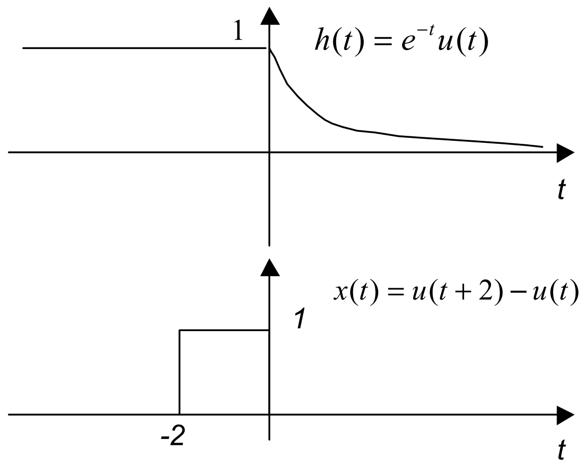
Sample Midterm Test 1 (mt1s01)
Covering Chapters 1-3 of *Fundamentals of Signals & Systems*

Problem 1 (25 marks)

(a) [20 marks] Compute the output $y(t)$ of the continuous-time LTI system with impulse response $h(t)$ for an input signal $x(t)$ as depicted below.



Answer:
SOLUTION 1: Let's time-reverse and shift the impulse response. This way the integral will always be evaluated over $-2 \le \tau \le 0$ The intervals of interest are:

$t < -2$: overlap with the left, constant part of $h(\tau)$, so

$$y(t) = \int_{-2}^{0} h(t-\tau)d\tau = \int_{-2}^{0} 1 d\tau = [\tau]_{-2}^{0} = 2 .$$

$t > 0$: overlap with the right, exponential part of $h(\tau)$, so

$$\begin{aligned} y(t) &= \int_{-2}^{0} h(t-\tau)d\tau = \int_{-2}^{0} e^{-(t-\tau)}d\tau = e^{-t}\int_{-2}^{0} e^{\tau}d\tau = e^{-t}\left[e^{\tau}\right]_{-2}^{0} . \\ &= \left(1-e^{-2}\right)e^{-t} \end{aligned}$$

$-2 < t < 0$: overlap with both parts of $h(\tau)$.

$$\begin{aligned} y(t) &= \int_{-2}^{0} h(t-\tau)d\tau = \int_{-2}^{t} e^{-(t-\tau)}d\tau + \int_{t}^{0} 1 d\tau = e^{-t}\int_{-2}^{t} e^{\tau}d\tau - t \\ &= e^{-t}\left[e^{\tau}\right]_{-2}^{t} - t = \left[1-e^{-(t+2)}\right] - t \end{aligned}$$

Finally, piecing all three intervals together, we get:

$$y(t) = \begin{cases} 2 & t < -2 \\ \left[1 - e^{-(t+2)}\right] - t & -2 \le t < 0 \\ \left(1 - e^{-2}\right)e^{-t} & t \ge 0 \end{cases}$$

SOLUTION 2: Let's break down the impulse response into two parts and use linear superposition.
$h(t) = h_1(t) + h_2(t),\;\; h_1(t) = e^{-t}u(t),\, h_2(t) = u(-t)$.
Response of $h_1(t)$: the intervals are

$t < -2$: no overlap with $x(\tau)$, so $y_1(t) = 0$.

$-2 < t < 0$: overlap with $x(\tau)$ for $-2 \le \tau \le t$.

$$y_1(t) = \int_{-2}^{t} h_1(t-\tau)d\tau = \int_{-2}^{t} e^{-(t-\tau)}d\tau = e^{-t}\int_{-2}^{t} e^{\tau}d\tau$$
$$= e^{-t}\left[e^{\tau}\right]_{-2}^{t} = \left(1 - e^{-(t+2)}\right)$$

$t > 0$: complete overlap with $x(\tau)$, so

$$y_1(t) = \int_{-2}^{0} h_1(t-\tau)d\tau = \int_{-2}^{0} e^{-(t-\tau)}d\tau = e^{-t}\int_{-2}^{0} e^{\tau}d\tau = e^{-t}\left[e^{\tau}\right]_{-2}^{0}.$$
$$= \left[1 - e^{-2}\right]e^{-t}$$

