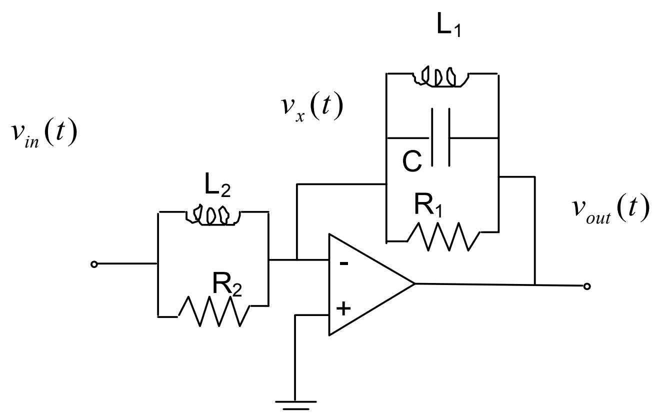
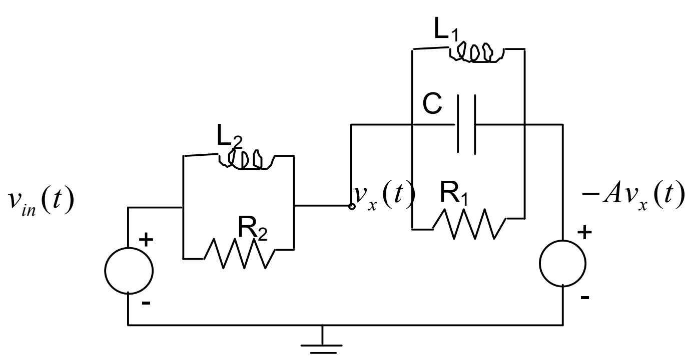
Sample Final Exam (finals03)
Covering Chapters 1-9 and part of Chapter 15 of *Fundamentals of Signals & Systems*

Problem 1 (20 marks)

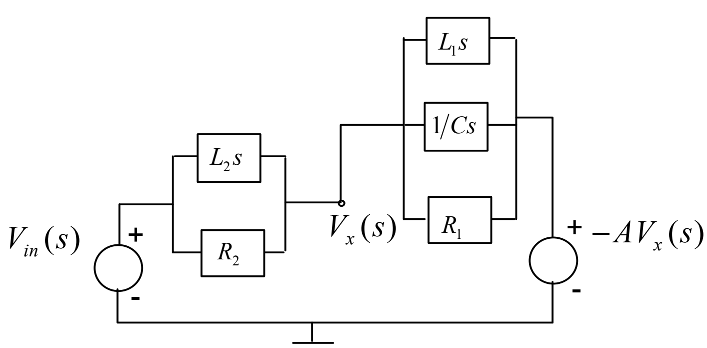
Consider the causal op-amp circuit initially at rest depicted below. Its LTI circuit model with a voltage-controlled source is also given below.

(a) [8 marks] Transform the circuit using the Laplace transform, and find the transfer function $H_A(s) = V_{out}(s)/V_{in}(s)$. Then, let the op-amp gain $A \to +\infty$ to obtain the ideal transfer function $H(s) = \lim_{A \to +\infty} H_A(s)$.





Answer:
The transformed circuit is



There are two supernodes for which the nodal voltages are given by the source voltages. The remaining nodal equation is

$$\frac{V_{in}(s)-V_x(s)}{R_2\|L_2 s}+\frac{-AV_x(s)-V_x(s)}{R_1\|\frac{1}{Cs}\|L_1 s}=0$$

where $R_1\|\frac{1}{Cs}\|L_1 s=\frac{1}{Cs+\frac{1}{R_1}+\frac{1}{L_1 s}}=\frac{R_1L_1s}{R_1L_1Cs^2+L_1s+R_1}$ and $R_2\|L_2 s=\frac{R_2L_2s}{R_2+L_2s}$. Simplifying the above equation, we get:

$$\frac{R_2+L_2s}{R_2L_2s}V_{in}(s)-\left[\frac{(A+1)(R_1L_1Cs^2+L_1s+R_1)}{R_1L_1s}+\frac{R_2+L_2s}{R_2L_2s}\right]V_x(s)=0$$

Thus, the transfer function between the input voltage and the node voltage is given by

$$\frac{V_x(s)}{V_{in}(s)}=\frac{\frac{R_2+L_2s}{R_2L_2s}}{\frac{(A+1)(R_1L_1Cs^2+L_1s+R_1)}{R_1L_1s}+\frac{R_2+L_2s}{R_2L_2s}}=\frac{R_1L_1s(R_2+L_2s)}{R_2L_2s(A+1)(R_1L_1Cs^2+L_1s+R_1)+R_1L_1s(R_2+L_2s)}$$

.

