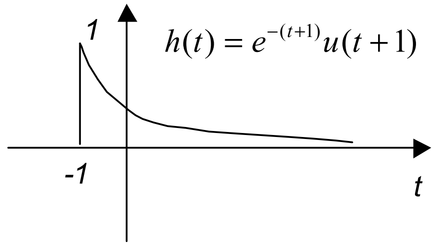
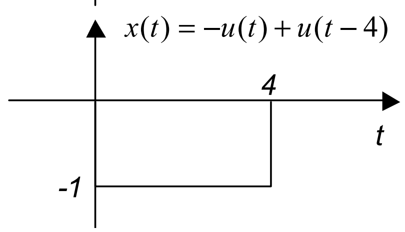
Sample Midterm Test 1 (mt1s03)
Covering Chapters 1-3 of *Fundamentals of Signals & Systems*

Problem 1 (25 marks)

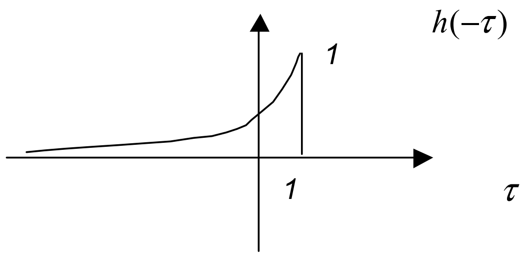
(a) [20 marks] Compute the output $y(t)$ of the continuous-time LTI system with impulse response $h(t)$ for an input signal $x(t)$ as depicted below.





Answer:

SOLUTION 1: Let's time-reverse and shift the impulse response. The intervals of time of interest are:



$t < -1$: no overlap, so
$y(t) = 0$.

$-1 \le t < 3$: overlap over $0 \le \tau < t+1$, so

$$y(t) = \int_0^{t+1} x(\tau)h(t-\tau)d\tau = -\int_0^{t+1} e^{-t+\tau-1}d\tau = -e^{-1-t}\left[e^{\tau}\right]_0^{t+1}.$$

$$= -e^{-1-t}\left[e^{t+1} - 1\right] = e^{-t-1} - 1$$

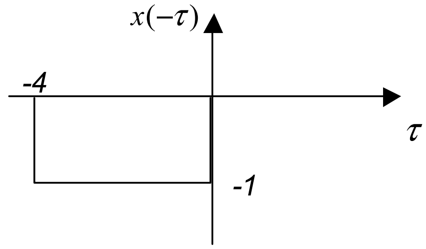
$t \ge 3$: overlap over $0 \le \tau < 4$, so

$$y(t) = \int_0^4 x(\tau)h(t-\tau)d\tau = -\int_0^4 e^{-t-1+\tau}d\tau = -e^{-t-1}\int_0^4 e^{\tau}d\tau = -e^{-t-1}\left[e^{\tau}\right]_0^4.$$
$$= -e^{-t-1}\left[e^4 - 1\right] = -e^{-(t-3)} + e^{-(t+1)} = \left(e^{-1} - e^3\right)e^{-t}$$

Finally, piecing all three intervals together, we get:

$$y(t) = \begin{cases} 0, & t < -1 \\ e^{-t-1} - 1, & -1 \le t < 3 \\ \left(e^{-1} - e^3\right)e^{-t}, & t \ge 3 \end{cases}$$

SOLUTION 2: Time reversing and shifting $x(t)$.



$t < -1$: no overlap, so
$y(t) = 0$.

$-1 \le t < 3$: overlap over $-1 \le \tau < t$, so
$$y(t) = \int_{-1}^{t} h(\tau)x(t-\tau)d\tau = -\int_{-1}^{t} e^{-\tau-1}d\tau = e^{-1}\left[e^{-\tau}\right]_{-1}^{t}.$$
$$= e^{-1}\left[e^{-t} - e\right] = e^{-t-1} - 1$$

$t \ge 3$: overlap over $t-4 \le \tau < t$, so
$$y(t) = \int_{t-4}^{t} h(\tau)x(t-\tau)d\tau = -\int_{t-4}^{t} e^{-\tau-1}d\tau = e^{-1}\left[e^{-\tau}\right]_{t-4}^{t}.$$
$$= e^{-t-1} - e^{-t+3}$$

Finally, piecing all three intervals together, we get:

$$y(t) = \begin{cases} 0, & t < -1 \\ e^{-t-1} - 1, & -1 \le t < 3 \\ \left(e^{-1} - e^3\right)e^{-t}, & t \ge 3 \end{cases}$$

(b) [5 marks] Is the LTI system S_0 in (a) stable? Is it causal? Justify your answers.