Response of $h_2(t)$: the intervals are

$t < -2$: complete overlap with $x(\tau)$.

$$y_2(t) = \int_{-2}^{0} h_2(t-\tau)d\tau = \int_{-2}^{0} 1 d\tau = 2$$

$-2 < t < 0$: overlap with $x(\tau)$ for $t \le \tau \le 0$.

$$y_2(t) = \int_{t}^{0} h_2(t-\tau)d\tau = \int_{t}^{0} 1 d\tau = -t$$

$t > 0$: no overlap with $x(\tau)$, so $y_2(t) = 0$.
Finally,

$$y(t) = y_1(t) + y_2(t) = \begin{cases} 2 & t < -2 \\ \left(1 - e^{-(t+2)}\right) - t & -2 \le t < 0 \\ \left[1 - e^{-2}\right]e^{-t} & t \ge 0 \end{cases}$$

(b) [5 marks] Is the system in (a) stable? Is it causal? Justify your answers.

Answer:
No it is not BIBO stable, because the impulse response is not absolutely integrable, as shown below.

$$\int_{-\infty}^{+\infty} |h(t)|\, dt = \int_{-\infty}^{+\infty} \left| e^{-t}u(t) + u(-t) \right| dt = \int_{-\infty}^{0} 1 dt + \int_{0}^{+\infty} e^{-t} dt$$
$$= +\infty$$

The system is noncausal because $h(t) \neq 0, t < 0$.

Problem 2 (25 marks)

Consider the following first-order, causal LTI differential system S initially at rest:

$$S: \qquad \frac{d^2 y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{d}{dt}x(t) + 2x(t)$$

(a) [20 marks] Compute and sketch the impulse response $h(t)$ of the system S.

Answer:
SOLUTION 1:

Step 1: Set up the problem to calculate the intermediate impulse response $h_a(t)$

$$\frac{d^2 h_a(t)}{dt^2} + 3\frac{dh_a(t)}{dt} + 2h_a(t) = \delta(t)$$

Step 2: Find the initial condition of the corresponding homogeneous equation at $t = 0^+$ by integrating the above differential equation from $t = 0^-$ to $t = 0^+$. Note that the impulse will be in the term $\frac{d^2 h_a(t)}{dt^2}$, so $\frac{dh_a(t)}{dt}$ will have a finite jump at most, and $h_a(t)$ will be continuous. Thus we have

$$\int_{0^-}^{0^+} \frac{d^2 h_a(\tau)}{d\tau^2} d\tau = \frac{dh_a}{d\tau}(0^+) = 1,$$

and $h_a(0^+) = 0$.These are our initial conditions for the homogeneous equation for $t > 0$

$$\frac{d^2 h_a(t)}{dt^2} + 3\frac{dh_a(t)}{dt} + 2h_a(t) = 0.$$

Step 3: The characteristic polynomial is $p(s) = s^2 + 3s + 2$ and it has two zeros; one at $s_1 = -1$ and one at $s_2 = -2$, which means that the homogeneous response has the form

$h_a(t) = Ae^{-t} + Be^{-2t}$ for $t > 0$. The initial condition allows us to determine the constants A and B:

$$h_a(0^+) = A + B = 0\,,$$

$$\frac{dh_a}{d\tau}(0^+) = -A - 2B = 1$$

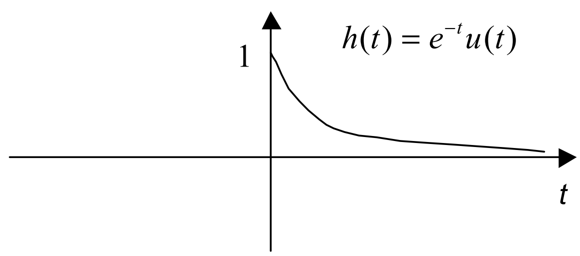
and we solve to get $A = 1, B = -1$. Thus, $h_a(t) = e^{-t} - e^{-2t}$ for $t > 0$.

