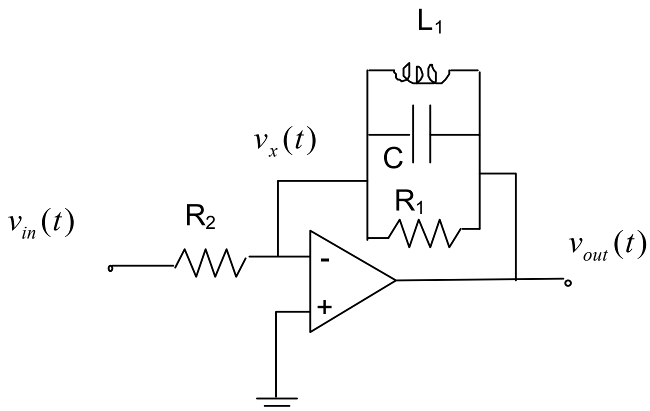
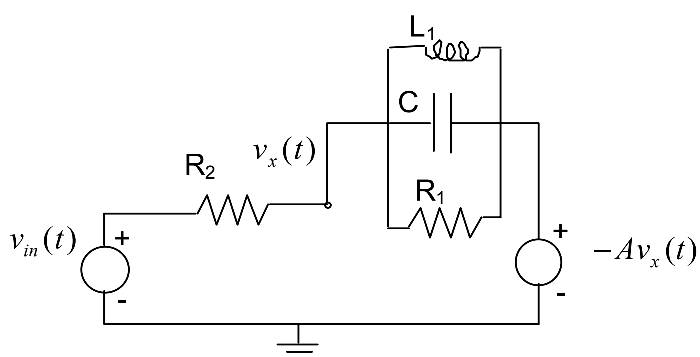
Sample Final Exam (finals03)
Covering Chapters 1-9 of *Fundamentals of Signals & Systems*

Problem 1 (20 marks)

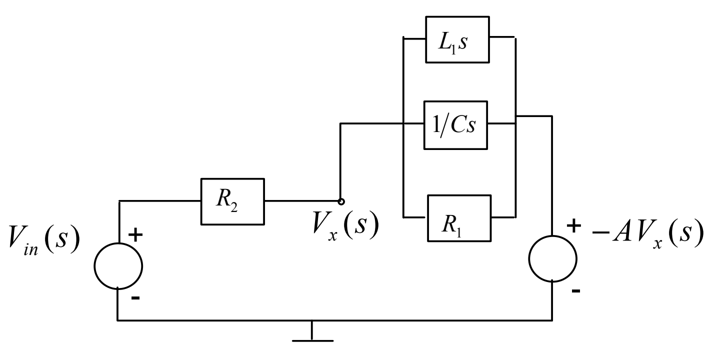
Consider the causal op-amp circuit initially at rest depicted below. Its LTI circuit model with a voltage-controlled source is also given below.

(a) [8 marks] Transform the circuit using the Laplace transform, and find the transfer function $H_A(s) = V_{out}(s)/V_{in}(s)$. Then, let the op-amp gain $A \to +\infty$ to obtain the ideal transfer function $H(s) = \lim_{A\to+\infty} H_A(s)$.





Answer:
The transformed circuit:



There are two supernodes for which the nodal voltages are given by the source voltages. The remaining nodal equation is

$$\frac{V_{in}(s)-V_x(s)}{R_2}+\frac{-AV_x(s)-V_x(s)}{R_1\left\|\frac{1}{Cs}\right\|L_1 s}=0$$

where $R_1\left\|\frac{1}{Cs}\right\|L_1 s=\frac{1}{Cs+\frac{1}{R_1}+\frac{1}{L_1 s}}=\frac{R_1L_1s}{R_1L_1Cs^2+L_1s+R_1}$. Simplifying the above equation, we get:

$$\frac{1}{R_2}V_{in}(s)-\left[\frac{(A+1)(R_1L_1Cs^2+L_1s+R_1)}{R_1L_1s}+\frac{1}{R_2}\right]V_x(s)=0$$

Thus, the transfer function between the input voltage and the node voltage is given by

$$\frac{V_x(s)}{V_{in}(s)}=\frac{\frac{1}{R_2}}{\frac{(A+1)(R_1L_1Cs^2+L_1s+R_1)}{R_1L_1s}+\frac{1}{R_2}}=\frac{R_1L_1s}{R_2(A+1)(R_1L_1Cs^2+L_1s+R_1)+R_1L_1s}$$

The transfer function between the input voltage and the output voltage is

$$H_A(s)=\frac{V_{out}(s)}{V_{in}(s)}=\frac{-AV_x(s)}{V_{in}(s)}=\frac{-AR_1L_1s}{R_2(A+1)(R_1L_1Cs^2+L_1s+R_1)+R_1L_1s}$$