The transfer function between the input voltage and the output voltage is

$$H_A(s)=\frac{V_{out}(s)}{V_{in}(s)}=\frac{-AV_x(s)}{V_{in}(s)}=\frac{-AR_1L_1s(R_2+L_2s)}{R_2L_2s(A+1)(R_1L_1Cs^2+L_1s+R_1)+R_1L_1s(R_2+L_2s)}$$

The ideal transfer function is the limit as the op-amp gain tends to infinity:

$$H(s)=\lim_{A\to\infty}H_A(s)=-\frac{R_1L_1(R_2+L_2s)}{R_2L_2(R_1L_1Cs^2+L_1s+R_1)}=-\frac{L_1(1+\frac{L_2}{R_2}s)}{L_2(L_1Cs^2+\frac{L_1}{R_1}s+1)}$$

(b) [5 marks] Assume that the circuit has a DC gain of -50, one zero at -1 and two complex conjugate poles with $\omega_n=10$ rd/s, $\zeta=0.5$. Let $L_1=10H$. Find the values of the remaining circuit components L_2, R_1, R_2, C.

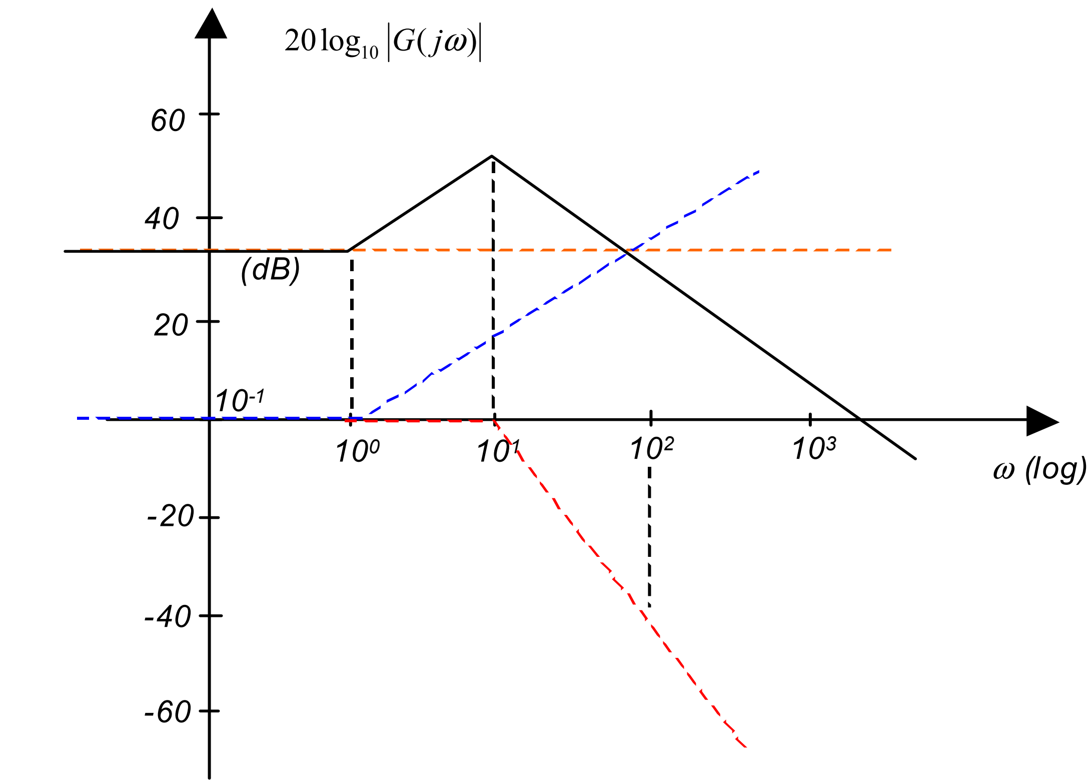
Component values are obtained by setting