Step 4:

Finally, we apply the RHS of the differential equation:

$$\begin{aligned} h(t) &= \frac{dh_a(t)}{dt} + 2h_a(t) \\ &= \frac{d}{dt}\left[\left(e^{-t} - e^{-2t}\right)u(t)\right] + 2\left(e^{-t} - e^{-2t}\right)u(t) \\ &= \left(-e^{-t} + 2e^{-2t}\right)u(t) + (1-1)\delta(t) + 2\left(e^{-t} - e^{-2t}\right)u(t) \\ &= e^{-t}u(t) \end{aligned}$$

Sketch:



SOLUTION 2: (step response approach)

Step 1: Set up the problem to calculate the step response of the left-hand side

$$\frac{d^2 s_a(t)}{dt^2} + 3\frac{ds_a(t)}{dt} + 2s_a(t) = u(t)$$

Step 2: Compute the step response as the sum of a forced response and a homogeneous response.

The characteristic polynomial is $p(s) = s^2 + 3s + 2$ and it has two zeros; one at $s_1 = -1$ and one at $s_2 = -2$, which means that the homogeneous response has the form $s_{ah}(t) = Ae^{-t} + Be^{-2t}$ for $t > 0$.

We look for a particular solution of the form $s_{ap}(t) = K$ for $t > 0$ when $x(t) = 1$. We find

$$s_{ap}(t) = \frac{1}{2}.$$

Adding the homogeneous and particular solutions, we obtain the overall step response for $t > 0$:

$$s_a(t) = Ae^{-t} + Be^{-2t} + \frac{1}{2}.$$

From initial rest, the initial conditions at $t = 0^-$ are $s_a(0^-) = 0$ and $\frac{ds_a}{dt}(0^-) = 0$. Thus,

$$s_a(0) = 0 = A + B + \frac{1}{2}.$$

$$\frac{ds_a}{dt}(0^+) = -A - 2B = 0$$

and we obtain: $A = -1,\ B = \frac{1}{2}$.

Which means that the intermediate step response of the system is:

$$s_a(t) = \left(-e^{-t} + \frac{1}{2}e^{-2t} + \frac{1}{2}\right)u(t).$$

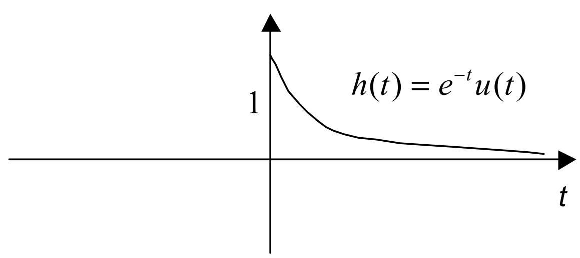
Step 3: Differentiating, we obtain the intermediate impulse response:

$$h_a(t) = \frac{d}{dt}s_a(t) = \left(e^{-t} - e^{-2t}\right)u(t)$$

Step 4: Use the right-hand side of the differential equation

$$\begin{aligned} h(t) &= \frac{dh_a(t)}{dt} + 2h_a(t) \\ &= \frac{d}{dt}\left[\left(e^{-t} - e^{-2t}\right)u(t)\right] + 2\left(e^{-t} - e^{-2t}\right)u(t) \\ &= \left(-e^{-t} + 2e^{-2t}\right)u(t) + (1-1)\delta(t) + 2\left(e^{-t} - e^{-2t}\right)u(t) \\ &= e^{-t}u(t) \end{aligned}$$

Sketch:



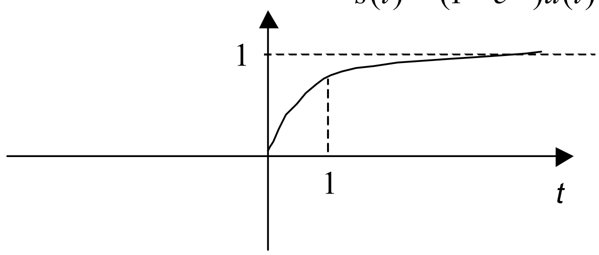
(b) [5 marks] Compute and sketch the step response of the system.

