Sample Midterm Test 1 (mt1s00)
Covering Chapters 1-3 of *Fundamentals of Signals & Systems*

Problem 1 (15 marks)

Determine if the discrete-time system described by $y[n]=\begin{cases}(n+1)x[n], & x[n]\geq 0\\ -(n+1)x[n], & x[n]<0\end{cases}$ is

(a) [5 marks] Time-invariant
(b) [5 marks] Linear
(c) [5 marks] Stable

Justify your answers.
Answers:

(a) It is <u>not time-invariant.</u> Let $y_1[n]=Sx[n-N]=\begin{cases}(n+1)x[n-N], & x[n-N]\geq 0\\ -(n+1)x[n-N], & x[n-N]<0\end{cases}$. It is easy to see that $y_1[n]\neq y[n-N]=\begin{cases}(n-N+1)x[n-N], & x[n-N]\geq 0\\ -(n-N+1)x[n-N], & x[n-N]<0\end{cases}$.

(b) It is <u>nonlinear.</u>

Method 1: Principle of Superposition, let $y_1[n]=Sx_1[n]=\begin{cases}(n+1)x_1[n], & x_1[n]\geq 0\\ -(n+1)x_1[n], & x_1[n]<0\end{cases}$ and

$y_2[n]=Sx_2[n]=\begin{cases}(n+1)x_2[n], & x_2[n]\geq 0\\ -(n+1)x_2[n], & x_2[n]<0\end{cases}$.

Then for $x[n]=ax_1[n]+bx_2[n]$, we have

$$\begin{aligned}y[n]&=\begin{cases}(n+1)(ax_1[n]+bx_2[n]), & ax_1[n]+bx_2[n]\geq 0\\ -(n+1)(ax_1[n]+bx_2[n]), & ax_1[n]+bx_2[n]<0\end{cases}\\ &\neq ay_1[n]+by_2[n]\\ &=\begin{cases}(n+1)ax_1[n], & x_1[n]\geq 0\\ -(n+1)ax_1[n], & x_1[n]<0\end{cases}+\begin{cases}(n+1)bx_2[n], & x_2[n]\geq 0\\ -(n+1)bx_2[n], & x_2[n]<0\end{cases}\end{aligned}$$

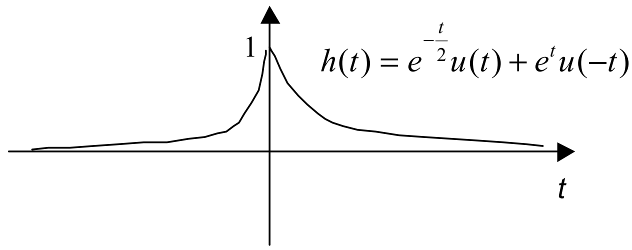
Method 2: We show that the system is not homogeneous. Consider the input $x_1[n]=-1$, then $y_1[n]=Sx_1[n]=(n+1)$. Now for the input $x_2[n]=(-1)x_1[n]=1$, we have $y_2[n]=Sx_2[n]=(n+1)\neq(-1)y_1[n]=-(n+1)$.

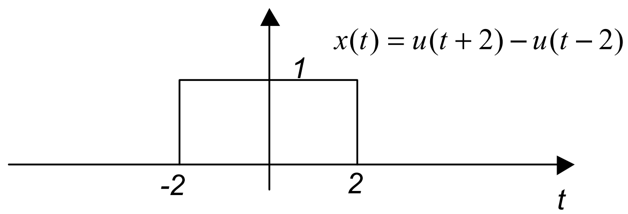
Method 3: We show that the system is not additive. Consider the inputs $x_1[n]=-2$ and $x_2[n]=1$, for which $y_1[n]=Sx_1[n]=2(n+1)$ and $y_2[n]=Sx_2[n]=(n+1)$. For the input $x[n]:=x_1[n]+x_2[n]=-1$ we have $y[n]=(n+1)\neq y_1[n]+y_2[n]=3(n+1)$.

