Sample Midterm Test 2 (mt2s00)
Covering Chapters 4-5 and part of Chapter 15 of *Fundamentals of Signals & Systems*

Problem 1 (5 marks)

State the Dirichlet conditions for a signal $x(t)$ to have a Fourier transform. Are these conditions necessary for $x(t)$ to have a Fourier transform, sufficient, or necessary and sufficient?

Answer:
The three Dirichlet conditions are *sufficient* for $x(t)$ to have a Fourier transform.
They are:
(1) $x(t)$ is absolutely integrable,
(2) $x(t)$ has a finite number of finite discontinuities in any finite interval of time,
(3) $x(t)$ has a finite number of maxima and minima in any finite interval of time.

Problem 2 (5 marks)

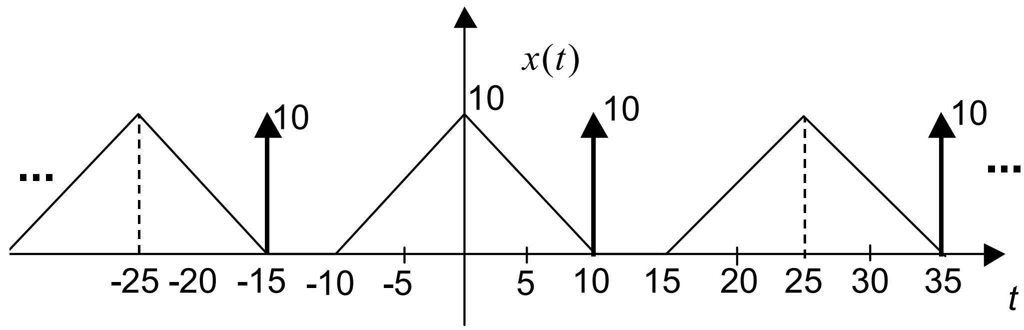
Suppose $x(t) \overset{\mathcal{FT}}{\leftrightarrow} X(j\omega)$ and $x(t) \in \mathbb{R}$. Show that $Ev\{x(t)\} \overset{\mathcal{FT}}{\leftrightarrow} \mathrm{Re}\{X(j\omega)\}$, where $Ev\{x(t)\}$ denotes the even part of $x(t)$.

Answer:

$$
\begin{aligned}
\mathcal{F}\{Ev\{x(t)\}\} &= \frac{1}{2}\int_{-\infty}^{+\infty}[x(t)+x(-t)]e^{-j\omega t}\,dt \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt + \frac{1}{2}\int_{-\infty}^{+\infty} x(-t)e^{-j\omega t}\,dt \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt - \frac{1}{2}\int_{+\infty}^{-\infty} x(\tau)e^{j\omega \tau}\,d\tau \\
&= \frac{1}{2}\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt + \frac{1}{2}\int_{-\infty}^{+\infty} x(\tau)e^{j\omega \tau}\,d\tau \\
&= \frac{1}{2}\left[X(j\omega)+X(-j\omega)\right] \\
&= \frac{1}{2}\left[X(j\omega)+X^*(j\omega)\right] \\
&= \mathrm{Re}\{X(j\omega)\}
\end{aligned}
$$

Problem 3 (25 marks)

(a) [10 marks] Find the Fourier series coefficients a_k of the following periodic signal $x(t)$. Express $x(t)$ as a Fourier series.