The ideal transfer function is the limit as the op-amp gain tends to infinity:

$$H(s)=\lim_{A\to\infty}H_A(s)=-\frac{R_1L_1s}{R_2R_1L_1Cs^2+R_2L_1s+R_2R_1}=-\frac{L_1/R_2\ s}{(L_1Cs^2+\frac{L_1}{R_1}s+1)}$$

(b) [5 marks] Assume that the transfer function $H_1(s)=\frac{H(s)}{s}$ has a DC gain of -50, and that $H(s)$ has one zero at 0 and two complex conjugate poles with $\omega_n=10$ rd/s, $\zeta=0.5$. Let $L_1=10H$. Find the values of the remaining circuit components R_1, R_2, C.

Answer:

DC gain of $H_1(s)=\frac{H(s)}{s}=-\frac{L_1/R_2}{(L_1Cs^2+\frac{L_1}{R_1}s+1)}$ is given by $-L_1/R_2=-50$.

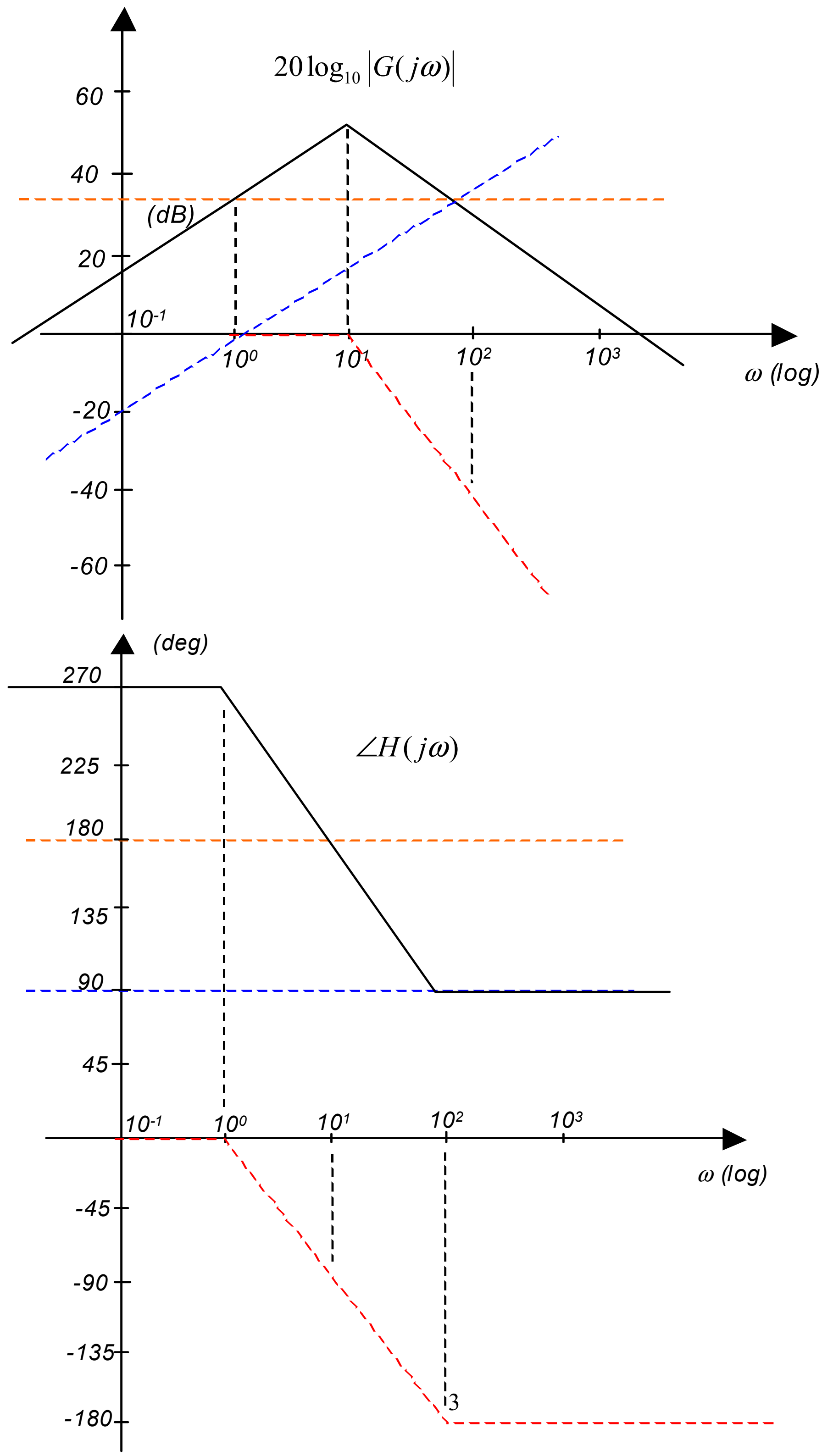
Component values are obtained by setting