Answer:

The step response of the system is the integral of the impulse response:

$$s(t)=\int_0^t e^{-\tau}d\tau=-\left[e^{-t}-1\right]u(t)=\left[1-e^{-t}\right]u(t)$$

Sketch:



Problem 3 (20 marks)

Determine if the discrete-time system described by $y[n]=\left(\frac{1}{2}\right)^n x[3n+2]$ is

(a) [5 marks] Time-invariant
(b) [5 marks] Linear
(c) [5 marks] Stable
(d) [5 marks] Causal

Justify your answers.

Answer:

(a) It is not time-invariant. Let $y_1[n]=Sx[n-N]=\left(\frac{1}{2}\right)^n x[3n-N+2]$. It is easy to see that

$$y_1[n]\neq y[n-N]=\left(\frac{1}{2}\right)^{n-N} x[3(n-N)+2].$$

(b) It is linear.

Principle of Superposition, let $y_1[n]=Sx_1[n]=\left(\frac{1}{2}\right)^n x_1[3n+2]$ and

$$y_2[n]=Sx_2[n]=\left(\frac{1}{2}\right)^n x_2[3n+2].$$

Then for $x[n]=ax_1[n]+bx_2[n]$, we have

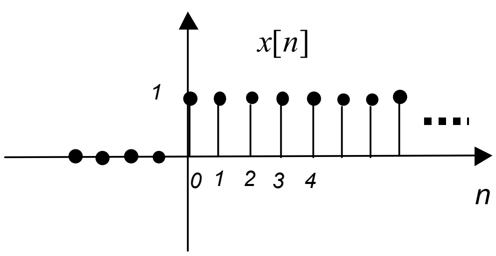
$$y[n]=\left(\frac{1}{2}\right)^n\left(ax_1[3n+2]+bx_2[3n+2]\right)=a\left(\frac{1}{2}\right)^n x_1[3n+2]+b\left(\frac{1}{2}\right)^n x_2[3n+2]$$
$$=ay_1[n]+by_2[n]$$

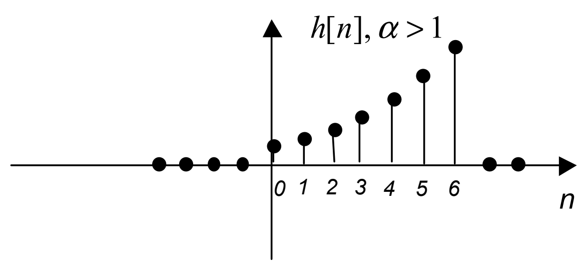
(c) It is <u>unstable:</u> for a given bound $|x[n]| < B$, the output can not be bounded for n negative going to $-\infty$. For example, $x_1[n]=1$ is a bounded input leading to the output $y_1[n] = Sx_1[n] = \left(\frac{1}{2}\right)^n$ which is unbounded for negative times.

(d) It is <u>not causal:</u> To compute $y[0]$, the system needs the future value of the input $x[2]$.

Problem 4 (20 marks)

Compute the response $y[n]$ of the discrete-time LTI system described by its impulse response $h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{otherwise} \end{cases}$ to the step input signal $x[n]=u[n]$. Give the numerical value of $y[100]$ for the case $\alpha = 2$.





Answer:

We break down the problem into 3 intervals for n.

<u>For</u> $n<0$: $h[n-k]$ is zero for k>0, hence $g[k] = h[k]x[n-k] = 0\ \forall k$ and $y[n]=0$.

