-1}^{t}$$
$$= (e^{1-t} - e^{-t}) = (e-1)e^{-t}$$

$3 \le t < 4$: overlap for $t-1 \le \tau < 3$

Then

$$y_0(t) = \int_{t-1}^{3} h(t-\tau)x_0(\tau)d\tau = \int_{t-1}^{3} e^{-\tau}d\tau = -e^{-\tau}\Big|_{t-1}^{3}$$
$$= (e^{1-t} - e^{-3})$$

$t \ge 4$: no overlap

Then

$y_0(t) = 0$

Thus

$$y_0(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-t} & 0 \le t < 1 \\ (e-1)e^{-t} & 1 \le t < 3 \\ (e^{1-t} - e^{-3}) & 3 \le t < 4 \\ 0 & t \ge 4 \end{cases}$$

and the periodic output signal is

$$y(t) = \sum_{m=-\infty}^{\infty} y_0(t - m5)$$

To compute the power ratio, we have to compute the total average power in the output signal. A time domain solution is adopted:

$$
\begin{aligned}
P_\infty &= \frac{1}{T}\int_0^T {y_0}^2(t)dt \\
&= \frac{1}{5}\left[\int_0^1 \left(1-e^{-t}\right)^2 dt + \int_1^3 \left((e-1)e^{-t}\right)^2 dt + \int_3^4 \left(e^{1-t}-e^{-3}\right)^2 dt\right] \\
&= \frac{1}{5}\left[\int_0^1 \left(1-e^{-t}+e^{-2t}\right)dt + (e-1)^2\int_1^3 e^{-2t}dt + \int_3^4 \left(e^{-2(t-1)}-2e^{-(t+2)}+e^{-6}\right)dt\right] \\
&= \frac{1}{5}\left[\left(t+e^{-t}-2e^{-2t}\right)_0^1 + (e-1)^2\left(-2e^{-2t}\right)_1^3 + \left(-2e^{-2(t-1)}+2e^{-(t+2)}+e^{-6}t\right)_3^4\right] \\
&= \frac{1}{5}\left[\left(1+e^{-1}-2e^{-2}-1+2\right)+(e^2-2e+1)\left(-2e^{-6}+2e^{-2}\right)+\left(-2e^{-6}+2e^{-6}+4e^{-6}+2e^{-4}-2e^{-5}-3e^{-6}\right)\right] \\
&= \frac{1}{5}\left[\left(2+e^{-1}-2e^{-2}\right)+\left(-2e^{-4}+4e^{-5}-2e^{-6}+2-4e^{-1}+2e^{-2}\right)+\left(e^{-6}+2e^{-4}-2e^{-5}\right)\right] \\
&= \frac{1}{5}\left[4+2e^{-5}-e^{-6}-3e^{-1}\right] = 0.5815
\end{aligned}
$$

The frequency response of the system is given by:

$$
\begin{aligned}
H(j\omega) &= \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt = \int_0^1 e^{-j\omega t}dt = -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_0^1 \\
&= -\frac{1}{j\omega}\left[e^{-j\omega}-1\right] = \frac{(1-e^{-j\omega})}{j\omega}
\end{aligned},
$$

then we obtain the Fourier series coefficients of the output:

$$
\begin{aligned}
d_k &= H(j\frac{2\pi k}{5})a_k = \frac{(1-e^{-j\frac{2\pi k}{5}})}{j\frac{2\pi k}{5}}\left[\frac{1-e^{-(3+jk\frac{2\pi}{5}3)}}{5+jk2\pi}\right],\ k\neq 0 \\
d_0 &= H(j0)a_0 = \underbrace{\left[\frac{(1-e^{-j\omega})}{j\omega}\right]_{\omega=0}}_{=1}\left[\frac{1-e^{-3}}{5}\right] = \frac{1-e^{-3}}{5}
\end{aligned}.
$$

Finally, the power in the fundamental components is given by

$$P_{1tot} = |d_{-1}|^2 + |d_1|^2 = 2|d_1|^2$$

$$= 2\left| \frac{(1-e^{-j\frac{2\pi}{5}})}{j\frac{2\pi}{5}} \left[\frac{1-e^{-(3+j\frac{2\pi}{5}3)}}{5+j2\pi} \right] \right|^2 = 2\left| \frac{(1-e^{-j\frac{2\pi}{5}} - e^{-(3+j\frac{6\pi}{5})} + e^{-(3+j\frac{8\pi}{5})})}{j2\pi - \frac{4\pi^2}{5}} \right|^2$$