(c) It is <u>unstable:</u> for a given bound $|x[n]|<B$, the output can not be bounded for n large. For example, $x_1[n]=1$ is a bounded input leading to the unbounded output $y_1[n]=Sx_1[n]=(n+1)$.

Problem 2 (30 marks)

(a) [20 marks] Compute the output $y(t)$ of the continuous-time LTI system S_0 with impulse response $h(t)$ for an input signal $x(t)$ as depicted below.





Answer:
SOLUTION 1: Let's time-reverse and shift the impulse response. This way the integral will always be evaluated over $-2 \le \tau \le 2$ The intervals of interest are:

$t < -2$: overlap with the left part of $h(\tau)$, so

$$y(t) = \int_{-2}^{2} h(t-\tau)d\tau = \int_{-2}^{2} e^{t-\tau} d\tau = e^t \int_{-2}^{2} e^{-\tau} d\tau = -e^t \left[e^{-\tau} \right]_{-2}^{2}.$$
$$= \left[e^2 - e^{-2} \right] e^t$$

$t > 2$: overlap with the right part of $h(\tau)$, so

$$y(t) = \int_{-2}^{2} h(t-\tau)d\tau = \int_{-2}^{2} e^{-\frac{t-\tau}{2}} d\tau = e^{-\frac{t}{2}} \int_{-2}^{2} e^{\frac{\tau}{2}} d\tau = 2e^{-\frac{t}{2}} \left[e^{\frac{\tau}{2}} \right]_{-2}^{2}.$$
$$= 2\left[e^1 - e^{-1} \right] e^{-\frac{t}{2}}$$

$-2 < t < 2$: overlap with both parts of $h(\tau)$.

$$y(t) = \int_{-2}^{2} h(t-\tau)d\tau = \int_{-2}^{t} e^{-\frac{t-\tau}{2}} d\tau + \int_{t}^{2} e^{t-\tau} d\tau = e^{-\frac{t}{2}} \int_{-2}^{t} e^{\frac{\tau}{2}} d\tau + e^{t} \int_{t}^{2} e^{-\tau} d\tau$$

$$= 2e^{-\frac{t}{2}} \left[e^{\frac{\tau}{2}} \right]_{-2}^{t} - e^{t} \left[e^{-\tau} \right]_{t}^{2}$$

$$= 2\left[e^{\frac{t}{2}} - e^{-1} \right] e^{-\frac{t}{2}} - e^{t} \left[e^{-2} - e^{-t} \right]$$

$$= 3 - 2e^{-1} e^{-\frac{t}{2}} - e^{-2} e^{t}$$

Finally, piecing all three intervals together, we get:

$$y(t) = \begin{cases} \left[e^{2} - e^{-2} \right] e^{t} & t < -2 \\ 3 - 2e^{-1} e^{-\frac{t}{2}} - e^{-2} e^{t} & -2 \le t < 2 \\ 2\left[e^{1} - e^{-1} \right] e^{-\frac{t}{2}} & t \ge 2 \end{cases}$$

SOLUTION 2: Let's break down the impulse response into two parts and use linear superposition.

$h(t) = h_1(t) + h_2(t),\ \ h_1(t) = e^{-\frac{t}{2}} u(t),\ h_2(t) = e^{t} u(-t)$.

Response of $h_1(t)$: the intervals are

$t < -2$: no overlap with $x(\tau)$, so $y_1(t) = 0$.