<u>For</u> $0 \le n \le 6$: Then $g[k] = h[k]x[n-k] \ne 0$ for $k = 0,\ldots,n$. We get

$$y[n] = \sum_{k=0}^{n} g[k] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}$$

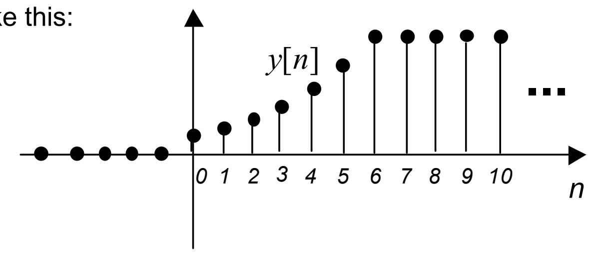
<u>For</u> $n>6$: Then $g[k] = h[k]x[n-k] \ne 0$ for $k = 0,\ldots,6$. We get

$$y[n] = \sum_{k=0}^{6} \alpha^k = \frac{1-\alpha^7}{1-\alpha}$$

In summary, the output signal of the LTI system is

$$y[n] = \begin{cases} 0, & n < 0 \\ \dfrac{1-\alpha^{n+1}}{1-\alpha}, & 0 \le n \le 6 \\ \dfrac{1-\alpha^{7}}{1-\alpha}, & n > 6 \end{cases}$$

Which, for $\alpha > 1$, looks something like this:



The numerical value of $y[100]$ for the case $\alpha = 2$ is $y[100] = \dfrac{1-\alpha^7}{1-\alpha} = \dfrac{1-128}{-1} = 127$

Problem 5 (10 marks)

Compute the impulse response $h[n]$ of the following causal LTI second-order difference system initially at rest:

$$y[n] - y[n-1] + 0.5y[n-2] = x[n]$$

Simplify your expression of $h[n]$ to obtain a real function of time.

Answer:

Write:

$$y[n] - y[n-1] + 0.5y[n-2] = \delta[n].$$

Initial conditions for the homogeneous equation for $n > 0$ are $y[0] = 1$, $y[-1] = 0$.

characteristic polynomial and zeros:

$$p(z) = z^2 - z + 0.5 = (z - 0.5 - j0.5)(z - 0.5 + j0.5)$$

The zeros are $z_1 = 0.5 + j0.5 = \dfrac{1}{\sqrt{2}} e^{j\frac{\pi}{4}}$, $z_2 = 0.5 - j0.5 = \dfrac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}$.

The homogeneous response for $n > 0$ is given by

$$h_a[n] = A\left(\frac{1}{\sqrt{2}} e^{j\frac{\pi}{4}}\right)^n + B\left(\frac{1}{\sqrt{2}} e^{-j\frac{\pi}{4}}\right)^n.$$

Use initial conditions to compute the coefficients A and B:

$$h_a[-1] = 0 = A(\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}})^{-1} + B(\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}})^{-1} = \sqrt{2}e^{-j\frac{\pi}{4}}A + \sqrt{2}e^{j\frac{\pi}{4}}B$$

$$h_a[0] = 1 = A + B$$

From the first equation, we get $A = -e^{j\frac{\pi}{2}}B = -jB$, and from the second equation:
$B = \frac{1}{1-j} = \frac{1}{2} + \frac{1}{2}j$. Thus $A = \frac{1}{2} - \frac{1}{2}j$, and

the homogeneous response is

$$\begin{aligned} h_a[n] &= \left[\left(\frac{1}{2} - \frac{1}{2}j\right)(\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}})^n + \left(\frac{1}{2} + \frac{1}{2}j\right)(\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}})^n\right]u[n] \\ &= \left[\left(\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\right)(\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}})^n + \left(\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}\right)(\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}})^n\right]u[n] \\ &= 2\,\mathrm{Re}\left\{\left(\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\right)(\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}})^n\right\}u[n] = 2\left(\frac{1}{\sqrt{2}}\right)^{n+1}\mathrm{Re}\left\{e^{j\frac{\pi}{4}(n-1)}\right\}u[n] \\ &= \left(\frac{1}{\sqrt{2}}\right)^{n-1}\cos\left[\frac{\pi}{4}(n-1)\right]u[n] \end{aligned}$$

The impulse response is the same: $h[n] = h_a[n] = \left(\frac{1}{\sqrt{2}}\right)^{n-1}\cos\left[\frac{\pi}{4}(n-1)\right]u[n]$