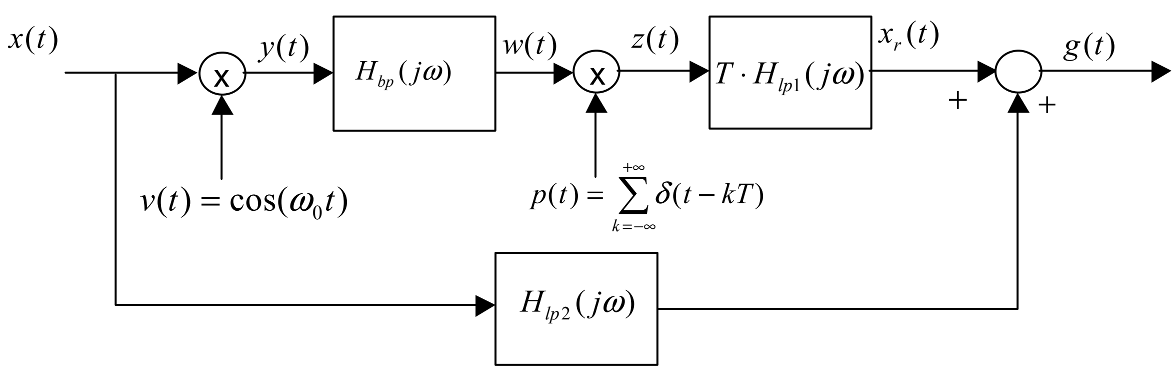
$$= \frac{2}{(\frac{4\pi^2}{5})^2 + 4\pi^2} \left[\left(1-\cos\frac{2\pi}{5} - e^{-3}\cos\frac{6\pi}{5} + e^{-3}\cos\frac{8\pi}{5}\right)^2 + \left(\sin\frac{2\pi}{5} + e^{-3}\sin\frac{6\pi}{5} - e^{-3}\sin\frac{8\pi}{5}\right)^2 \right]$$

$$= 0.0196[0.5575 + 0.9392] = 0.0294$$

and the ratio of powers is $ratio = \frac{P_{1tot}}{P_\infty} = \frac{0.0294}{0.5815} = 0.0506 = 5.06\%$

Problem 2 (25 marks)

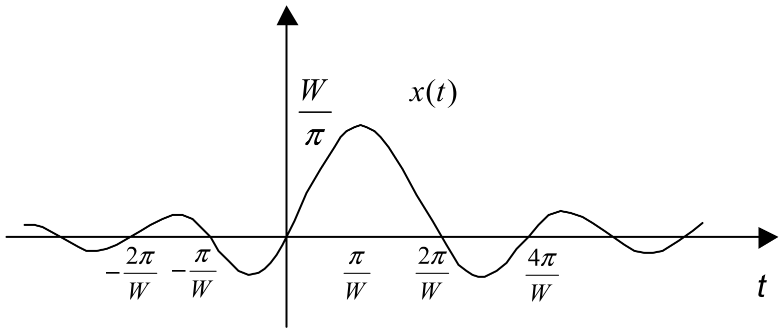
Consider the following system where the sampling frequency is $\omega_s = \frac{2\pi}{T}$.



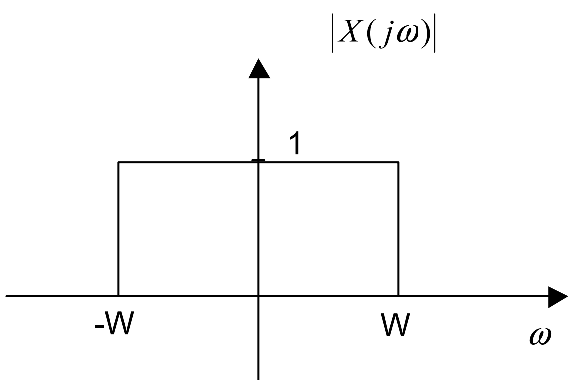
The input signal is $x(t) = \frac{W}{\pi}\text{sinc}\left(\frac{W}{\pi}(t-\frac{\pi}{W})\right)$, and $\omega_0 = 2W$.