$$H(s)=-50\frac{s}{0.01s^2+0.1s+1}=-\frac{\frac{L_1}{R_2}s}{(L_1Cs^2+\frac{L_1}{R_1}s+1)}$$

which yields $\Leftrightarrow R_1 = 100\Omega,\ R_2 = 0.2\Omega,\ C = 0.001F$

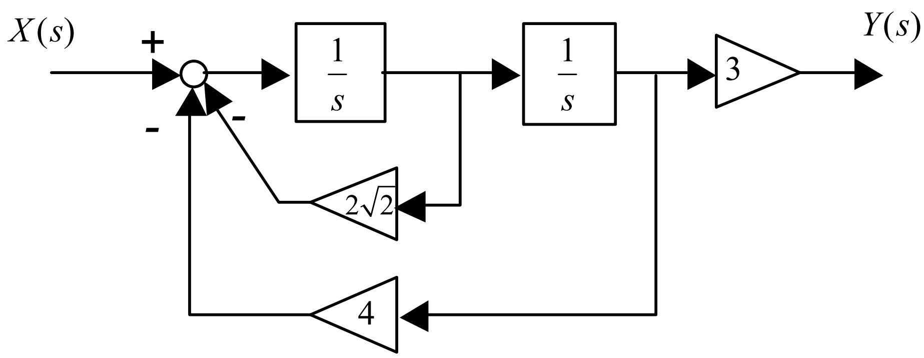
(c) [7 marks] Give the frequency response of $H(s)$ and sketch its Bode plot.

Answer: Frequency response is $H(j\omega) = -50\dfrac{j\omega}{0.01(j\omega)^2 + 0.1(j\omega) + 1}$. Bode plot:



Problem 2 (20 marks)

Consider the causal differential system described by its direct form realization shown below,



and with initial conditions $\frac{dy(0^-)}{dt} = -1, \quad y(0^-) = 2$. Suppose that this system is subjected to the unit step input signal $x(t) = u(t)$.

(a) [8 marks] Write the differential equation of the system. Find the system's damping ratio ζ and undamped natural frequency ω_n. Give the transfer function of the system and specify its ROC. Sketch its pole-zero plot. Is the system stable? Justify.

Answer:

Differential equation: $\frac{d^2 y(t)}{dt^2} + 2\sqrt{2}\frac{dy(t)}{dt} + 4y(t) = 3x(t)$

Let's take the unilateral Laplace transform on both sides of the differential equation.

$$\left[s^2\mathcal{Y}(s) - sy(0^-) - \frac{dy(0^-)}{dt} \right] + 2\sqrt{2}\left[s\mathcal{Y}(s) - y(0^-) \right] + 4\mathcal{Y}(s) = 3\mathcal{X}(s)$$

Collecting the terms containing $\mathcal{Y}(s)$ on the left-hand side and putting everything else on the right-hand side, we can solve for $\mathcal{Y}(s)$.

$$\left(s^2 + 2\sqrt{2}s + 4\right)\mathcal{Y}(s) = 3\mathcal{X}(s) + sy(0^-) + 2\sqrt{2}y(0^-) + \frac{dy(0^-)}{dt}$$

$$\mathcal{Y}(s) = \underbrace{\frac{3\mathcal{X}(s)}{s^2 + 2\sqrt{2}s + 4}}_{\text{zero-state resp.}} + \underbrace{\frac{(s + 2\sqrt{2})y(0^-) + \frac{dy(0^-)}{dt}}{s^2 + 2\sqrt{2}s + 4}}_{\text{zero-input resp.}}$$

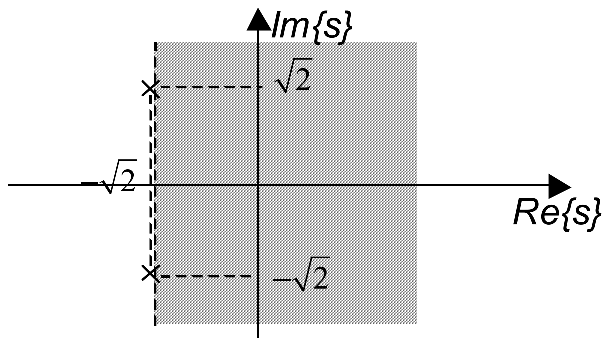
The transfer function is $\mathcal{H}(s) = \dfrac{3}{s^2 + 2\sqrt{2}s + 4}$,

and since the system is causal, the ROC is an open RHP to the right of the rightmost pole.

