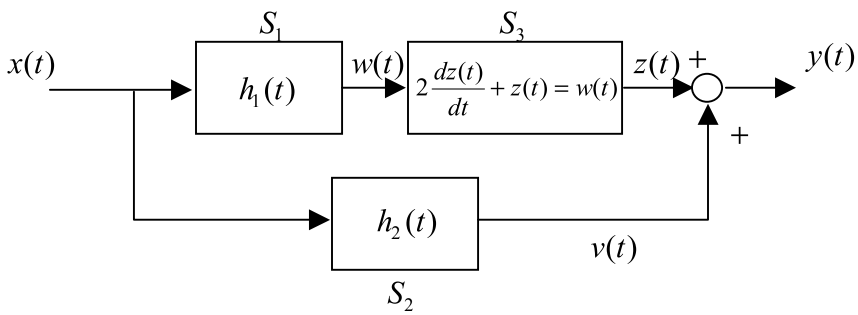
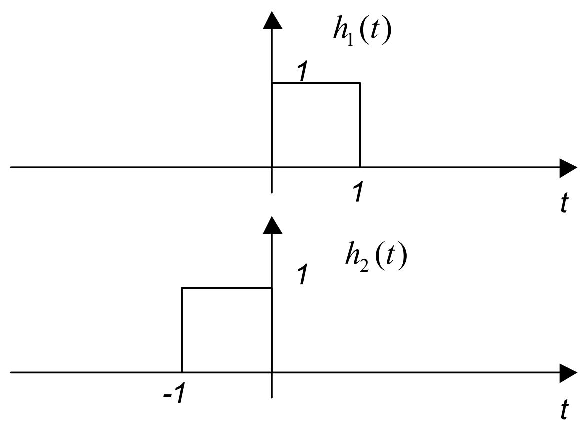
Sample Midterm Test 1 (mt1s02)
Covering Chapters 1-3 of *Fundamentals of Signals & Systems*

Problem 1 (30 marks)

Consider the LTI system initially at rest shown below:



Where the differential system S_3 is causal, and the impulse responses $h_1(t)$ and $h_2(t)$ are as shown below.



(a) [25 marks] Compute and sketch the overall impulse response $h(t)$ of the system S, i.e., from $x(t)$ to $y(t)$. Justify all calculations.

Answer:

First, we compute the impulse response $h_3(t)$ of S_3.

$$S_3: \qquad 2\frac{dz(t)}{dt}+z(t)=w(t).$$

Step 1: Set up the problem of calculating the impulse response of the differential equation.

$$2\frac{dh_3(t)}{dt}+h_3(t)=\delta(t)$$

Step 2: Find the initial condition of the homogeneous equation at $t=0^+$ by integrating from $t=0^-$ to $t=0^+$.

$$\int_{0^-}^{0^+} 2\frac{dh_3(\tau)}{dt}d\tau = 2h_3(0^+) = 1,$$

hence $h_3(0^+) = \frac{1}{2}$ is our initial condition for the homogeneous equation

$$2\frac{dh_3(t)}{dt} + h_3(t) = 0.$$

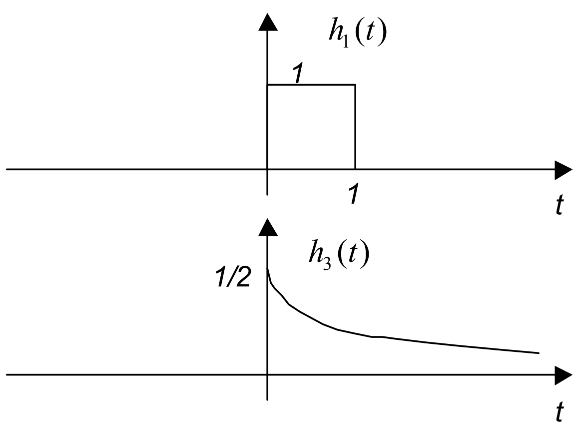
Step 3: The characteristic polynomial is $p(s) = 2s+1$ and it has one zero at $s = \frac{-1}{2}$, which means that the homogeneous response has the form $h_3(t) = Ae^{\frac{-t}{2}}$ for $t > 0$. The initial condition allows us to determine the constant A:

$$h_3(0^+) = A = \frac{1}{2},$$

so that

$$h_3(t) = \frac{1}{2}e^{\frac{-t}{2}}u(t).$$

Next, we compute the impulse response of the upper cascade of systems S_1, S_3, which is given by $h_1(t) * h_3(t)$:



SOLUTION 1 for $h_1(t) * h_3(t)$: Let's time-reverse and shift the impulse response $h_1(t)$. The intervals of interest are:

$t < 0$: no overlap so $h_1(t) * h_3(t) = 0$.

$0 \le t < 1$: overlap over $0 < \tau < t$:

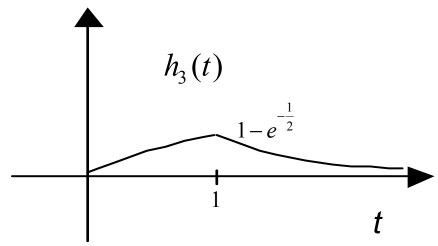
$$h_1(t) * h_3(t) = \int_0^t h_3(\tau)d\tau = \frac{1}{2}\int_0^t e^{-\frac{1}{2}\tau}d\tau = \left[-e^{-\frac{1}{2}\tau}\right]_0^t = \left(1 - e^{-\frac{1}{2}t}\right).$$

$t \geq 1$: overlap over $t-1 < \tau < t$:

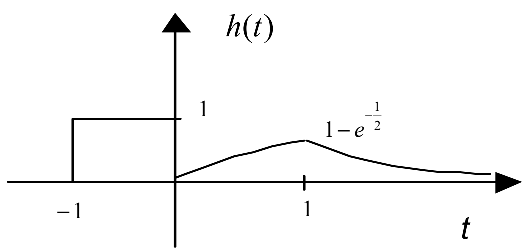
$$h_1(t) * h_3(t) = \int_{t-1}^{t} h_3(\tau)d\tau = \frac{1}{2}\int_{t-1}^{t} e^{-\frac{1}{2}\tau} d\tau = \left[-e^{-\frac{1}{2}\tau}\right]_{t-1}^{t} = \left(e^{-\frac{1}{2}(t-1)} - e^{-\frac{1}{2}t}\right) = \left(e^{\frac{1}{2}} - 1\right)e^{-\frac{1}{2}t}$$

Finally, piecing all three intervals together, we get:

$$h_1(t) * h_3(t) = \begin{cases} 0, & t < 0 \\ 1 - e^{-\frac{1}{2}t}, & 0 \leq t < 1 \\ \left(e^{\frac{1}{2}} - 1\right)e^{-\frac{1}{2}t}, & t \geq 1 \end{cases}$$



$$h(t) = h_1(t) * h_3(t) + h_2(t) = \begin{cases} 0, & t < -1 \\ 1, & -1 \leq t < 0 \\ 1 - e^{-\frac{1}{2}t}, & 0 \leq t < 1 \\ \left(e^{\frac{1}{2}} - 1\right)e^{-\frac{1}{2}t} & t \geq 1 \end{cases}$$



(b) [5 marks] Is the overall system S in (a) stable? Is it causal? Justify your answers.

Answer:

No, it is not causal as its impulse response $h(t) \neq 0$ at negative times.

Yes it is BIBO stable, because the impulse response is absolutely integrable, as shown below.

First the part of $h(t)$ between -1 and 1 has a finite area, so we only need to check for t>1:

$$\int_{1}^{+\infty} |h(t)|\,dt = (1-e^{-\frac{1}{2}})\int_{1}^{+\infty} \left|e^{-\frac{1}{2}t}\right| dt = (e^{\frac{1}{2}}-1)\int_{1}^{+\infty} e^{-\frac{1}{2}t}\,dt$$

$$= (e^{\frac{1}{2}}-1)\left[-2e^{-\frac{1}{2}t}\right]_{1}^{+\infty} = 2e^{-\frac{1}{2}}(e^{\frac{1}{2}}-1) = 2(1-e^{-\frac{1}{2}}) < +\infty$$

The system is noncausal because $h(t) \neq 0,\, t < 0$.

