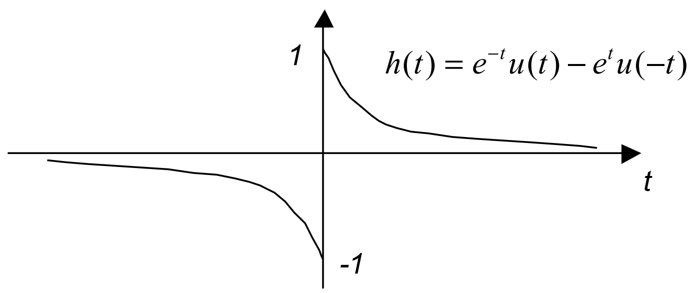
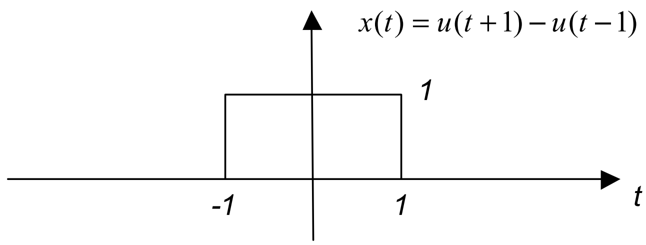
# Sample Midterm Test 1 (mt1s04)
## Covering Chapters 1-3 of *Fundamentals of Signals & Systems*

### *Problem 1 (25 marks)*

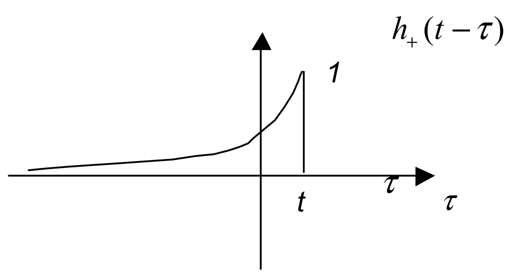
(a) [20 marks] Compute the output $y(t)$ of the continuous-time LTI system with impulse response $h(t)$ for an input signal $x(t)$ as depicted below.




*Answer:*
Let's time-reverse and shift the impulse response after splitting it up into two parts $h_+(t) := e^{-t}u(t), h_-(t) := -e^{t}u(-t)$.

For $h_+(t)$, the intervals of time of interest are:




$t < -1$ : no overlap, so
$y_+(t) = 0$.

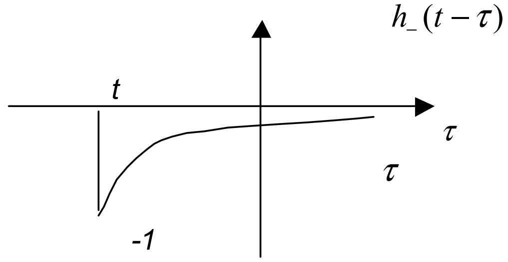
$-1 \le t < 1$ : overlap over $-1 \le \tau < t$, so

$$y_+(t) = \int_{-1}^{t} x(\tau)h_+(t-\tau)d\tau = \int_{-1}^{t} e^{\tau-t}d\tau = \left[e^{\tau-t}\right]_{-1}^{t}.$$
$$= 1 - e^{-t-1}$$

$1 \le t$ : overlap over $-1 \le \tau \le 1$, so

$$y_+(t) = \int_{-1}^{1} x(\tau)h_+(t-\tau)d\tau = \int_{-1}^{1} e^{\tau-t}d\tau = \left[e^{\tau-t}\right]_{-1}^{1}.$$
$$= e^{-t+1} - e^{-t-1} = (e^{1} - e^{-1})e^{-t}$$

For $h_-(t)$, the intervals of time of interest are:




$t \ge 1$: no overlap, so
$y_-(t) = 0$.

$-1 \le t < 1$: overlap over $t \le \tau \le 1$, so

$$y_-(t) = \int_{t}^{1} x(\tau)h_-(t-\tau)d\tau = \int_{t}^{1} -e^{t-\tau}d\tau = \left[e^{t-\tau}\right]_{t}^{1}.$$
$$= e^{t-1} - 1$$

$t < -1$: overlap over $-1 \le \tau \le 1$, so

$$y_-(t) = \int_{-1}^{1} x(\tau)h_-(t-\tau)d\tau = \int_{-1}^{1} -e^{t-\tau}d\tau = \left[e^{t-\tau}\right]_{-1}^{1}.$$
$$= e^{t-1} - e^{t+1} = (e^{-1} - e^{1})e^{t}$$

Finally, piecing the two responses on all three intervals together, we get:

$$y(t) = \begin{cases} (e^{-1} - e^{1})e^{t}, & t < -1 \\ e^{t-1} - e^{-t-1} & -1 \le t < 1 \\ \left(e^{1} - e^{-1}\right)e^{-t}, & t \ge 1 \end{cases}$$

(b) [5 marks] Is the LTI system in (a) stable? Is it causal? Justify your answers.