The undamped natural frequency is $\omega_n = 2$ and the damping ratio is $\zeta = \dfrac{1}{\sqrt{2}}$. The poles are

$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sqrt{2} \pm j2\sqrt{1-\dfrac{1}{2}} = -\sqrt{2} \pm j\sqrt{2}$.

Therefore the ROC is $\mathrm{Re}\{s\} > -\sqrt{2}$. System is *stable* as jw-axis is contained in ROC. Pole-zero plot:



(b) [8 marks] Compute the step response of the system (including the effect of initial conditions), its steady-state response $y_{ss}(t)$ and its transient response $y_{tr}(t)$ for $t \geq 0$. Identify the zero-state response and the zero-input response in the Laplace domain.

Answer:

The unilateral LT of the input is given by

$$\mathcal{X}(s) = \frac{1}{s}, \quad \mathrm{Re}\{s\} > 0,$$

thus, $$\mathcal{Y}(s) = \underbrace{\frac{3}{(s^2 + 2\sqrt{2}s + 4)s}}_{\substack{\mathrm{Re}\{s\}>0 \\ \text{zero-state resp.}}} + \underbrace{\frac{2(s + 2\sqrt{2}) - 1}{s^2 + 2\sqrt{2}s + 4}}_{\substack{\mathrm{Re}\{s\}>-1 \\ \text{zero-input resp.}}} = \frac{2s^2 + (4\sqrt{2} - 1)s + 3}{\left(s^2 + 2\sqrt{2}s + 4\right)s}$$

Let's compute the overall response:

$$\mathcal{Y}(s)=\frac{2s^2+(4\sqrt{2}-1)s+3}{\left(s^2+2\sqrt{2}s+4\right)s},\quad \mathrm{Re}\{s\}>0$$

$$=\underbrace{\frac{A\sqrt{2}+B(s+\sqrt{2})}{\left(s+\sqrt{2}\right)^2+2}}_{\mathrm{Re}\{s\}>-\sqrt{2}}+\underbrace{\frac{C}{s}}_{\mathrm{Re}\{s\}>0}$$

$$=\underbrace{\frac{A\sqrt{2}+B(s+\sqrt{2})}{\left(s+\sqrt{2}\right)^2+2}}_{\mathrm{Re}\{s\}>-\sqrt{2}}+\underbrace{\frac{0.75}{s}}_{\mathrm{Re}\{s\}>0}$$

Let $s=-\sqrt{2}$ to compute

$$\frac{2(2)+(4\sqrt{2}-1)(-\sqrt{2})+3}{2(-\sqrt{2})}=\frac{1}{\sqrt{2}}A+\frac{0.75}{-\sqrt{2}}$$

$$\frac{-1+\sqrt{2}}{-2\sqrt{2}}=\frac{1}{\sqrt{2}}A+\frac{0.75}{-\sqrt{2}},$$

$$\Rightarrow A=\frac{1-\sqrt{2}}{2}+0.75=0.5429$$

then multiply both sides by s and let $s\to\infty$ to get $2=B+0.75\Rightarrow B=1.25$:

$$\mathcal{Y}(s)=\underbrace{\frac{0.5429\sqrt{2}}{\left(s+\sqrt{2}\right)^2+2}}_{\mathrm{Re}\{s\}>-\sqrt{2}}+\underbrace{\frac{1.25(s+\sqrt{2})}{\left(s+\sqrt{2}\right)^2+2}}_{\mathrm{Re}\{s\}>-\sqrt{2}}+\underbrace{\frac{0.75}{s}}_{\mathrm{Re}\{s\}>0}$$

Notice that the second term $\frac{1}{s}$ is the steady-state response, and thus $y_{ss}(t)=0.75u(t)$.

Taking the inverse Laplace transform using the table yields

$$y(t)=0.5429e^{-\sqrt{2}t}\sin(\sqrt{2}t)u(t)+1.25e^{-\sqrt{2}t}\cos(\sqrt{2}t)u(t)+0.75u(t).$$

Thus, the transient response is

$$y_{tr}(t)=0.5429e^{-\sqrt{2}t}\sin(\sqrt{2}t)u(t)+1.25e^{-\sqrt{2}t}\cos(\sqrt{2}t)u(t).$$

(c) [4 marks] Compute the percentage of the first overshoot in the step response of the system assumed this time to be initially at rest.

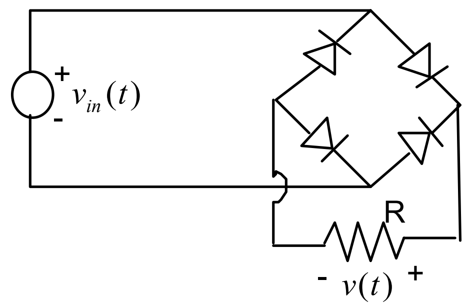
Answer:

Transfer function is $\mathcal{H}(s)=\dfrac{3}{s^2+2\sqrt{2}s+4},\mathrm{Re}\{s\}>\sqrt{2}$ with damping ratio $\zeta=\dfrac{1}{\sqrt{2}}$:

$$OS = 100e^{-\frac{\varsigma\pi}{\sqrt{1-\varsigma^2}}}\% = 100e^{-\frac{0.707\pi}{0.707}}\% = 100e^{-\pi}\% = 4.3\%$$

Problem 3 (20 marks)

The following nonlinear circuit is an ideal full-wave rectifier.