Answer:

This signal is the sum of an impulse train $x_1(t) \overset{\mathcal{FS}}{\leftrightarrow} b_k$ and a triangular wave $x_2(t) \overset{\mathcal{FS}}{\leftrightarrow} c_k$, both of period $T = 25$:

$$x(t) = x_1(t) + x_2(t) \overset{\mathcal{FS}}{\leftrightarrow} a_k = b_k + c_k$$

The FS coefficients b_k of the impulse train are given by

$$x_1(t) = \sum_{n=-\infty}^{+\infty} 10\delta[(t-10) - n25] \overset{\mathcal{FS}}{\leftrightarrow} b_k = \frac{10}{25} e^{-jk\frac{2\pi}{25}10} = \frac{2}{5} e^{-jk\frac{4\pi}{5}}$$

For the triangular wave, the average value $c_0 = 0$ over one period is given by:

$$c_0 = \frac{1}{T}\int_T x_2(t)dt = \frac{1}{25}\int_{-10}^{10} x_2(t)dt = \frac{1}{25}\frac{2(10)(10)}{2} = 4$$. For $k \neq 0$,

$$\begin{aligned}
c_k &= \frac{1}{T}\int_T x_2(t) e^{-jk\frac{2\pi}{T}t} dt \\
&= \frac{1}{25}\left[\int_{-10}^{0} (t+10)e^{-jk\frac{2\pi}{25}t} dt + \int_{0}^{10} (10-t)e^{-jk\frac{2\pi}{25}t} dt\right] \\
&= \frac{-1}{jk2\pi}\left[(t+10)e^{-jk\frac{2\pi}{25}t}\right]_{-10}^{0} + \frac{1}{jk2\pi}\int_{-10}^{0} e^{-jk\frac{2\pi}{25}t} dt - \frac{1}{jk2\pi}\left[(10-t)e^{-jk\frac{2\pi}{25}t}\right]_{0}^{10} - \frac{1}{jk2\pi}\int_{0}^{10} e^{-jk\frac{2\pi}{25}t} dt \\
&= \frac{-10}{jk2\pi} - \frac{25}{(jk2\pi)^2}\left[e^{-jk\frac{2\pi}{25}t}\right]_{-10}^{0} + \frac{10}{jk2\pi} + \frac{25}{(jk2\pi)^2}\left[e^{-jk\frac{2\pi}{25}t}\right]_{0}^{10} \\
&= \frac{25}{(jk2\pi)^2}\left[-1 + e^{jk\frac{4\pi}{5}} + e^{-jk\frac{4\pi}{5}} - 1\right] = \frac{25}{2(k\pi)^2}\left[1 - \cos(k\frac{4\pi}{5})\right]
\end{aligned}$$

Note that the coefficients are real and even, which is consistent with our real triangular waveform. The Fourier series representation of $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{25}t} = 4 + \frac{2}{5} + \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \left[\frac{25}{2(k\pi)^2}\left(1-\cos k\frac{4\pi}{5}\right) + \frac{2}{5}e^{-jk\frac{4\pi}{5}} \right] e^{jk\frac{2\pi}{25}t}.$$

(b) [5 marks] Find the coefficients B_k, C_k of the real form of the Fourier series of $x(t)$:

$$x(t) = a_0 + 2\sum_{k=1}^{+\infty}[B_k \cos(k\omega_0 t) - C_k \sin(k\omega_0 t)]$$

Answer:

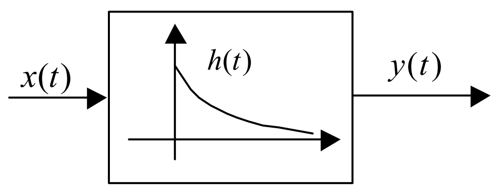
It is easy to show that

$B_k = \mathrm{Re}\{a_k\}, C_k = \mathrm{Im}\{a_k\}, k \geq 1$, thus

$$B_k = \frac{25}{2(k\pi)^2}\left(1-\cos(k\frac{4\pi}{5})\right) + \frac{2}{5}\cos k\frac{4\pi}{5}$$

$$C_k = -\frac{2}{5}\sin(k\frac{4\pi}{5})$$

(c) [10 marks] Suppose that $x(t)$ is the input to an LTI system with impulse response $h(t) = e^{-t}u(t)$ as shown below:



Compute the total average power in the fundamental component of $y(t) \overset{\mathcal{FS}}{\leftrightarrow} d_k$.