*Answer:*
Yes it is BIBO stable, because the impulse response is absolutely integrable, as shown below.

$$\int_{-\infty}^{+\infty} |h(t)|\,dt = \int_{-\infty}^{0} e^{t}\,dt + \int_{0}^{+\infty} e^{-t}\,dt$$
$$= \left[e^{t}\right]_{-\infty}^{0} + \left[-e^{-t}\right]_{0}^{+\infty} = 1 + 1 = 2 < +\infty$$

The system is noncausal because $h(t) \neq 0,\ t < 0$.

## *Problem 2 (25 marks)*

Determine if the system $S$ described by $y(t) = \int_{-\infty}^{+\infty} e^{-\tau} u(-\tau) x(t-\tau)\,d\tau$ is

(a) [5 marks] Time-invariant
(b) [5 marks] Linear
(c) [5 marks] Memoryless
(d) [5 marks] Causal
(e) [5 marks] BIBO stable

Justify your answers.

*Answer:*
Note that the system is an LTI system described by the convolution equation.
(a) The system $S$ is time-invariant since

$$y_1(t) := Sx(t-T) = y(t) = \int_{-\infty}^{+\infty} e^{-\tau} u(-\tau) x(t-T-\tau)\,d\tau = y(t-T)$$

(b) The system $S$ is linear since it has the superposition property:

For $x_1(t), x_2(t)$, let $y_1(t) = \int_{-\infty}^{+\infty} e^{-\tau} u(-\tau) x_1(t-\tau)\,d\tau,\ \ y_2(t) = \int_{-\infty}^{+\infty} e^{-\tau} u(-\tau) x_2(t-\tau)\,d\tau$

Define $x(t) = ax_1(t) + bx_2(t)$.

Then $y(t) = \int_{-\infty}^{+\infty} e^{-\tau} u(-\tau)[ax_1(t-\tau) + bx_2(t-\tau)]\,d\tau$

$$= a\int_{-\infty}^{+\infty} e^{-\tau} u(-\tau) x_1(t-\tau)\,d\tau + b\int_{-\infty}^{+\infty} e^{-\tau} u(-\tau) x_2(t-\tau)\,d\tau = ay_1(t) + by_2(t)$$

(c) The system's input-output relationship can be written as $y(t) = \int_{-\infty}^{0} e^{-\tau} x(t-\tau)\,d\tau$. The system has memory because the computation of $y(t)$ uses all future values of the input.

(d) The system is <u>noncausal</u> as the output $y(t)=\int_{-\infty}^{0} e^{-\tau}x(t-\tau)d\tau$ is a function of future and current values of the input.

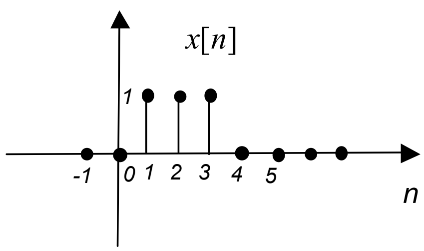
(e) The system is <u>unstable.</u> To show this consider the input $x(t)=1$, then

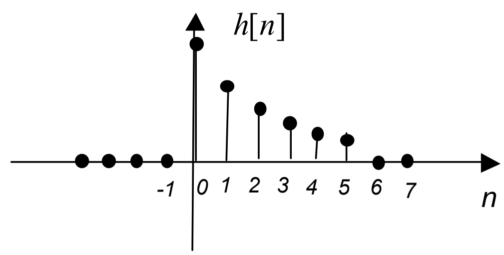
$y(t)=\int_{-\infty}^{0} e^{-\tau}d\tau=+\infty$ is unbounded.

## *Problem 3 (25 marks)*

Compute the response of an LTI system described by its impulse response

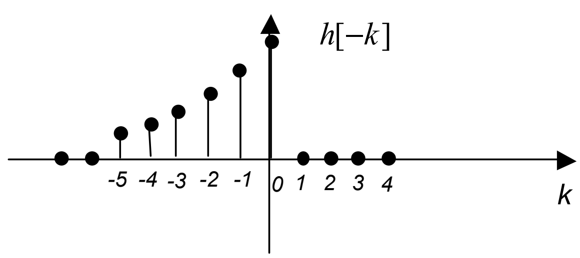
$h[n]=\begin{cases}(0.8)^n, & 0\le n\le 5\\ 0, & \text{otherwise}\end{cases}$ to the input signal shown below.




*Answer:*

We break down the problem into 5 intervals for n.




<u>For</u> $n<1$: $x[k]h[n-k]$ is zero for all k, hence $y[n]=0$.