$-2 < t < 2$: overlap with $x(\tau)$ for $-2 \le \tau \le t$.

$$y_1(t) = \int_{-2}^{t} h_1(t-\tau)d\tau = \int_{-2}^{t} e^{-\frac{t-\tau}{2}} d\tau = e^{-\frac{t}{2}} \int_{-2}^{t} e^{\frac{\tau}{2}} d\tau$$

$$= 2e^{-\frac{t}{2}} \left[e^{\frac{\tau}{2}} \right]_{-2}^{t} = 2\left[e^{\frac{t}{2}} - e^{-1} \right] e^{-\frac{t}{2}}$$

$$= 2 - 2e^{-1} e^{-\frac{t}{2}}$$

$t > 2$: overlap with $x(\tau)$, so

$$y_1(t) = \int_{-2}^{2} h_1(t-\tau)d\tau = \int_{-2}^{2} e^{-\frac{t-\tau}{2}} d\tau = e^{-\frac{t}{2}} \int_{-2}^{2} e^{\frac{\tau}{2}} d\tau = 2e^{-\frac{t}{2}} \left[e^{\frac{\tau}{2}} \right]_{-2}^{2}.$$

$$= 2\left[e^{1} - e^{-1} \right] e^{-\frac{t}{2}}$$

Response of $h_2(t)$: the intervals are

$t < -2$: overlap with $x(\tau)$.

$$y_2(t) = \int_{-2}^{2} h_2(t-\tau)d\tau = \int_{-2}^{2} e^{t-\tau}d\tau = e^t \int_{-2}^{2} e^{-\tau}d\tau = -e^t\left[e^{-\tau}\right]_{-2}^{2}$$
$$= \left[e^2 - e^{-2}\right]e^t$$

$-2 < t < 2$: overlap with $x(\tau)$ for $t \le \tau \le 2$.

$$y_2(t) = \int_{-2}^{2} h_2(t-\tau)d\tau = \int_{t}^{2} e^{t-\tau}d\tau = e^t \int_{t}^{2} e^{-\tau}d\tau = -e^t\left[e^{-\tau}\right]_{t}^{2}$$
$$= -e^t\left[e^{-2} - e^{-t}\right] = 1 - e^{-2}e^t$$

$t > 2$: no overlap with $x(\tau)$, so $y_2(t) = 0$.

Finally,

$$y(t) = y_1(t) + y_2(t) = \begin{cases} \left[e^2 - e^{-2}\right]e^t & t < -2 \\ 3 - 2e^{-1}e^{-\frac{t}{2}} - e^{-2}e^t & -2 \le t < 2 \\ 2\left[e^1 - e^{-1}\right]e^{-\frac{t}{2}} & t \ge 2 \end{cases}$$

(b) [10 marks] Is the system S_0 in (a) stable? Is it causal? Justify your answers.

Yes it is BIBO stable, because the impulse response is absolutely integrable, as shown below.

$$\int_{-\infty}^{+\infty} |h(t)|\,dt = \int_{-\infty}^{+\infty} \left| e^{-\frac{t}{2}}u(t) + e^t u(-t) \right| dt \le \int_{-\infty}^{0} e^t dt + \int_{0}^{+\infty} e^{-\frac{t}{2}}dt$$
$$= \left[e^t\right]_{-\infty}^{0} - 2\left[e^{-\frac{t}{2}}\right]_{0}^{+\infty} = 1 + 2 = 3 < +\infty$$

The system is noncausal because $h(t) \ne 0,\ t < 0$.

Problem 3 (15 marks)

Consider the following first-order, causal LTI differential system S initially at rest:

$$S: \qquad 2\frac{dy(t)}{dt} + 3y(t) = 5\frac{d}{dt}x(t) - x(t)$$

Compute and sketch the impulse response $h(t)$ of the system S.

Answer:

SOLUTION 1:

Step 1: Set up the problem to calculate the impulse response

$$2\frac{dh_a(t)}{dt} + 3h_a(t) = \delta(t)$$

Step 2: Find the initial condition of the corresponding homogeneous equation at $t = 0^+$ by integrating the above differential equation from $t = 0^-$ to $t = 0^+$. Note that the impulse will be in the term $\frac{dh_a(t)}{dt}$, so $h_a(t)$ will have a finite jump at most. Thus we have

$$\int_{0^-}^{0^+} \frac{dh_a(\tau)}{d\tau} d\tau = h_a(0^+) = 0.5,$$

hence $h_a(0^+) = 0.5$ is our initial condition for the homogeneous equation for $t > 0$