Answer:

First we compute the frequency response of the system

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt = \int_{0}^{+\infty} e^{-t}e^{-j\omega t}dt = \frac{1}{j\omega+1},$$

then we obtain the Fourier series coefficients of the output:

$$d_k = H(j\frac{2\pi k}{25})a_k = \frac{1}{j\frac{2\pi k}{25}+1}\left[\frac{25}{2(k\pi)^2}\left(1-\cos k\frac{4\pi}{5}\right) + \frac{2}{5}e^{-jk\frac{4\pi}{5}}\right], \ k \neq 0.$$

Finally, the power in the fundamental component is given by

$$P_1 = |d_{-1}|^2 + |d_1|^2 = 2|d_1|^2$$

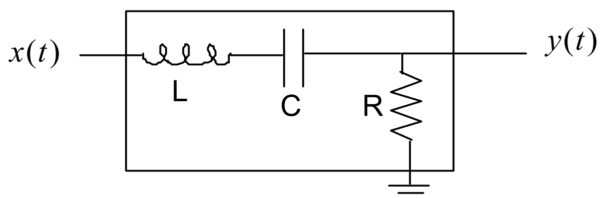
$$= \left| \frac{2}{j\frac{2\pi}{25}+1} \left[\frac{25}{2\pi^2}\left(1-\cos\frac{4\pi}{5}\right) + \frac{2}{5}e^{-j\frac{4\pi}{5}} \right] \right|^2$$

$$= \frac{4}{(\frac{2\pi}{25})^2+1} \left[\left[\frac{25}{2\pi^2}\left(1-\cos\frac{4\pi}{5}\right) + \frac{2}{5}\cos\frac{4\pi}{5} \right]^2 + \frac{4}{25}(\sin\frac{4\pi}{5})^2 \right]$$

$$= \frac{4}{(\frac{2\pi}{25})^2+1} \left[\left(\frac{25}{2\pi^2}\right)^2 \left(1-\cos\frac{4\pi}{5}\right)^2 + \frac{5}{\pi^2}\cos\frac{4\pi}{5}\left(1-\cos\frac{4\pi}{5}\right) + \frac{4}{25} \right]$$

Problem 4 (20 marks)

Consider the second-order RLC filter depicted below. The input voltage is $x(t)$ and the output voltage is $y(t)$.



The differential equation relating the input and output voltages of this RLC filter is

$$LC\frac{d^2y(t)}{dt^2} + RC\frac{dy(t)}{dt} + y(t) = RC\frac{dx(t)}{dt}. \quad (0.1)$$

(a) [10 marks] Find the frequency response $H(j\omega) = Y(j\omega)/X(j\omega)$ of the filter. Give its damping ratio ς and undamped natural frequency ω_n when expressed as follows.

$$H(j\omega) = \frac{A\omega_n^2}{(j\omega)^2 + 2\varsigma\omega_n(j\omega) + \omega_n^2}.$$

Answer:

The frequency response of the filter is

$$H(j\omega) := \frac{Y(s)}{X(s)} = \frac{RCj\omega}{LC(j\omega)^2 + RC(j\omega) + 1}$$

$$= \frac{\frac{R}{L}j\omega}{(j\omega)^2 + \frac{R}{L}(j\omega) + \frac{1}{LC}} = \frac{\frac{R}{L}j\omega}{(j\omega)^2 + 2\varsigma\omega_n(j\omega) + \omega_n^2}$$

Thus, $\omega_n = \dfrac{1}{\sqrt{LC}}$ and $\varsigma = \dfrac{R}{2\frac{1}{\sqrt{LC}}L} = \dfrac{R\sqrt{C}}{2\sqrt{L}}$.