<u>For</u> $1\le n\le 3$: Then $g[k]=x[k]h[n-k]\ne 0$ for $k=1,\ldots,n$. We get

$$y[n]=\sum_{k=1}^{n} g[k]=\sum_{k=1}^{n}(0.8)^{n-k}=(0.8)^n\sum_{m=0}^{n-1}(0.8)^{-(m+1)}=(0.8)^{n-1}\sum_{m=0}^{n-1}(0.8)^{-m}$$

$$=(0.8)^{n-1}\left(\frac{1-(0.8^{-1})^n}{1-(0.8)^{-1}}\right)=\frac{(0.8)^n-1}{-0.2}=5-5(0.8)^n$$

<u>For</u> $4\le n\le 6$: Then $g[k]=x[k]h[n-k]\ne 0$ for $k=1,\ldots,3$. We get

$$y[n] = \sum_{k=1}^{3} g[k] = \sum_{k=1}^{3} (0.8)^{n-k} = (0.8)^n \sum_{m=0}^{2} (0.8)^{-(m+1)} = (0.8)^{n-1} \sum_{m=0}^{2} (0.8)^{-m}$$

$$= (0.8)^{n-1} \left( \frac{1-(0.8^{-1})^3}{1-(0.8)^{-1}} \right) = \frac{(0.8)^n - (0.8)^{n-3}}{-0.2} = 5(0.8)^{n-3} - 5(0.8)^n = 4.7656(0.8)^n$$

<u>For</u> $7 \le n \le 8$: Then $g[k] = x[k]h[n-k] \neq 0$ for $n-5 \le k \le 3$. We have

$$y[n] = \sum_{k=n-5}^{3} g[k] = \sum_{k=n-5}^{3} (0.8)^{n-k} = \sum_{m=0}^{8-n} (0.8)^{n-(m+n-5)} = (0.8)^5 \sum_{m=0}^{8-n} (0.8)^{-m}$$

$$= (0.8)^5 \left( \frac{1-(0.8^{-1})^{9-n}}{1-(0.8)^{-1}} \right) = \frac{(0.8)^6 - (0.8)^{n-3}}{-0.2} = 5(0.8)^{n-3} - 5(0.8)^6$$

<u>Finally for</u> $n \ge 9$ the two signals $x[k], h[n-k]$ don't overlap, so $y[n] = 0$.

In summary, the output signal of the LTI system is

$$y[n] = \begin{cases} 0, & n \le 0 \\ 5 - 5(0.8)^n, & 1 \le n \le 3 \\ \left[5(0.8)^{-3} - 5\right](0.8)^n, & 4 \le n \le 6 \\ 5(0.8)^{n-3} - 5(0.8)^6, & 7 \le n \le 8 \\ 0, & n \ge 9 \end{cases}$$

## *Problem 4 (25 marks)*

Compute and sketch the impulse response $h(t)$ of the following causal LTI first-order differential system initially at rest:

$$S: \qquad 2\frac{dy(t)}{dt} + 4y(t) = 3\frac{d}{dt}x(t) + 2x(t)$$

*Answer:*

Step 1: Set up the problem to calculate the impulse response

$$2\frac{dh_a(t)}{dt} + 4h_a(t) = \delta(t)$$

Step 2: Find the initial condition of the corresponding homogeneous equation at $t = 0^+$ by integrating the above differential equation from $t = 0^-$ to $t = 0^+$. Note that the impulse will be in the term $\frac{dh_a(t)}{dt}$, so $h_a(t)$ will have a finite jump at most. Thus we have

$$\int_{0^-}^{0^+} \frac{dh_a(\tau)}{d\tau} d\tau = h_a(0^+) = 0.5 ,$$

hence $h_a(0^+) = 0.5$ is our initial condition for the homogeneous equation for $t > 0$

$$2\frac{dh_a(t)}{dt} + 4h_a(t) = 0 .$$

Step 3: The characteristic polynomial is $p(s) = 2s + 4$ and it has one zero at $s = -2$, which means that the homogeneous response has the form $h_a(t) = Ae^{-2t}$ for $t > 0$. The initial condition allows us to determine the constant $A$:

$$h_a(0^+) = A = 0.5 ,$$

so that
$$h_a(t) = 0.5e^{-2t} .$$

Step 4:

$$\begin{aligned} h(t) &= 3\frac{dh_a(t)}{dt} + 2h_a(t) \\ &= 3\frac{d}{dt}\left(0.5e^{-2t}u(t)\right) + e^{-2t}u(t) \\ &= -3e^{-2t}u(t) + 1.5\delta(t) + e^{-2t}u(t) \\ &= -2e^{-2t}u(t) + 1.5\delta(t) \end{aligned}$$