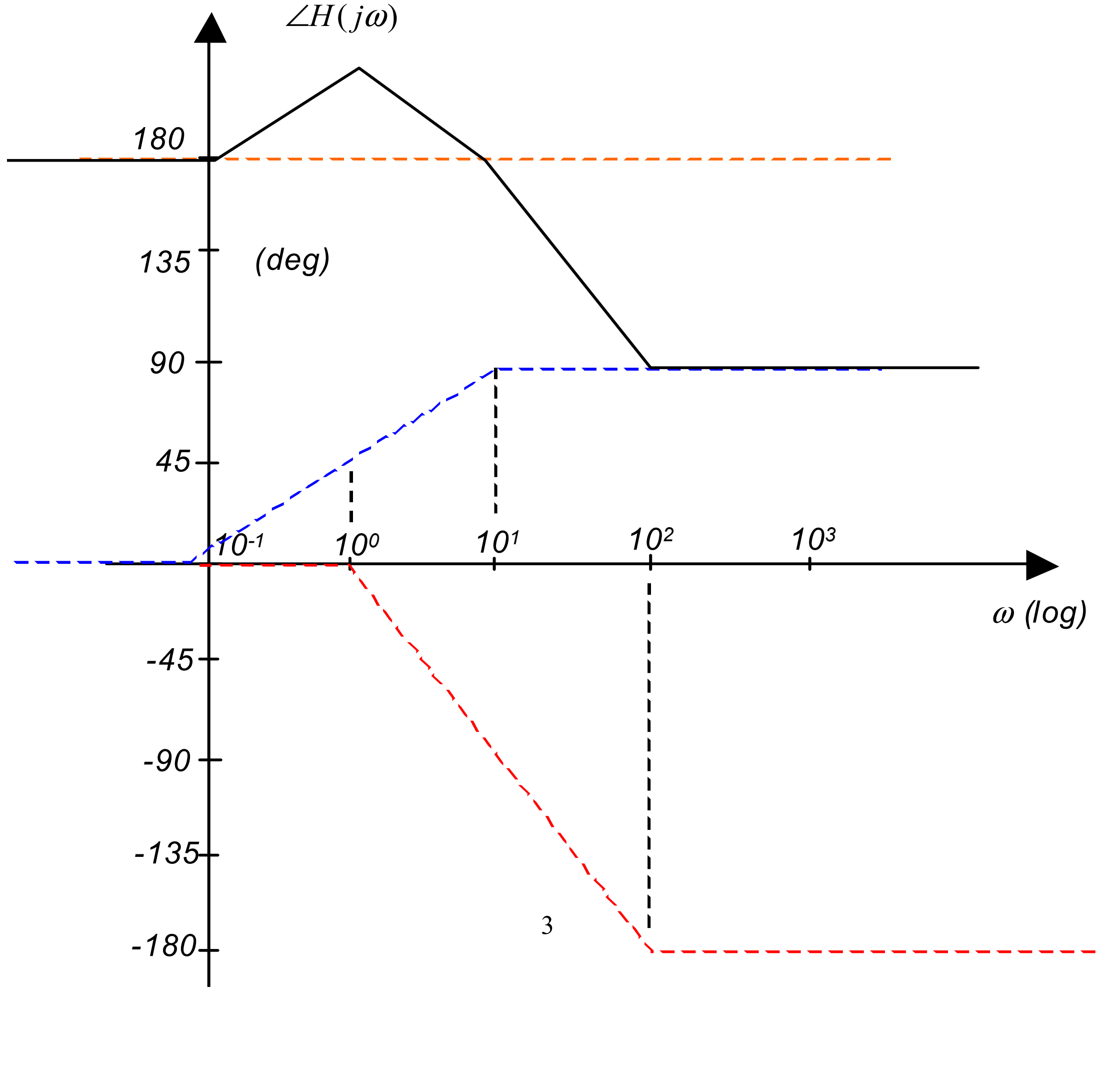
$$H(s)=-50\frac{s+1}{0.01s^2+0.1s+1}=-\frac{L_1}{L_2}\frac{(\frac{L_2}{R_2}s+1)}{(L_1Cs^2+\frac{L_1}{R_1}s+1)}$$

which yields $L_2=0.2H,\ R_1=100\Omega,\ R_2=0.2\Omega,\ C=0.001F$

(c) [7 marks] Give the frequency response of $H(s)$ and sketch its Bode plot.
Answer:

Frequency response is $H(j\omega)=-50\frac{j\omega+1}{0.01(j\omega)^2+0.1(j\omega)+1}$. Bode plot:


$20\log_{10}|G(j\omega)|$
60
40
(dB)
20
10^{-1}
10^0
10^1
10^2
10^3
ω (log)
-20
-40
-60


$\angle H(j\omega)$
180
135
(deg)
90
45
10^{-1}
10^0
10^1
10^2
10^3
ω (log)
-45
-90
-135
-180

Problem 2 (20 marks)

Consider the causal differential system described by

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = 2x(t)$$

and with initial conditions $\frac{dy(0^-)}{dt} = -1, \quad y(0^-) = 2$. Suppose that this system is subjected to the unit step input signal $x(t) = u(t)$.

(a) [8 marks] Find the system's damping ratio ζ and undamped natural frequency ω_n. Give the transfer function of the system and specify its ROC. Sketch its pole-zero plot. Is the system stable? Justify.

Ans:

Let's take the unilateral Laplace transform on both sides of the differential equation.

$$\left[s^2\mathcal{Y}(s) - sy(0^-) - \frac{dy(0^-)}{dt}\right] + 2\left[s\mathcal{Y}(s) - y(0^-)\right] + 2\mathcal{Y}(s) = 2\mathcal{X}(s)$$

Collecting the terms containing $\mathcal{Y}(s)$ on the left-hand side and putting everything else on the right-hand side, we can solve for $\mathcal{Y}(s)$.

$$\left(s^2 + 2s + 2\right)\mathcal{Y}(s) = 2\mathcal{X}(s) + sy(0^-) + 2y(0^-) + \frac{dy(0^-)}{dt}$$

$$\mathcal{Y}(s) = \underbrace{\frac{2\mathcal{X}(s)}{s^2 + 2s + 2}}_{\text{zero-state resp.}} + \underbrace{\frac{(s+2)y(0^-) + \frac{dy(0^-)}{dt}}{s^2 + 2s + 2}}_{\text{zero-input resp.}}$$

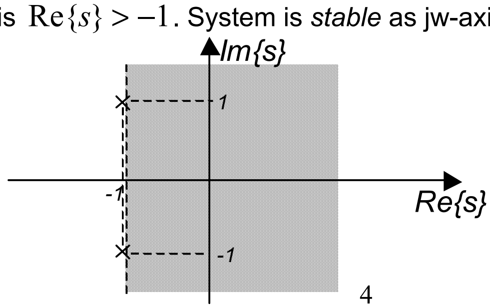
The transfer function is $\mathcal{H}(s) = \frac{2}{s^2 + 2s + 2}$,

and since the system is causal, the ROC is an open RHP to the right of the rightmost pole.

The undamped natural frequency is $\omega_n = \sqrt{2}$ and the damping ratio is $\zeta = \frac{1}{\sqrt{2}}$. The poles are

$$p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -1 \pm j\sqrt{2}\sqrt{1-\frac{1}{2}} = -1 \pm j.$$

Therefore the ROC is $\mathrm{Re}\{s\} > -1$. System is *stable* as jw-axis is contained in ROC. Pole-zero plot:



(b) [8 marks] Compute the step response of the system (including the effect of initial conditions), its steady-state response $y_{ss}(t)$ and its transient response $y_{tr}(t)$ for $t \geq 0$. Identify the zero-state response and the zero-input response in the Laplace domain.