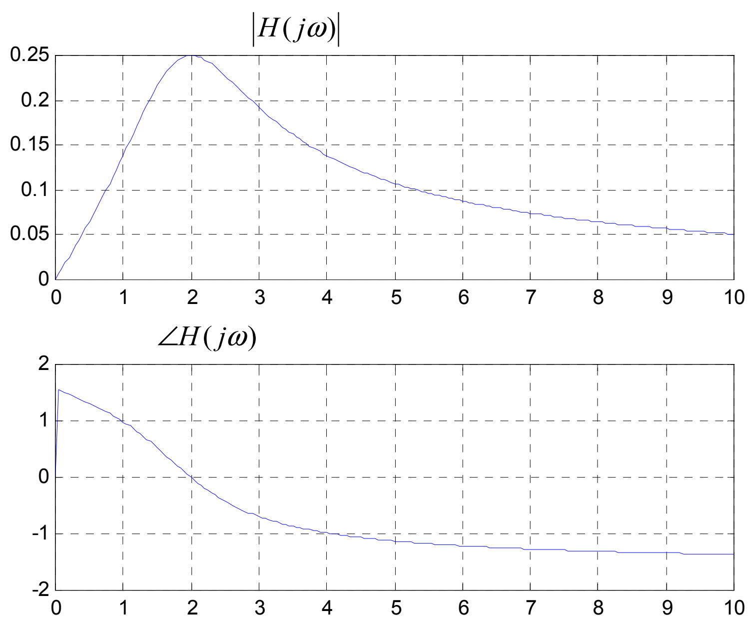
(b) [15 marks] Let $R = 10\,\Omega$, $L = 5\mathrm{H}$, and $C = 0.05\,\mathrm{F}$. Give the numerical values of the damping ratio ς and the undamped natural frequency ω_n. Compute and sketch the magnitude of the frequency response $|H(j\omega)|$ using linear scales and for frequencies ranging from -10 rd/s to 10 rd/s. What type of filter is it (lowpass, highpass or bandpass)?

Answer:

$$\varsigma = \frac{10\sqrt{0.05}}{2\sqrt{5}} = \frac{1}{2}, \quad \omega_n = \frac{1}{\sqrt{5(0.05)}} = 2$$

The frequency response of the filter is:

$$\begin{aligned} |H(j\omega)| &= \left| \frac{0.5 j\omega}{(j\omega)^2 + 2(j\omega) + 4} \right| \\ &= \frac{0.5|\omega|}{\sqrt{(4-\omega^2)^2 + 4\omega^2}} \end{aligned}$$



This is essentially a *bandpass* filter.

(c) Calculate and sketch the response of the circuit to a 10-second 1-Volt pulse in input voltage, i.e., $x(t) = u(t) - u(t-10)V$.

Answer:

The frequency response of the filter is:

$$H(j\omega) = \frac{0.5 j\omega}{(j\omega)^2 + 2(j\omega) + 4}$$
$$= \frac{0.5 j\omega}{(j\omega + 1 - j\sqrt{3})(j\omega + 1 + j\sqrt{3})}$$

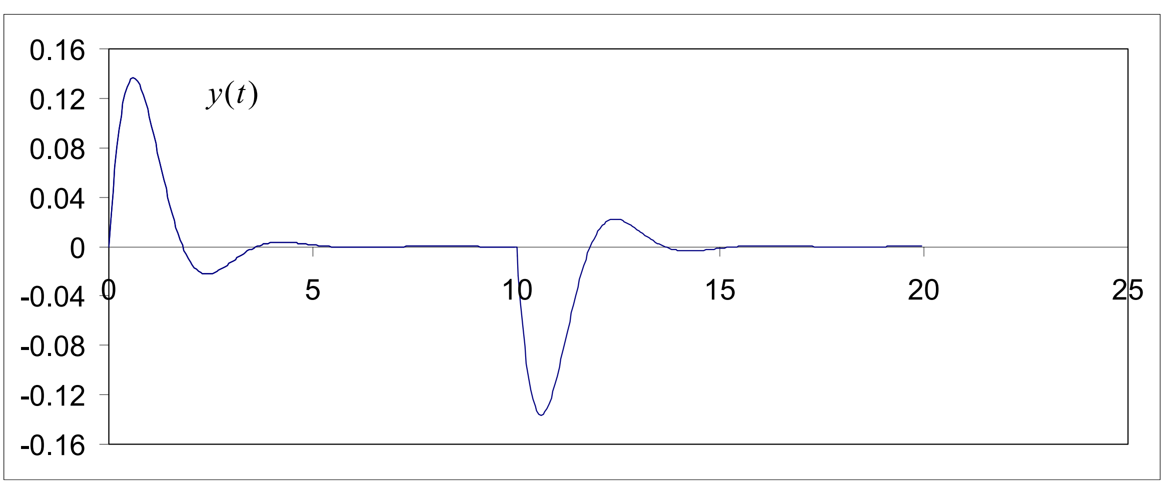
The input voltage has the following Fourier transform:

Hence, the Fourier transform of the output voltage is given by:

$$H(j\omega) = \frac{0.5 j\omega}{(j\omega + 1 - j\sqrt{3})(j\omega + 1 + j\sqrt{3})}\left(\frac{1 - e^{-j\omega 10}}{j\omega}\right) = \frac{0.5(1 - e^{-j\omega 10})}{(j\omega + 1 - j\sqrt{3})(j\omega + 1 + j\sqrt{3})}$$
$$= \left(\frac{-j\frac{1}{4\sqrt{3}}}{j\omega + 1 - j\sqrt{3}} + \frac{j\frac{1}{4\sqrt{3}}}{j\omega + 1 + j\sqrt{3}}\right)(1 - e^{-j\omega 10})$$

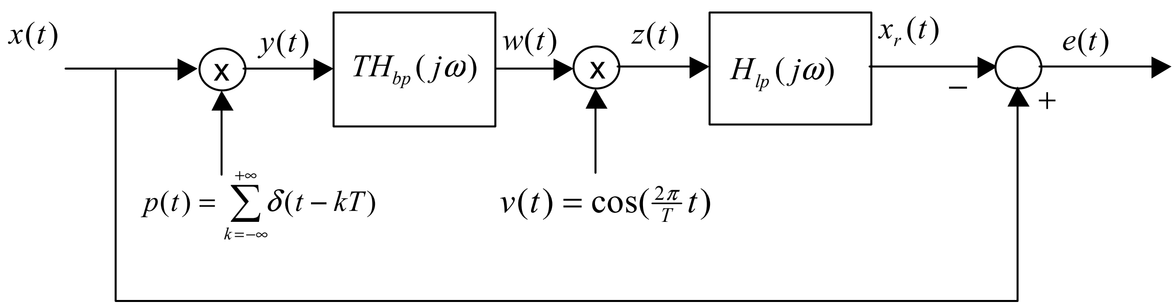
and taking the inverse transform, we get

$$y(t) = \left[-j\tfrac{1}{4\sqrt{3}} e^{(-1+j\sqrt{3})t} + j\tfrac{1}{4\sqrt{3}} e^{(-1-j\sqrt{3})t}\right]u(t) - \left[-j\tfrac{1}{4\sqrt{3}} e^{(-1+j\sqrt{3})(t-10)} + j\tfrac{1}{4\sqrt{3}} e^{(-1-j\sqrt{3})(t-10)}\right]u(t-10)$$
$$= \frac{1}{4\sqrt{3}} e^{-t}\left[e^{j(\sqrt{3}t - \frac{\pi}{2})} + e^{-j(\sqrt{3}t - \frac{\pi}{2})}\right]u(t) - \frac{1}{4\sqrt{3}} e^{-(t-10)}\left[e^{j(\sqrt{3}(t-10) - \frac{\pi}{2})} + e^{-j(\sqrt{3}(t-10) - \frac{\pi}{2})}\right]u(t-10)$$
$$= \frac{1}{2\sqrt{3}} e^{-t}\cos(\sqrt{3}t - \frac{\pi}{2})u(t) - \frac{1}{2\sqrt{3}} e^{-(t-10)}\cos(\sqrt{3}(t-10) - \frac{\pi}{2})u(t-10) \;\; V$$