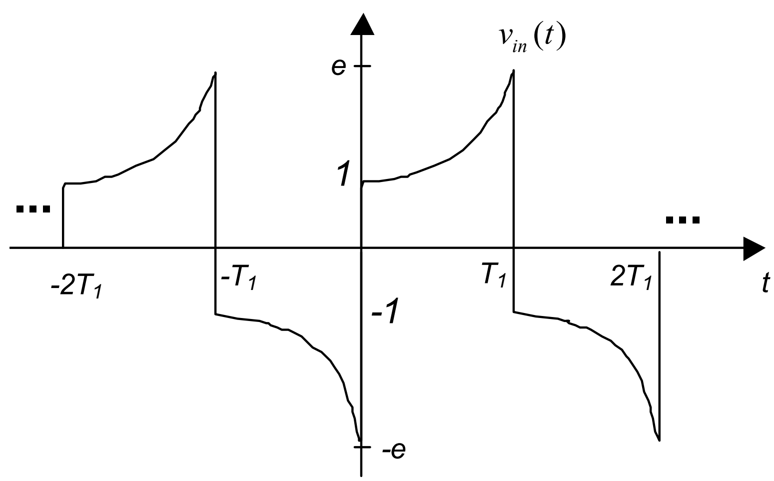


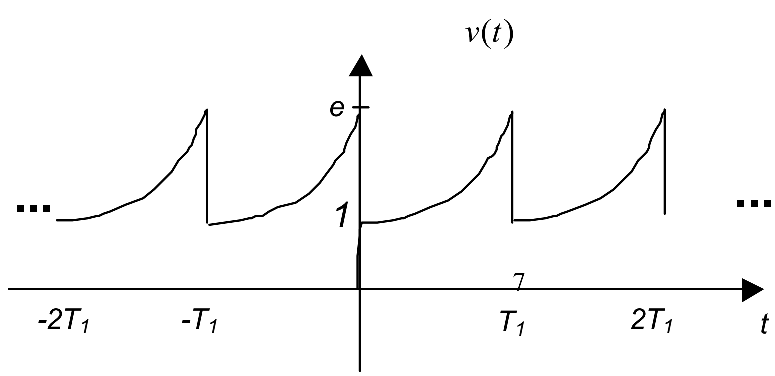
The voltages are $v_{in}(t) = e^{\alpha t}\left[u(t) - u(t-T_1)\right] * \sum_{k=-\infty}^{+\infty}\left[\delta(t-2kT_1) - \delta(t-(2k-1)T_1)\right]$ where $\alpha \in \mathbb{R}, \alpha > 0$, and $v(t) = \left|v_{in}(t)\right|$.

(a) [5 marks] Find the fundamental period T of the input voltage. Sketch the input and output voltages $v_{in}(t), v(t)$ for $\alpha = 1/T_1$.

Answer:

We have $T = 2T_1$





(b) [8 marks] Compute the Fourier series coefficients a_k of the input voltage $v_{in}(t)$ for any positive values of α and T_1. Write $v_{in}(t)$ as a Fourier series.

Answer:
DC component :

$$a_0 = \frac{1}{2T_1}\int_{-T_1}^{T_1} x(t)dt = \frac{1}{2T_1}\int_0^{T_1} e^{\alpha t}dt - \frac{1}{2T_1}\int_{-T_1}^{0} e^{\alpha(t+T_1)}dt$$

$$= \frac{1}{2T_1}\int_0^{T_1} e^{\alpha t}dt - \frac{1}{2T_1}\int_0^{T_1} e^{\alpha \tau}d\tau = 0$$

for $k \neq 0$:

$$a_k = \frac{1}{2T_1}\int_{-T_1}^{T_1} x(t)e^{-jk\frac{2\pi}{2T_1}t}dt$$

$$= \frac{1}{2T_1}\int_0^{T_1} e^{\alpha t}e^{-jk\frac{\pi}{T_1}t}dt - \frac{1}{2T_1}\int_{-T_1}^{0} e^{\alpha(t+T_1)}e^{-jk\frac{\pi}{T_1}t}dt$$

$$= \frac{1}{2T_1}\int_0^{T_1} e^{(\alpha - jk\frac{\pi}{T_1})t}dt - \frac{1}{2T_1}e^{\alpha T_1}\int_{-T_1}^{0} e^{(\alpha - jk\frac{\pi}{T_1})t}dt$$