Ans:

The unilateral LT of the input is given by

$$\mathcal{X}(s) = \frac{1}{s}, \quad \mathrm{Re}\{s\} > 0,$$

thus

$$\mathcal{Y}(s) = \underbrace{\frac{2}{\left(s^2+2s+2\right)s}}_{\substack{\mathrm{Re}\{s\}>0 \\ \text{zero-state resp.}}} + \underbrace{\frac{2s+3}{s^2+2s+2}}_{\substack{\mathrm{Re}\{s\}>-1 \\ \text{zero-input resp.}}} = \frac{2s^2+3s+2}{\left(s^2+2s+2\right)s}$$

Let's compute the overall response:

$$\begin{aligned}
\mathcal{Y}(s) &= \frac{2s^2+3s+2}{\left(s^2+2s+2\right)s}, \quad \mathrm{Re}\{s\} > 0 \\
&= \underbrace{\frac{A+B(s+1)}{(s+1)^2+1}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{C}{s}}_{\mathrm{Re}\{s\}>0} \\
&= \underbrace{\frac{A+B(s+1)}{(s+1)^2+1}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{1}{s}}_{\mathrm{Re}\{s\}>0}
\end{aligned}$$

Let $s=-1$ to compute $\frac{1}{-1} = \frac{1}{1}A + \frac{1}{-1} \Rightarrow A = 0$, then multiply both sides by s and let $s \to \infty$ to get $2 = B + 1 \Rightarrow B = 1$:

$$\mathcal{Y}(s) = \underbrace{\frac{(s+1)}{(s+1)^2+1}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{1}{s}}_{\mathrm{Re}\{s\}>0}$$

Notice that the second term $\frac{1}{s}$ is the steady-state response, and thus $y_{ss}(t) = u(t)$.

Taking the inverse Laplace transform using the table yields

$$y(t) = e^{-t}\cos(t)u(t) + u(t).$$

Thus, the transient response is $y_{tr}(t) = e^{-t}\cos(t)u(t)$

(c) [4 marks] Compute the percentage of the first overshoot in the step response of the system assumed this time to be initially at rest.

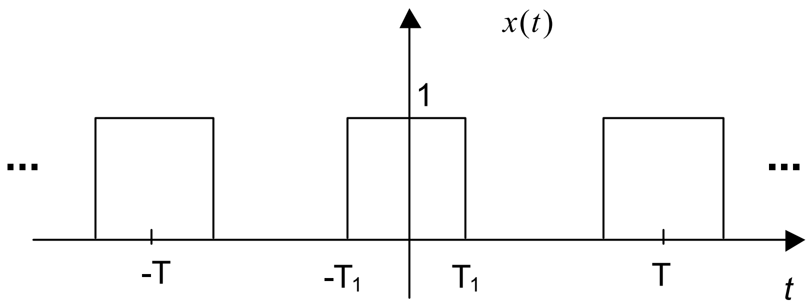
Answer:

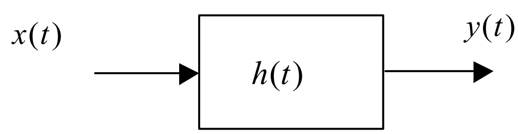
Transfer function is $\mathcal{H}(s) = \dfrac{2}{s^2 + 2s + 2}$, with damping ratio $\zeta = \dfrac{1}{\sqrt{2}}$:

$$OS = 100 e^{-\frac{\varsigma\pi}{\sqrt{1-\varsigma^2}}}\% = 100 e^{-\frac{0.707\pi}{0.707}}\% = 100 e^{-\pi}\% = 4.3\%$$

Problem 3 (15 marks)

Consider the rectangular waveform $x(t)$ of period T and duty cycle $\dfrac{2T_1}{T}$. This signal is the input to an LTI system with impulse response $h(t) = te^{-5t}u(t)$ and output $y(t)$.





(a) [5 marks] Compute the frequency response $H(j\omega)$ of the LTI system using the Fourier integral. Give expressions for its magnitude $|H(j\omega)|$ and phase $\angle H(j\omega)$ as functions of ω.