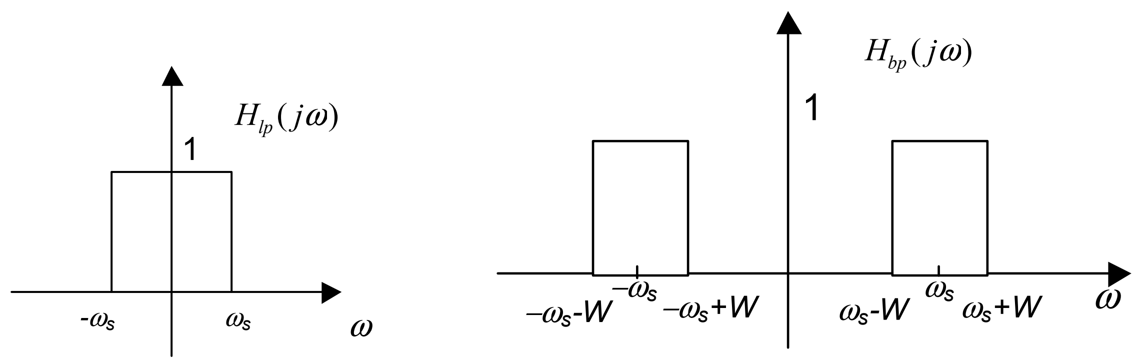


Problem 5 (15 marks)

Consider the following sampling system where the sampling frequency is $\omega_s = \dfrac{2\pi}{T}$.



The input signal is $x(t) = \dfrac{W^2}{2\pi}\operatorname{sinc}^2\left(\dfrac{W}{2\pi}t\right)$ and the spectra of the ideal bandpass filter and the ideal lowpass filters are shown below.



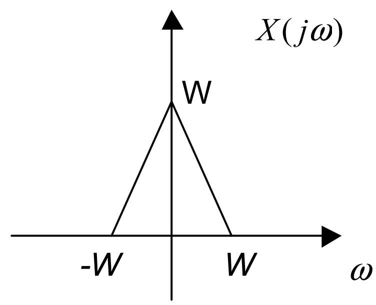
(a) [5 marks] Compute and sketch $X(j\omega)$, the Fourier transform of the input signal. For what range of sampling frequencies $\omega_s = \dfrac{2\pi}{T}$ is the sampling theorem satisfied for the first sampler?

Answer:

$$x(t) = \frac{W^2}{2\pi}\operatorname{sinc}^2\left(\frac{W}{2\pi}t\right) = 2\pi\frac{W^2}{(2\pi)^2}\operatorname{sinc}^2\left(\frac{W}{2\pi}t\right) \overset{\mathcal{FT}}{\leftrightarrow} R(j\omega) * R(j\omega)$$

$$\text{where } R(j\omega) := \begin{cases} 1, & |\omega| \le \dfrac{W}{2} \\ 0, & |\omega| > \dfrac{W}{2} \end{cases}$$

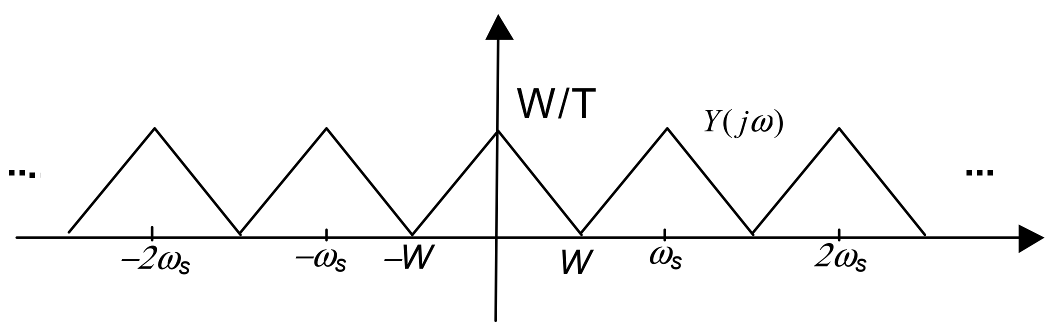
Hence, in the frequency domain $X(j\omega)$ is the convolution of the rectangular window $R(j\omega)$ with itself, yielding the triangular spectrum of bandwidth W shown below.