$$= \frac{1}{2T_1(\alpha - jk\frac{\pi}{T_1})}\left(e^{\alpha T_1}e^{-jk\pi} - 1\right) - \frac{e^{\alpha T_1}}{2T_1(\alpha - jk\frac{\pi}{T_1})}\left(1 - e^{-\alpha T_1}e^{jk\pi}\right)$$

$$= \frac{1}{2T_1(\alpha - jk\frac{\pi}{T_1})}\left(e^{\alpha T_1}e^{-jk\pi} - 1\right) - \frac{e^{\alpha T_1}}{2T_1(\alpha - jk\frac{\pi}{T_1})}\left(1 - e^{-\alpha T_1}e^{jk\pi}\right)$$

$$= \frac{e^{\alpha T_1}\left((-1)^k - 1\right) + (-1)^k - 1}{2T_1\alpha - j2k\pi}$$

$$= \frac{(e^{\alpha T_1}+1)\left((-1)^k - 1\right)}{2T_1\alpha - j2k\pi}$$

Fourier series:

$$v_{in}(t) = \sum_{k=-\infty}^{+\infty} \frac{(e^{\alpha T_1}+1)\left((-1)^k - 1\right)}{2T_1\alpha - j2k\pi} e^{jk\frac{\pi}{T_1}t}$$

(c) [5 marks] Compute the Fourier series coefficients b_k of $v(t)$ again for any positive values of α and T_1.

Answer: Here the fundamental period is $T = T_1$.

DC component :

$$b_0 = \frac{1}{T_1}\int_0^{T_1} x(t)dt = \frac{1}{T_1}\int_0^{T_1} e^{\alpha t}dt$$
$$= \frac{1}{\alpha T_1}\left[e^{\alpha t}\right]_0^{T_1} = \frac{1}{\alpha T_1}\left[e^{\alpha T_1} - 1\right]$$

for $k \neq 0$:

$$b_k = \frac{1}{T_1}\int_0^{T_1} x(t)e^{-jk\frac{2\pi}{T_1}t}dt = \frac{1}{T_1}\int_0^{T_1} e^{(\alpha - jk\frac{2\pi}{T_1})t}dt$$
$$= \frac{1}{T_1(\alpha - jk\frac{2\pi}{T_1})}\left(e^{\alpha T_1} - 1\right) = \frac{e^{\alpha T_1} - 1}{\alpha T_1 - j2k\pi}$$

(d) [2 marks] Compute the Fourier series coefficients of the output voltage signal $v(t)$ for the case $\alpha \to 0$ with T_1 held constant. What time-domain signal $v(t)$ do you obtain in this case?

Answer:

When $\alpha \to 0$ we get a constant signal for $v(t)$.

$$\lim_{\alpha \to 0} b_0 = \lim_{\alpha \to 0} \frac{1}{\alpha T_1}\left[e^{\alpha T_1} - 1\right] = 1$$
$$\lim_{\alpha \to 0} b_k = \lim_{\alpha \to 0} \frac{e^{\alpha T_1} - 1}{\alpha T_1 - j2k\pi} = \frac{1-1}{-j2k\pi} = 0$$

Problem 4 (15 marks)

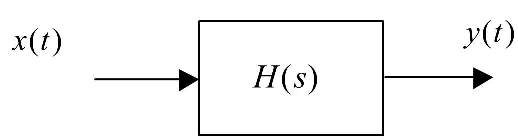
System identification

Suppose we know that the input of a differential LTI system is

$$x(t) = te^{-2t}u(t),$$

and we measured the output to be

$$y(t) = e^{-t}\left[\cos t + \sin t\right]u(t) - e^{-2t}u(t).$$



(a) [10 marks] Find the transfer function $H(s)$ of the system and its region of convergence. Is the system causal? Is it stable? Justify your answers.

Answer:

First take the Laplace transforms of the input and output signals using the table:

$$X(s) = \frac{1}{(s+2)^2}, \quad \mathrm{Re}\{s\} > -2$$

$$\begin{aligned} Y(s) &= \underbrace{\frac{s+1}{(s+1)^2+1^2}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{1}{(s+1)^2+1^2}}_{\mathrm{Re}\{s\}>-1} - \frac{1}{\underbrace{s+2}_{\mathrm{Re}\{s\}>-2}} \\ &= \underbrace{\frac{s+2}{(s+1)^2+1^2}}_{\mathrm{Re}\{s\}>-1} - \frac{1}{\underbrace{s+2}_{\mathrm{Re}\{s\}>-2}} \\ &= \frac{(s^2+4s+4)-(s^2+2s+2)}{(s^2+2s+2)(s+2)}, \ \mathrm{Re}\{s\} > -1 \\ &= \frac{2(s+1)}{(s^2+2s+2)(s+2)}, \ \mathrm{Re}\{s\} > -1 \end{aligned}$$