Problem 2 (25 marks)

Determine if the discrete-time system described by $y[n] = \sum_{k=-\infty}^{n} \frac{x[k]}{n-k}$ is

(a) [5 marks] Time-invariant
(b) [5 marks] Linear
(c) [5 marks] Stable
(d) [5 marks] Causal

Justify your answers.

Answer:

(a) It is time-invariant. Let $y_1[n] = Sx[n-N] = \sum_{k=-\infty}^{n} \frac{x[k-N]}{n-k}$. We have

$$y[n-N] = \sum_{k=-\infty}^{n-N} \frac{x[k]}{n-N-k} = \sum_{m=-\infty}^{n} \frac{x[m-N]}{n-m} = y_1[n].$$

(b) It is linear.

Principle of Superposition, let $y_1[n] = Sx_1[n] = \sum_{k=-\infty}^{n} \frac{x_1[k]}{n-k}$ and $y_1[n] = Sx_2[n] = \sum_{k=-\infty}^{n} \frac{x_2[k]}{n-k}$.

Then for $x[n] = ax_1[n] + bx_2[n]$, we have

$$y[n] = \sum_{k=-\infty}^{n} \frac{ax_1[k]+bx_2[k]}{n-k} = a\sum_{k=-\infty}^{n} \frac{x_1[k]}{n-k} + b\sum_{k=-\infty}^{n} \frac{x_2[k]}{n-k}$$
$$= ay_1[n] + by_2[n]$$

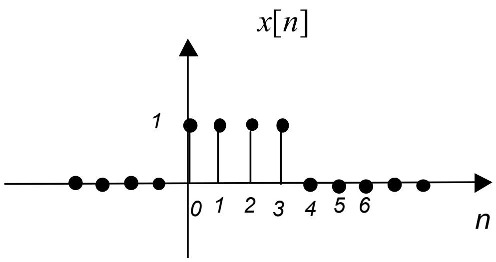
(c) It is unstable: for a given bound $|x[n]| < B$, the output can not be bounded for n negative going to $-\infty$. For example, $x_1[n] = 1$ is a bounded input leading to the output

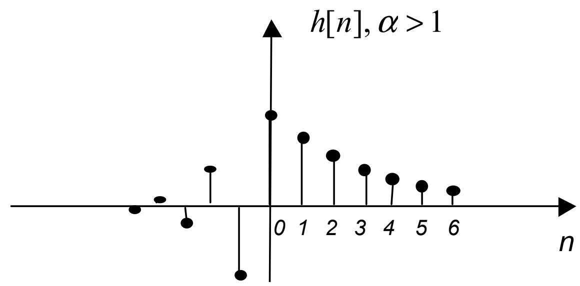
$$y[n] = \sum_{k=-\infty}^{n} \frac{1}{n-k} = \sum_{m=0}^{+\infty} \frac{1}{m} = +\infty \text{ which is unbounded.}$$

(d) It is causal: To compute $y[n]$, the system only needs past and current values of the input..

Problem 3 (20 marks)

Compute the response $y[n]$ of the discrete-time LTI system described by its impulse response $h[n]=(-2)^n u[-n-1]+(0.8)^n u[n]$ to the input signal $x[n]=u[n]-u[n-4]$. Sketch $x[n], h[n]$ and $y[n]$.





We break down the problem into 2 subproblems, each with 3 intervals for n.

Let $h_+[n]=(0.8)^n u[n]$, and $h_-[n]=(-2)^n u[-n-1]$, so that

$$y[n]=h_+[n]*x[n]+h_-[n]*x[n]=y_+[n]+y_-[n]$$

Starting with $y_+[n]=h_+[n]*x[n]$:

<u>For</u> $n<0$: $x[n-k]$ is zero for k>0, hence $g[k]=h[k]x[n-k]=0\ \forall k$ and $y[n]=0$.