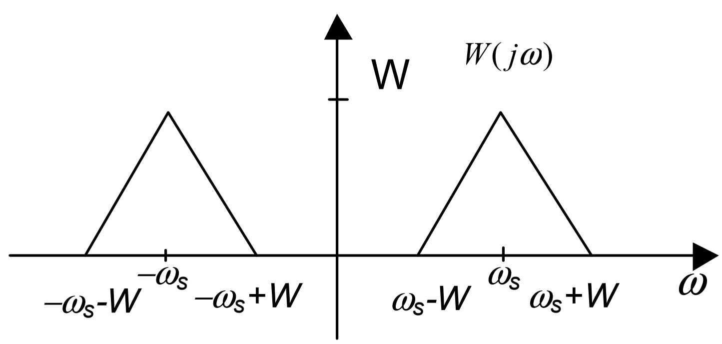


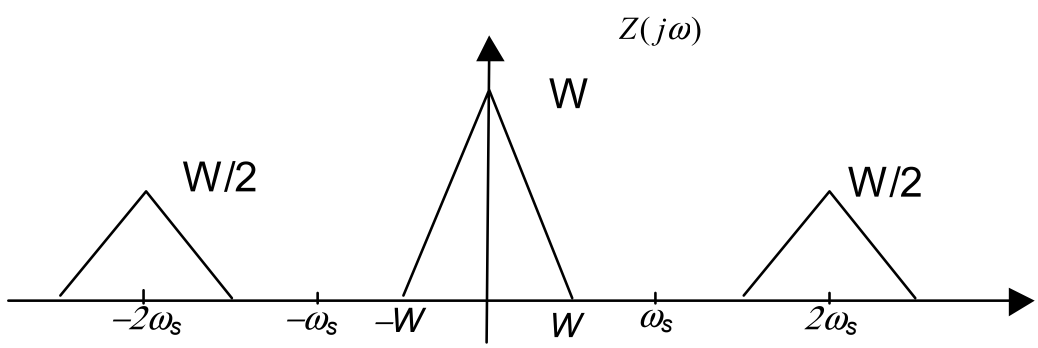
The sampling theorem is satisfied for $\omega_s > 2W$ for the sampler.

(b) [5 marks] Assume that the sampling theorem is satisfied for the slowest sampling frequency ω_s. Sketch the spectra $Y(j\omega)$, $W(j\omega)$ and $Z(j\omega)$.

Answer:

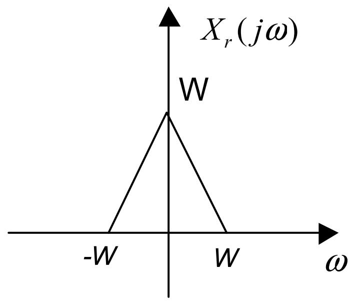




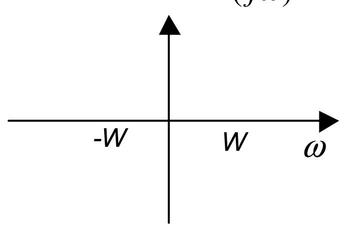


(c) [5 marks] Sketch the Fourier transforms $X_r(j\omega)$, $E(j\omega)$. Compute the total energy in the error signal $e(t) := x(t) - x_r(t)$.

Answer:



$E(j\omega) = X(j\omega) - X_r(j\omega) = 0$



$$E_e = \int_{-\infty}^{+\infty} |e(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |E(j\omega)|^2 \, d\omega = 0 .$$