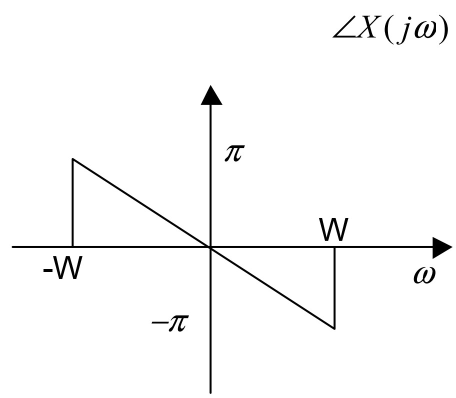
(a) [10 marks] Sketch the input signal $x(t)$. Find and sketch $X(j\omega)$, the Fourier transform of the input signal. Assuming that $\omega_{bp1} < \omega_{bp2}$ are the cutoff frequencies of the unit-magnitude ideal bandpass filter $H_{bp}(j\omega)$, find values for the cutoff frequencies such that $\omega_{bp2} - \omega_{bp1}$ is minimized, and the total average power of $w(t)$ is maximized. Sketch the Fourier transform $W(j\omega)$ of $w(t)$.

Answer:

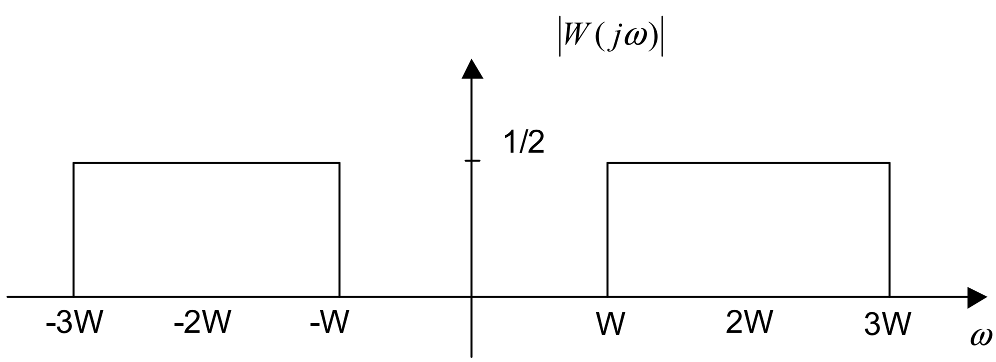


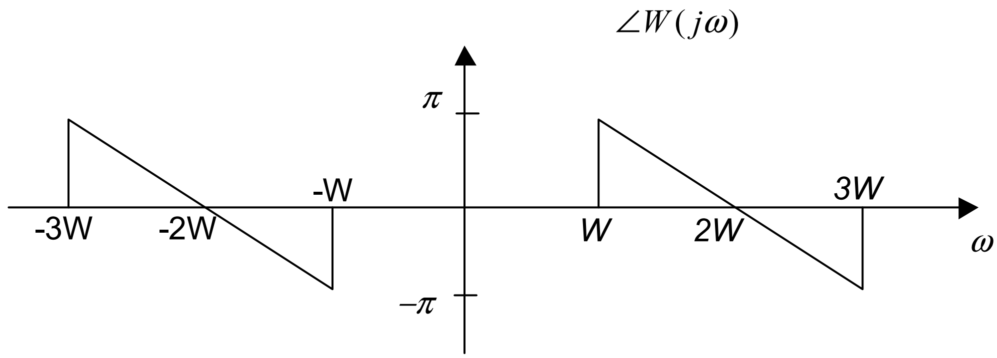
Fourier transform $X(j\omega)=\begin{cases} e^{-j\omega\frac{\pi}{W}}, & |\omega|<W \\ 0, & |\omega|>W \end{cases}$





The cutoff frequencies such that $\omega_{bp2}-\omega_{bp1}$ is minimized, and the total average power of $w(t)$ is maximized are: $\omega_{bp1}=W$, $\omega_{bp2}=3W$.