$$2\frac{dh_a(t)}{dt} + 3h_a(t) = 0.$$

Step 3: The characteristic polynomial is $p(s) = 2s + 3$ and it has one zero at $s = -\frac{3}{2}$, which means that the homogeneous response has the form $h_a(t) = Ae^{-\frac{3}{2}t}$ for $t > 0$. The initial condition allows us to determine the constant A:

$$h_a(0^+) = A = 0.5,$$

so that $$h_a(t) = 0.5e^{-\frac{3}{2}t}.$$

Step 4:

$$\begin{aligned} h(t) &= 5\frac{dh_a(t)}{dt} - h_a(t) \\ &= 5\frac{d}{dt}\left(0.5e^{-\frac{3}{2}t}u(t)\right) - 0.5e^{-\frac{3}{2}t}u(t) \\ &= -\frac{17}{4}e^{-\frac{3}{2}t}u(t) + 2.5\delta(t) \end{aligned}$$

SOLUTION 2: (step response approach)

Step 1: Set up the problem to calculate the step response of the left-hand side

$$2\frac{ds_a(t)}{dt} + 3s_a(t) = u(t)$$

Step 2: Compute the step response as the sum of a forced response and a homogeneous response.

The characteristic polynomial of this system is

$$p(s) = 2s + 3 = 2(s + 1.5)\ ,$$

Hence, the homogeneous solution has the form

$$y_h(t) = Ae^{-1.5t}\ .$$

We look for a particular solution of the form $y_p(t) = K$ for $t > 0$ when $x(t) = 1$. We find

$$y_p(t) = \frac{1}{3}\ .$$

Adding the homogeneous and particular solutions, we obtain the overall step response for $t > 0$:

$$s_a(t) = Ae^{-1.5t} + \frac{1}{3}.$$

The initial conditions at $t = 0^-$ is $s_a(0^-) = 0$. Thus,

$$\begin{aligned} s_a(0) = 0 = A + \frac{1}{3} \\ \Rightarrow \quad A = -\frac{1}{3} \end{aligned}.$$

Which means that the intermediate step response of the system is:

$$s_a(t) = \left(-\frac{1}{3}e^{-1.5t} + \frac{1}{3}\right)u(t)\ .$$

Step 3: Differentiating, we obtain the intermediate impulse response:

$$h_a(t) = \frac{d}{dt}s_a(t) = \left(\frac{1}{2}e^{-1.5t}\right)u(t)$$

Step 4: Use the right-hand side of the differential equation

$$\begin{aligned} h(t) &= 5\frac{dh_a(t)}{dt} - h_a(t) \\ &= 5\frac{d}{dt}\left(0.5e^{-\frac{3}{2}t}u(t)\right) - 0.5e^{-\frac{3}{2}t}u(t) \\ &= -\frac{17}{4}e^{-\frac{3}{2}t}u(t) + 2.5\delta(t) \end{aligned}$$

Problem 4 (10 marks)

Compute the response of the following homogeneous causal second-order difference equation to the initial conditions $y[-1]=1,\ y[-2]=2$:

$$2y[n]-3y[n-1]-2y[n-2]=0$$

Answer:

characteristic polynomial and zeros:

$$p(z)=2z^2-3z-2=2(z+0.5)(z-2)$$

The zeros are $z_1=-0.5,\ \ z_2=2$.

The homogeneous response is given by

$$y[n]=A(-0.5)^n+B(2)^n,\ \ n\geq 0.$$

Use initial conditions to compute the coefficients A and B:

$$y[-1]=1=A(-0.5)^{-1}+B(2)^{-1}$$

$$y[-2]=2=A(-0.5)^{-2}+B(2)^{-2}$$

Hence

$$A=\frac{3}{10},\quad B=\frac{16}{5},$$

and the homogeneous response is

$$y[n]=\left[\frac{3}{10}(-0.5)^n+\frac{16}{5}(2)^n\right]u[n].$$