<u>For</u> $0\le n\le 3$: Then $g[k]=h[k]x[n-k]\neq 0$ for $k=0,\ldots,n$. We get

$$y[n]=\sum_{k=0}^{n}h[k]=\sum_{k=0}^{n}0.8^k=\frac{1-(0.8)^{n+1}}{1-0.8}=5[1-(0.8)^{n+1}]$$

<u>For</u> $n>3$: Then $g[k]=h[k]x[n-k]\neq 0$ for $k=n-3,\ldots,n$. We get

$$y[n]=\sum_{k=n-3}^{n}0.8^k=\sum_{m=0}^{3}0.8^{m+n-3}=0.8^{n-3}\frac{1-0.8^4}{1-0.8}=5[0.8^{-3}-0.8]0.8^n$$

Now for $y_-[n]=h_-[n]*x[n]$:

<u>For</u> $n>2$: $x[n-k]$ is zero for k<0, hence $g[k]=h[k]x[n-k]=0\ \forall k$ and $y[n]=0$.

<u>For</u> $0\le n\le 2$: Then $g[k]=h[k]x[n-k]\neq 0$ for $k=n-3,\ldots,-1$. We get

$$y[n]=\sum_{k=n-3}^{-1}h[k]=\sum_{k=n-3}^{-1}(-2)^k=\sum_{m=0}^{2-n}(-2)^{m+n-3}=(-2)^{n-3}\frac{1-(-2)^{3-n}}{1-(-2)}=\frac{1}{3}[(-2)^{n-3}-1]$$

<u>For</u> $n\le -1$: Then $g[k]=h[k]x[n-k]\neq 0$ for $k=n-3,\ldots,n$. We get

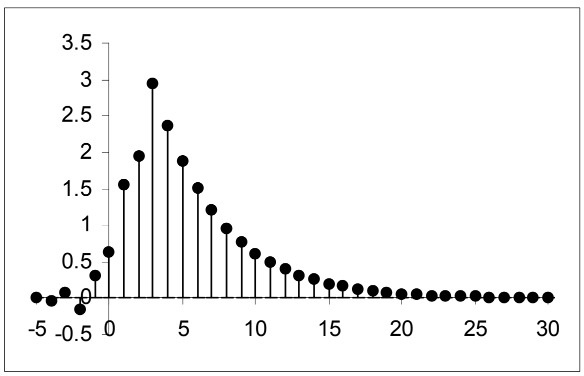
$$y[n] = \sum_{k=n-3}^{n} (-2)^k = \sum_{m=0}^{3} (-2)^{m+n-3} = (-2)^{n-3} \frac{1-(-2)^4}{1-(-2)}$$
$$= \frac{1}{3}[(-2)^{-3} - (-2)](-2)^n = -\frac{5}{8}(-2)^n$$

Combining the response, we find the output signal of the LTI system to be

$$y[n] = \begin{cases} -\frac{5}{8}(-2)^n, & n \le -1 \\ 5[1-(0.8)^{n+1}] + \frac{1}{3}[(-2)^{n-3} - 1], & 0 \le n \le 2 \\ 5[0.8^{-3} - 0.8]0.8^n, & n \ge 3 \end{cases}$$

Which looks like this:

$y[n]$



Problem 4 (25 marks)

(a) [20 marks] Compute the impulse response $h[n]$ of the following causal LTI second-order difference system initially at rest:

$$y[n] - 0.8\sqrt{2}y[n-1] = -0.64y[n-2] + x[n-1] - x[n-2]$$

Simplify your expression of $h[n]$ to obtain a real function of time.

Answer:

$$y[n] - 0.8\sqrt{2}y[n-1] + 0.64y[n-2] = x[n-1] - x[n-2]$$

Write:

$$y[n] - 0.8\sqrt{2}y[n-1] + 0.64y[n-2] = \delta[n].$$

Initial conditions for the homogeneous equation for $n > 0$ are $y[0] = 1$, $y[-1] = 0$.

characteristic polynomial and zeros:

$$p(z) = z^2 - 0.8\sqrt{2}z + 0.64 = (z - 0.8e^{j\frac{3\pi}{4}})(z - 0.8e^{-j\frac{3\pi}{4}})$$

The zeros are $z_1 = 0.8e^{j\frac{3\pi}{4}}$, $z_2 = 0.8e^{-j\frac{3\pi}{4}}$.