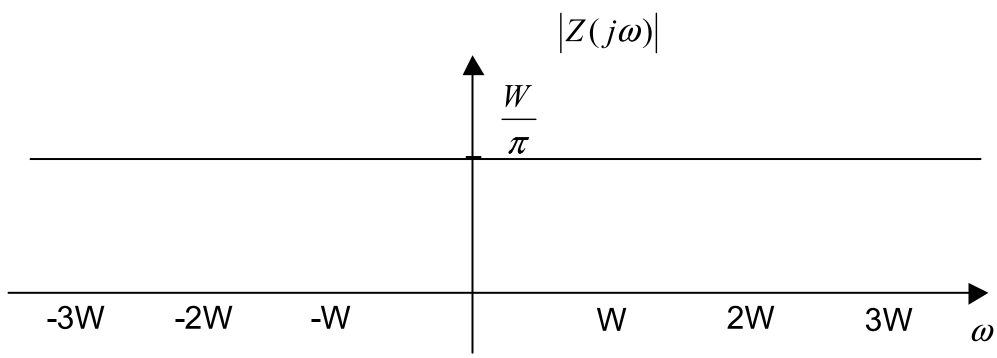


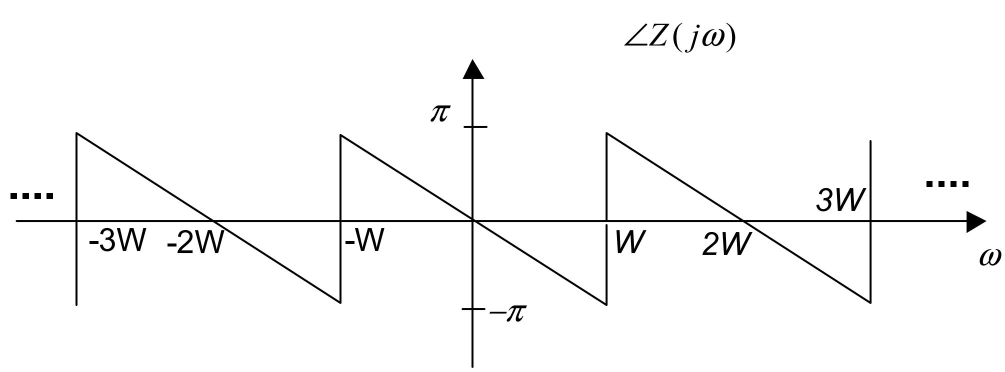


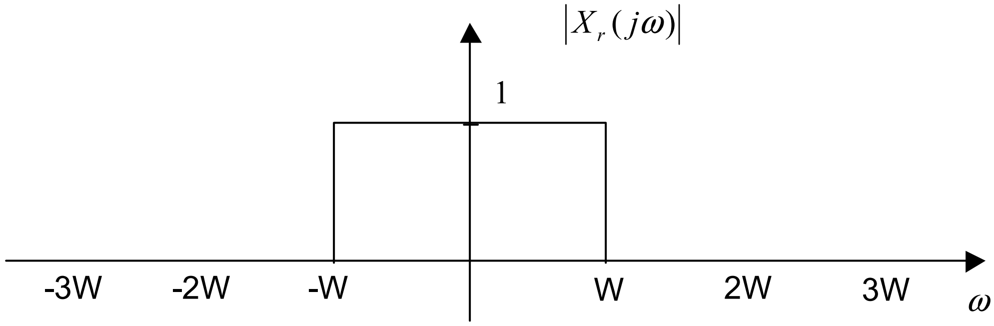
(b) [10 marks] Assume that the ideal unit-magnitude lowpass filter $H_{lp1}(j\omega)$ has a cutoff frequency $\omega_{lp1} = W$. Find the sampling frequency ω_s that will allow recovery of the spectrum of the input signal. Does it meet the requirement of the sampling theorem? Sketch the spectra $Z(j\omega)$ and $X_r(j\omega)$. Give an expression for $X_r(j\omega)$.