Answer:
Yes it is BIBO stable, because the impulse response is absolutely integrable, as shown below.

$$\int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} \left|e^{-t-1}u(t+1)\right| dt = \int_{-1}^{+\infty} e^{-t-1} dt$$
$$= -e^{-1}\left[e^{-t}\right]_{-1}^{+\infty} = -e^{-1}\left[-e\right] = 1 < +\infty$$

The system is noncausal because $h(t) \neq 0, -1 \leq t < 0$.

Problem 2 (20 marks)

Determine if the system S described by $y(t) = \dfrac{x(t)}{1 + x(t-1)}$ is

(a) [5 marks] Time-invariant
(b) [5 marks] Linear
(c) [5 marks] Memoryless
(d) [5 marks] Causal

Justify your answers.

Answer:
(a) The system S is time-invariant since

$$y_1(t) := Sx(t-T) = \frac{x(t-T)}{1 + x(t-1-T)} = y(t-T)$$

(b) The system S is nonlinear since it doesn't have the superposition property:

$$\text{For } x_1(t), x_2(t), \quad \text{let } y_1(t) = \frac{x_1(t)}{1 + x_1(t-1)}, \; y_2(t) = \frac{x_2(t)}{1 + x_2(t-1)}$$

$$\text{Define } x(t) = ax_1(t) + bx_2(t).$$

$$\text{Then } y(t) = \frac{ax_1(t) + bx_2(t)}{1 + ax_1(t-1) + bx_2(t-1)} \neq \frac{ax_1(t)}{1 + x_1(t-1)} + \frac{bx_2(t)}{1 + x_2(t-1)} = ay_1(t) + by_2(t)$$

(c) The system has memory since for time t it uses the past value of the input $x(t-1)$.

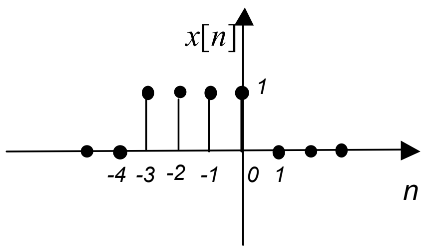
(d) The system is causal as the output is a function of past and current values of the input only.

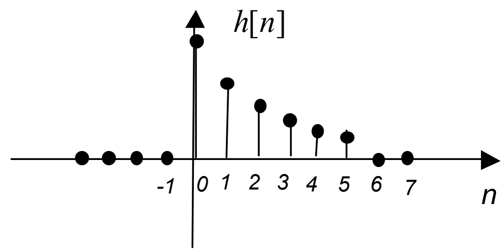
Problem 3 (30 marks)

Compute the response of an LTI system described by its impulse response

$$h[n] = \begin{cases} (0.8)^n, & 0 \le n \le 5 \\ 0, & \text{otherwise} \end{cases}$$

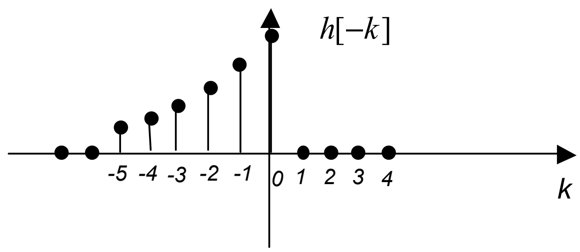
to the input signal shown below.





Answer:

We break down the problem into 5 intervals for n.



<u>For</u> $n < -3$: $h[n-k]$ is zero for k>0, hence $y[n] = 0$.

<u>For</u> $-3 \le n \le 0$: Then $g[k] = x[k]h[n-k] \ne 0$ for $k = -3, \ldots, n$. We get

$$y[n] = \sum_{k=-3}^{n} g[k] = \sum_{k=-3}^{n} (0.8)^{n-k} = (0.8)^n \sum_{m=0}^{n+3} (0.8)^{-(m-3)} = (0.8)^{n+3} \sum_{m=0}^{n+3} (0.8)^{-m}$$

$$= (0.8)^{n+3} \left(\frac{1-(0.8^{-1})^{n+4}}{1-(0.8)^{-1}} \right)$$

$$= \frac{(0.8)^{n+4} - 1}{-0.2} = 5 - 5(0.8)^{n+4}$$

<u>For</u> $1 \le n \le 2$: Then $g[k] = x[k]h[n-k] \ne 0$ for $k = -3, \ldots, 0$. We get