The homogeneous response for $n > 0$ is given by

$$h_a[n] = A(0.8e^{j\frac{3\pi}{4}})^n + B(0.8e^{-j\frac{3\pi}{4}})^n.$$

Use initial conditions to compute the coefficients A and B:

$$h_a[-1] = 0 = A(0.8e^{j\frac{3\pi}{4}})^{-1} + B(0.8e^{-j\frac{3\pi}{4}})^{-1} = \frac{5}{4}e^{-j\frac{3\pi}{4}}A + \frac{5}{4}e^{j\frac{3\pi}{4}}B$$

$$h_a[0] = 1 = A + B$$

From the first equation, we get $B = -e^{j\frac{6\pi}{4}}A = jA$, and from the second equation: $A = \frac{1}{1+j} = \frac{1}{2} - \frac{1}{2}j$. Thus $B = \frac{1}{2} + \frac{1}{2}j$, and

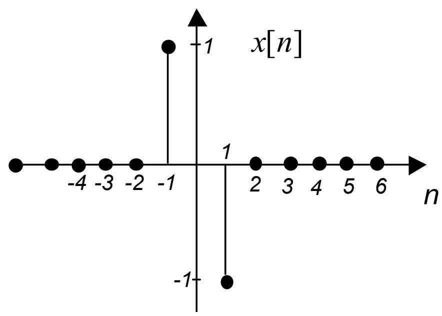
the homogeneous response is

$$\begin{aligned} h_a[n] &= \left[\left(\frac{1}{2} - \frac{1}{2}j\right)(0.8e^{j\frac{3\pi}{4}})^n + \left(\frac{1}{2} + \frac{1}{2}j\right)(0.8e^{-j\frac{3\pi}{4}})^n\right]u[n] \\ &= \left[\left(\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\right)(0.8e^{j\frac{3\pi}{4}})^n + \left(\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}\right)(0.8e^{-j\frac{3\pi}{4}})^n\right]u[n] \\ &= 2\,\mathrm{Re}\left\{\left(\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\right)(0.8e^{j\frac{3\pi}{4}})^n\right\}u[n] = 2\frac{1}{\sqrt{2}}(0.8)^n\,\mathrm{Re}\left\{e^{j(\frac{3\pi}{4}n - \frac{\pi}{4})}\right\}u[n] \\ &= \sqrt{2}(0.8)^n\cos\left[\frac{3\pi}{4}n - \frac{\pi}{4}\right]u[n] \end{aligned}$$

The impulse response is:

$$\begin{aligned} h[n] &= h_a[n-1] - h_a[n-2] \\ &= \sqrt{2}(0.8)^{n-1}\cos\left[\frac{3\pi}{4}(n-1) - \frac{\pi}{4}\right]u[n-1] - \sqrt{2}(0.8)^{n-2}\cos\left[\frac{3\pi}{4}(n-2) - \frac{\pi}{4}\right]u[n-2] \end{aligned}$$

(b) [5 marks] Compute the response $y[n]$ of the system in (a) to the input signal $x[n]$ shown below.



Answer:

The input signal is composed of two time-shifted impulses, i.e., $x[n] = \delta[n+1] - \delta[n-1]$, thus

$$y[n] = h[n+1] - h[n-1] = \sqrt{2}\left(0.8\right)^{n} \cos\left[\frac{3\pi}{4}(n) - \frac{\pi}{4}\right]u[n] - \sqrt{2}\left(0.8\right)^{n-1} \cos\left[\frac{3\pi}{4}(n-1) - \frac{\pi}{4}\right]u[n-1]$$

$$-\sqrt{2}\left(0.8\right)^{n-2} \cos\left[\frac{3\pi}{4}(n-2) - \frac{\pi}{4}\right]u[n-2] + \sqrt{2}\left(0.8\right)^{n-3} \cos\left[\frac{3\pi}{4}(n-3) - \frac{\pi}{4}\right]u[n-3]$$