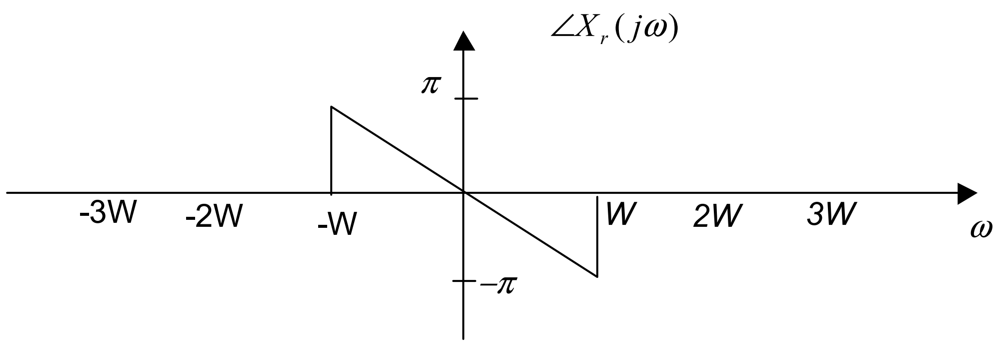
Answer:

The spectrum can be recovered if the sampling frequency is $\omega_s = 2W$ (sampling period $T = \frac{\pi}{W}$). This frequency does not satisfy the sampling theorem, but it allows us to bring the spectrum back around DC.





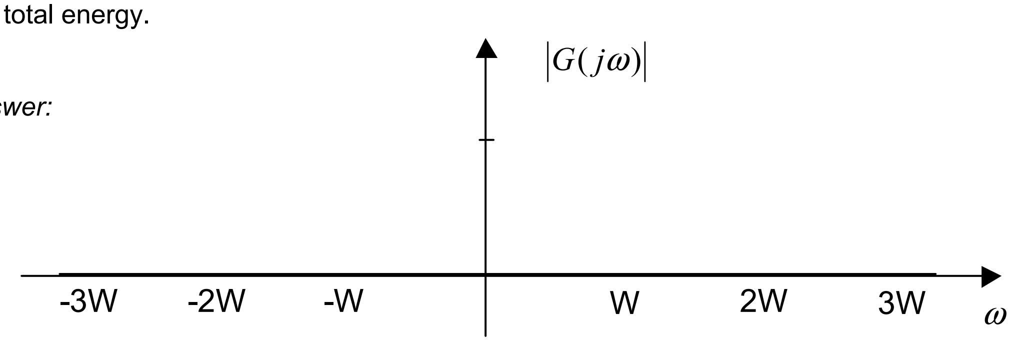


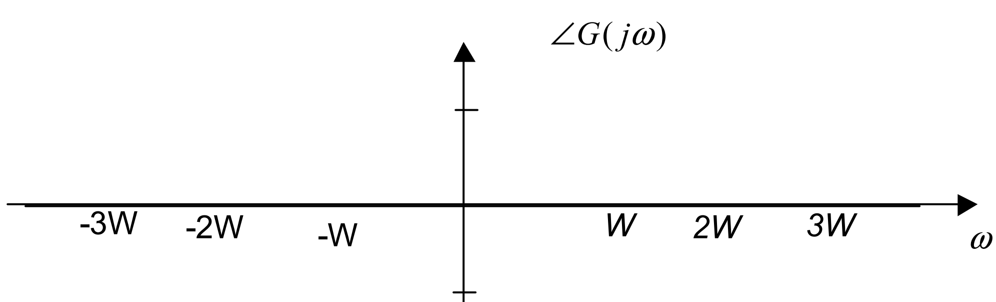


Fourier transform of $x_r(t)$ is $X_r(j\omega) = X(j\omega) = \begin{cases} e^{-j\omega\frac{\pi}{W}}, & |\omega| < W \\ 0, & |\omega| > W \end{cases}$.

(c) [5 marks] Assume that the second ideal unit-magnitude lowpass filter $H_{lp2}(j\omega)$ has a cutoff frequency $\omega_{lp2} = 2W$. Sketch the spectrum of the output signal $G(j\omega)$ and compute its total energy.

Answer:





Total energy in $g(t)$ is computed as:

$$E_\infty = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left|G(j\omega)\right|^2 d\omega = 0$$

Problem 3 (10 marks)

Suppose that $x(t)$ is periodic of period T and $x(t) \overset{\mathcal{FS}}{\leftrightarrow} a_k$. Prove Parseval's relation:

$P = \frac{1}{T}\int_T \left|x(t)\right|^2 dt = \sum_{k=-\infty}^{\infty} \left|a_k\right|^2$.