Then, the transfer function is simply

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\dfrac{2(s+1)}{(s^2+2s+2)(s+2)}}{\dfrac{1}{(s+2)^2}} = \frac{2(s+1)(s+2)}{(s^2+2s+2)} = 2\frac{s^2+3s+2}{s^2+2s+2}$$

To determine the ROC, first note that the ROC of $Y(s)$ should contain the intersection of the ROC's of $H(s)$ and $X(s)$. There are two possible ROC's for $H(s)$: (a) an open left half-plane to the left of $\mathrm{Re}\{s\} = -1$, (b) an open right half-plane to the right of $\mathrm{Re}\{s\} = -1$. But since the ROC of $X(s)$ is an open right half-plane to the right of $s = -2$, the only possible choice is (b). Hence, the ROC of $H(s)$ is $\mathrm{Re}\{s\} > -1$.

The system is causal as the transfer function is rational and the ROC is a right half-plane. It is also stable as both complex poles $p_{1,2} = -1 \pm j$ are in the open left half-plane.

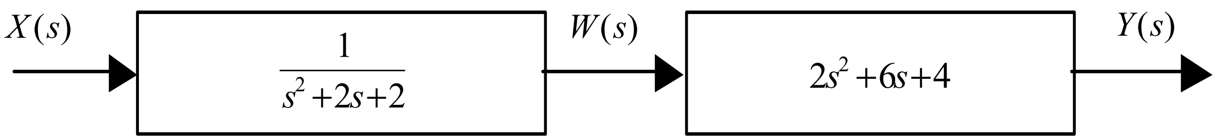
(b) [2 marks] Find an LTI differential equation representing the system.

Answer:
It can be derived from the transfer function obtained in (a):

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = 2\frac{d^2x(t)}{dt^2} + 6\frac{dx(t)}{dt} + 4x(t)$$

(c) [3 marks] Find the direct form realization of the transfer function $H(s)$.

Answer:
The transfer function can be split up into two systems as follows:



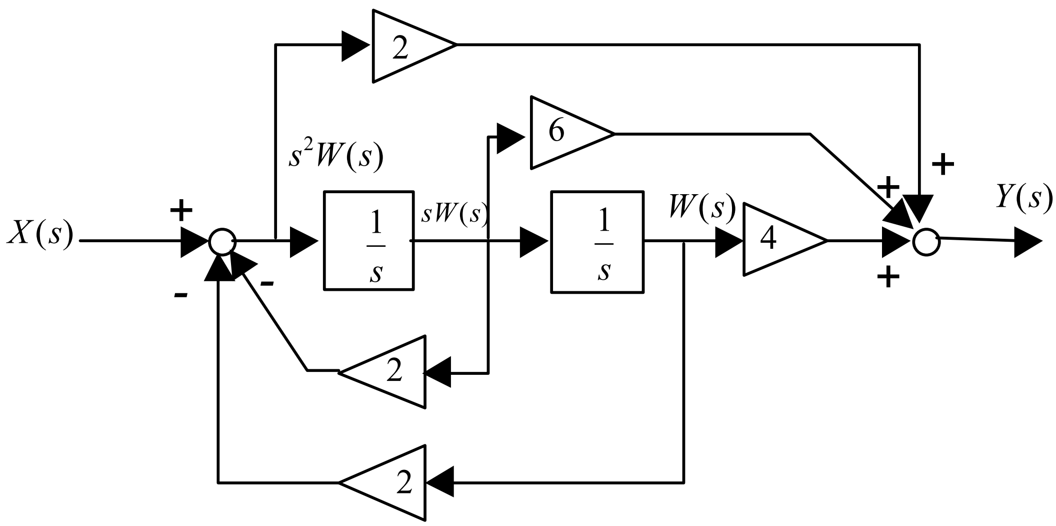
The input-output system equation of the first subsystem is

$$s^2W(s) = -2sW(s) - 2W(s) + X(s),$$

and for the second subsystem we have

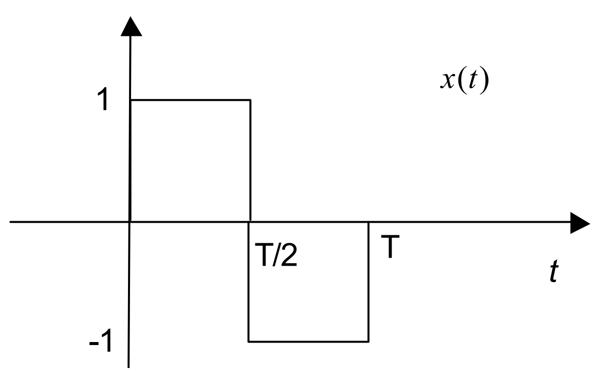
$$Y(s) = 2s^2W(s) + 6sW(s) + 4W(s).$$

The direct form realization of the system is given below:



Problem 5 (15 marks)

(a) [10 marks] Compute the Fourier transform $X(j\omega)$ of the following aperiodic signal $x(t)$ and give its magnitude and phase.