$$y[n]=\sum_{k=-3}^{0} g[k]=\sum_{k=-3}^{0}(0.8)^{n-k} =(0.8)^{n}\sum_{m=0}^{3}(0.8)^{-(m-3)} =(0.8)^{n+3}\sum_{m=0}^{3}(0.8)^{-m}$$

$$=(0.8)^{n+3}\left(\frac{1-(0.8^{-1})^{4}}{1-(0.8)^{-1}}\right)$$

$$=\frac{(0.8)^{n+4}-(0.8)^{n}}{-0.2}=\left[5-5(0.8)^{4}\right](0.8)^{n}$$

<u>For</u> $3\le n\le 5$: Then $g[k]=x[k]h[n-k]\neq 0$ for $n-5\le k\le 0$. We have

$$y[n]=\sum_{k=n-5}^{0} g[k]=\sum_{k=n-5}^{0}(0.8)^{n-k} =\sum_{m=0}^{5-n}(0.8)^{n-(m+n-5)} =(0.8)^{5}\sum_{m=0}^{5-n}(0.8)^{-m}$$

$$=(0.8)^{5}\left(\frac{1-(0.8^{-1})^{6-n}}{1-(0.8)^{-1}}\right)$$

$$=\frac{(0.8)^{6}-(0.8)^{n}}{-0.2}=5(0.8)^{n}-5(0.8)^{6}$$

<u>Finally for</u> $n\ge 6$ the two signals $x[k],h[n-k]$ don't overlap, so $y[n]=0$.

In summary, the output signal of the LTI system is

$$y[n]=\begin{cases}0, & n\le -4\\ 5-5(0.8)^{n+4}, & -3\le n\le 0\\ \left[5-5(0.8)^{4}\right](0.8)^{n}, & 1\le n\le 2\\ 5(0.8)^{n}-5(0.8)^{6}, & 3\le n\le 5\\ 0, & n\ge 6\end{cases}$$

Problem 4 (25 marks)

Compute the impulse response $h[n]$ of the following causal LTI second-order difference system initially at rest:

$$y[n]+\sqrt{3}y[n-1]+y[n-2]=x[n-1]+x[n-2]$$

Simplify your expression of $h[n]$ to obtain a real function of time.

Answer:

$$y[n]+\sqrt{3}y[n-1]+y[n-2]=x[n-1]-x[n-2]$$

Write:

$$y[n]+\sqrt{3}y[n-1]+y[n-2]=\delta[n].$$

Initial conditions for the homogeneous equation for $n > 0$ are $y[0] = 1$, $y[-1] = 0$.

characteristic polynomial and zeros:

$$p(z) = z^2 + \sqrt{3}z + 1 = (z - e^{j\frac{5\pi}{6}})(z - e^{-j\frac{5\pi}{6}})$$

The zeros are $z_1 = e^{j\frac{5\pi}{6}}$, $z_2 = e^{-j\frac{5\pi}{6}}$.

The homogeneous response for $n > 0$ is given by

$$h_a[n] = A(e^{j\frac{5\pi}{6}})^n + B(e^{-j\frac{5\pi}{6}})^n.$$

Use initial conditions to compute the coefficients *A* and *B*:

$$h_a[-1] = 0 = Ae^{-j\frac{5\pi}{6}} + Be^{j\frac{5\pi}{6}}$$

$$h_a[0] = 1 = A + B$$

We get $A = e^{j\frac{\pi}{3}}$, $B = e^{-j\frac{\pi}{3}}$,
the homogeneous response is

$$h_a[n] = \left[e^{j\frac{\pi}{3}}(e^{j\frac{5\pi}{6}})^n + e^{-j\frac{\pi}{3}}(e^{-j\frac{5\pi}{6}})^n \right] u[n]$$

$$= 2\,\mathrm{Re}\left\{ e^{j(\frac{5\pi}{6}n + \frac{\pi}{3})} \right\} u[n] = 2\cos(\frac{5\pi}{6}n + \frac{\pi}{3})u[n]$$

The impulse response is:

$$h[n] = h_a[n-1] + h_a[n-2]$$

$$= 2\cos(\frac{5\pi}{6}(n-1) + \frac{\pi}{3})u[n-1] + 2\cos(\frac{5\pi}{6}(n-2) + \frac{\pi}{3})u[n-2]$$