Answer:

$$P = \frac{1}{T}\int_T \left|x(t)\right|^2 dt = \frac{1}{T}\int_T x(t)x^*(t)dt = \frac{1}{T}\int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \sum_{m=-\infty}^{\infty} a_m{}^* e^{-jm\omega_0 t} dt$$

$$= \frac{1}{T}\sum_{k=-\infty}^{\infty} a_k \sum_{m=-\infty}^{\infty} a_m{}^* \underbrace{\int_T e^{j(k-m)\omega_0 t} dt}_{\substack{=0,\ k\neq m \\ =T,\ k=m}}$$

$$= \frac{T}{T}\sum_{k=-\infty}^{\infty} a_k a_k{}^* = \sum_{k=-\infty}^{\infty} \left|a_k\right|^2$$

Problem 4 (15 marks)

The differential equation of a second-order lowpass Butterworth filter has the form:

$$\frac{d^2 y(t)}{dt^2} + \omega_c\sqrt{2}\frac{dy(t)}{dt} + \omega_c^2 y(t) = \omega_c^2 x(t).$$

(a) [3 marks] What is ω_c?

Answer:

It is the cutoff frequency of the filter, for which $\left|H(j\omega_c)\right| = \frac{1}{\sqrt{2}}$, also called the -3dB cutoff.

(b) [6 marks] Find the frequency response of this filter, then compute its magnitude.

Answer:

Frequency response is

$$H_B(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\omega_c^2}{(j\omega)^2 + \omega_c\sqrt{2}j\omega + \omega_c^2}$$

$$= \frac{1}{1 + (\frac{j\omega}{\omega_c})^2 + \frac{\sqrt{2}j\omega}{\omega_c}} = \frac{1}{1 - (\frac{\omega}{\omega_c})^2 + \frac{\sqrt{2}j\omega}{\omega_c}}$$

Magnitude:

$$|H_B(j\omega)| = \frac{1}{\sqrt{\left[1 - (\frac{\omega}{\omega_c})^2\right]^2 + \frac{2\omega^2}{\omega_c^2}}}$$

$$= \frac{1}{\sqrt{\left[1 - (\frac{\omega}{\omega_c})^2\right]^2 + \frac{2\omega^2}{\omega_c^2}}}$$

$$= \frac{1}{\sqrt{1 + (\frac{\omega}{\omega_c})^4}}$$

(c) [6 marks] Design a highpass filter starting with the above Butterworth filter, i.e., give its frequency response and its differential equation.

Answer:

A highpass filter can be obtained as follows:

$$H_{Bhp}(j\omega) = 1 - H_B(j\omega) = 1 - \frac{\omega_c^2}{(j\omega)^2 + \omega_c\sqrt{2}j\omega + \omega_c^2}$$

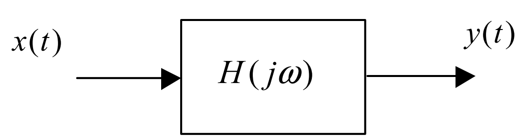
$$= \frac{(j\omega)^2 + \omega_c\sqrt{2}j\omega}{(j\omega)^2 + \omega_c\sqrt{2}j\omega + \omega_c^2}$$

Differential equation of highpass filter:

$$\frac{d^2y(t)}{dt^2} + \omega_c\sqrt{2}\frac{dy(ct)}{dt} + \omega_c^2 y(t) = \frac{d^2x(t)}{dt^2} + \omega_c\sqrt{2}\frac{dx(t)}{dt}$$

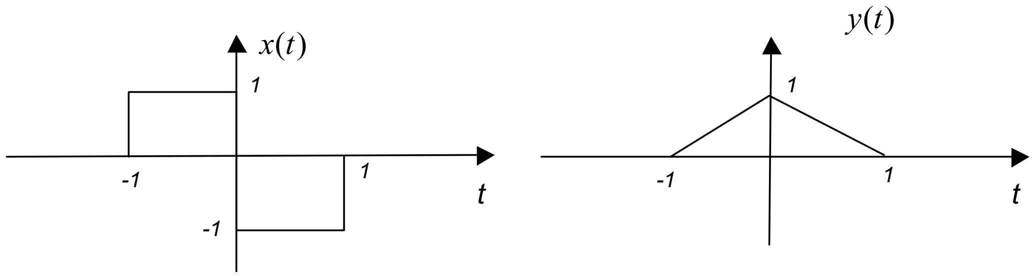
Problem 5 (20 marks)

The input signal of the LTI differential system shown below (initially at rest) is $x(t) = u(t+1) - 2u(t) + u(t-1)$ and its corresponding output is $y(t) = (t+1)u(t+1) - 2tu(t) + (t-1)u(t-1)$.