Answer:

$$
\begin{aligned}
X(j\omega) &= \int_0^T x(t)e^{-j\omega t}\,dt \\
&= \int_0^{T/2} e^{-j\omega t}\,dt - \int_{T/2}^{T} e^{-j\omega t}\,dt \\
&= \frac{1}{-j\omega}\left(e^{-j\omega\frac{T}{2}} - 1\right) + \frac{1}{j\omega}\left(e^{-j\omega T} - e^{-j\omega\frac{T}{2}}\right) \\
&= \frac{1}{j\omega}\left(e^{-j\omega T} - 2e^{-j\omega\frac{T}{2}} + 1\right) \\
&= \frac{1}{j\omega}\left(1 - e^{-j\omega\frac{T}{2}}\right)^2 = \frac{1}{j\omega}\left(e^{-j\omega\frac{T}{4}}(e^{j\omega\frac{T}{4}} - e^{-j\omega\frac{T}{4}})\right)^2 \\
&= j\frac{4}{\omega}\sin^2(\omega\frac{T}{4})e^{-j\omega\frac{T}{2}}
\end{aligned}
$$

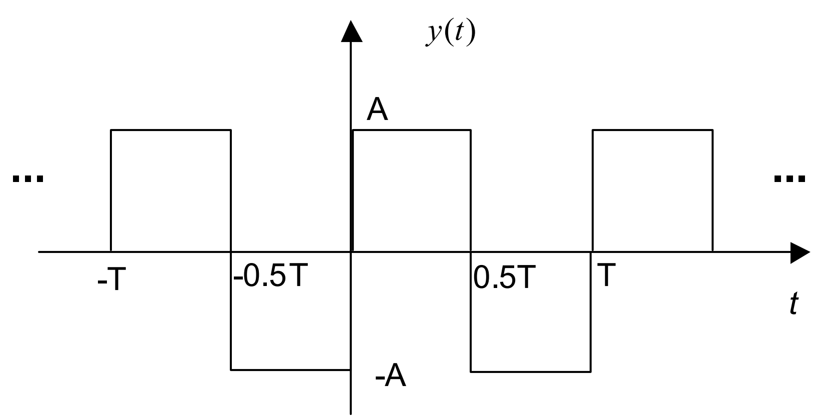
Magnitude:

$$
|X(j\omega)| = \frac{4}{|\omega|}\sin^2(\omega\frac{T}{4})
$$

Phase:

$$
\angle X(j\omega) = \begin{cases} -\omega\frac{T}{2} + \frac{\pi}{2}, & \omega > 0 \\ -\omega\frac{T}{2} - \frac{\pi}{2}, & \omega < 0 \\ 0, & \omega = 0 \end{cases}
$$

(b) [5 marks] Write the Fourier series coefficients a_k of the following rectangular waveform $y(t)$ in terms of $X(j\omega)$ that you obtained in (a) and compute them.



Answer:

We have

$$
\begin{aligned}
a_k &= A\frac{1}{T}X(jk\frac{2\pi}{T}) \\
&= A\frac{1}{T}j\frac{4}{k\frac{2\pi}{T}}\sin^2(k\frac{2\pi}{T}\frac{T}{4})e^{-jk\frac{2\pi}{T}\frac{T}{2}} \\
&= j\frac{2A}{k\pi}\sin^2(k\frac{\pi}{2})e^{-jk\pi} = j\frac{2A}{k\pi(-2)}((-1)^k-1)(-1)^k \\
&= jA\frac{((-1)^k-1)}{k\pi}
\end{aligned}
$$

Problem 6 (10 marks)

Just answer true or false.

(a) The Fourier transform $Z(j\omega)$ of the product of a real even signal $x(t)$ and a real odd signal $y(t)$ is imaginary.

Answer: True.

(b) The system defined by $y(t) = x(t+1)$ is causal.

Answer: False.

(c) The Fourier series coefficients a_k of a purely imaginary even periodic signal $x(t)$ have the following property: $a_k^* = a_k$.

Answer: False.

(d) The causal linear discrete-time system defined by $y[n-2]+0.4y[n-1]-0.45y[n]=x[n-1]$ is stable.

Answer: False.

(e) The fundamental period of the signal $x[n]=\sin(\frac{3\pi}{5}n)$ is 10.

Answer: True.

END OF EXAMINATION