(a) [6 marks] Sketch the input signal $x(t)$ and the output signal $y(t)$.

Answer:



(b) [7 marks] Compute the Fourier transform of the output $Y(j\omega)$.

Answer:

Method 1: (direct)

$$Y(j\omega)=\int_{-\infty}^{+\infty} y(t)e^{-j\omega t}\,dt$$
$$=\int_{-1}^{0}(1+t)e^{-j\omega t}\,dt+\int_{0}^{1}(1-t)e^{-j\omega t}\,dt$$
$$=\int_{0}^{1}\tau e^{-j\omega(\tau-1)}\,d\tau+\int_{0}^{1}(1-\tau)e^{-j\omega\tau}\,d\tau \quad (\tau=t+1 \text{ in first int., } \tau=t \text{ in second int.})$$
$$=\int_{0}^{1}e^{-j\omega\tau}\,d\tau+(e^{j\omega}-1)\int_{0}^{1}\tau e^{-j\omega\tau}\,d\tau$$
$$=\frac{1}{-j\omega}(e^{-j\omega}-1)+\frac{(e^{j\omega}-1)}{-j\omega}\left[\left[te^{-j\omega t}\right]_0^1-\int_{0}^{1}e^{-j\omega\tau}\,d\tau\right]$$
$$=\frac{1}{-j\omega}(e^{-j\omega}-1)+\frac{(e^{j\omega}-1)}{-j\omega}\left[e^{-j\omega}+\frac{(e^{-j\omega}-1)}{j\omega}\right]$$
$$=\frac{1}{-j\omega}(e^{-j\omega}-1)+\frac{(1-e^{-j\omega})}{-j\omega}+\frac{(2-e^{-j\omega}-e^{j\omega})}{-(j\omega)^2}$$
$$=\frac{2-2\cos(\omega)}{(\omega)^2}=\frac{4\sin^2(\omega/2)}{(\omega)^2}=\mathrm{sinc}^2(\omega/2\pi)$$

Method 2: convolution of two pulses to get y(t).
The output signal can be represented as the convolution of two pulses of width 0.5:

$$y(t)=\left[u(t+0.5)-u(t-0.5)\right]*\left[u(t+0.5)-u(t-0.5)\right]$$
$$=(t+1)u(t+1)-2tu(t)+(t-1)u(t-1)$$

Thus, its FT is a "sinc" squared:

$$Y(j\omega) = (2T_1)^2 \text{sinc}^2(\frac{T_1\omega}{\pi}) = \text{sinc}^2(\frac{\omega}{2\pi}) = \frac{4\sin^2(\omega/2)}{(\omega)^2}$$

(c) [7 marks] Compute the Frequency response $H(j\omega)$ of the system.

Answer:

Noticing that $y(t)$ is simply the running integral of $x(t)$, we find that the FR of the system is $H(j\omega) = \frac{1}{j\omega}$.

We can also compute the FT of the input and take the ratio $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$:

$$\begin{aligned} X(j\omega) &= (2T_1)\text{sinc}(\frac{T_1\omega}{\pi})e^{T_1 j\omega} - (2T_1)\text{sinc}(\frac{T_1\omega}{\pi})e^{-T_1 j\omega} \\ &= \text{sinc}(\frac{\omega}{2\pi})e^{0.5j\omega} - \text{sinc}(\frac{\omega}{2\pi})e^{-0.5j\omega} \\ &= 2j\sin(\frac{\omega}{2})\text{sinc}(\frac{\omega}{2\pi}) = j\omega\text{sinc}^2(\frac{\omega}{2\pi}) \end{aligned}$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\text{sinc}^2(\frac{\omega}{2\pi})}{j\omega\text{sinc}^2(\frac{\omega}{2\pi})} = \frac{1}{j\omega}$$