The frequency response of the system is given by

$$\begin{aligned} H(j\omega) &= \int_0^{+\infty} te^{-5t} e^{-j\omega t}\, dt = \int_0^{+\infty} te^{-(5+j\omega)t}\, dt \\ &= \underbrace{\frac{1}{-(5+j\omega)}\left[te^{-(5+j\omega)t} \right]_0^{+\infty}}_{0} - \frac{1}{-(5+j\omega)} \int_0^{+\infty} e^{-(5+j\omega)t}\, dt \\ &= \frac{1}{-(5+j\omega)^2}\left[e^{-(5+j\omega)t} \right]_0^{+\infty} \\ &= \frac{1}{(5+j\omega)^2} \end{aligned}$$

Magnitude:

$$|H(j\omega)| = \frac{1}{25+\omega^2}$$

Phase:

$$\angle H(j\omega) = 2\,\text{atan}\,2\left(-\omega, 5\right)$$

(b) [3 marks] Find the Fourier series coefficients a_k of the input voltage $x(t)$ for $T = 1s$ and a 40% duty cycle.

The period given corresponds to a signal frequency of 1Hz, and the 40% duty cycle means that $T_1 = \frac{2T}{10}$.

From the lecture notes, we get:

$$a_k = \frac{2}{5}\text{sinc}(\frac{2k}{5})$$

(c) [3 marks] Compute the Fourier series coefficients b_k of the output signal $y(t)$ (for the input described in (b) above), and compute its power spectrum.

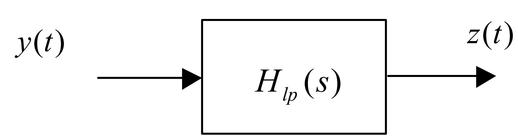
We have $\omega_0 = 2\pi$:

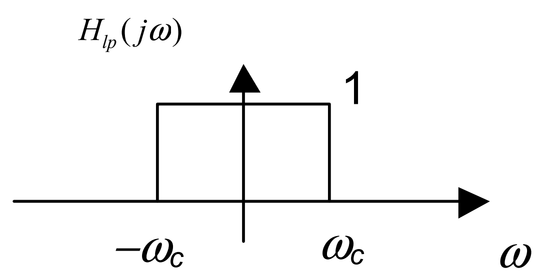
$$b_k = H(jk\omega_0)a_k = \frac{2}{5}\text{sinc}(\frac{2k}{5})\frac{1}{(5+jk\omega_0)^2} = \frac{2}{5}\text{sinc}(\frac{2k}{5})\frac{1}{(5+jk2\pi)^2}$$

The power spectrum of the output signal is

$$|b_k|^2 = \frac{4\text{sinc}^2(\frac{2k}{5})}{25(25+(k2\pi)^2)^2}$$

(d) [4 marks] Compute the total average power P of the output signal $z(t)$ of the following ideal lowpass filter $H_{lp}(s)$, whose input is $y(t)$ of the above system. The frequency response of $H_{lp}(s)$ is shown below, and its cutoff frequency is $\omega_c = 5\pi$ rd/s.





Answer:
The filter keeps only the dc components, the first harmonic components at $\pm 2\pi$ rd/s and the second harmonic components at $\pm 4\pi$ rd/s . Thus,

$$
\begin{aligned}
P &= |b_0|^2 + 2|b_1|^2 + 2|b_2|^2 \\
&= \left.\frac{4\text{sinc}^2(\frac{2k}{5})}{25(25+(k\omega_0)^2)^2}\right|_{k=0} + 2\left.\frac{4\text{sinc}^2(\frac{2k}{5})}{25(25+(k\omega_0)^2)^2}\right|_{k=1} + 2\left.\frac{4\text{sinc}^2(\frac{2k}{5})}{25(25+(k\omega_0)^2)^2}\right|_{k=2} \\
&= \frac{4}{25^3} + 2\frac{4\text{sinc}^2(\frac{2}{5})}{25(25+(2\pi)^2)^2} + 2\frac{4\text{sinc}^2(\frac{4}{5})}{25(25+(4\pi)^2)^2} \\
&= \frac{4}{25^3} + 2\frac{4\sin^2(\frac{2\pi}{5})}{25(\frac{2\pi}{5})^2(25+(2\pi)^2)^2} + 2\frac{4\sin^2(\frac{4\pi}{5})}{25(\frac{4\pi}{5})^2(25+(4\pi)^2)^2} \\
&= \frac{4}{25^3} + 2\frac{\sin^2(\frac{2\pi}{5})}{\pi^2(25+(2\pi)^2)^2} + 2\frac{\sin^2(\frac{4\pi}{5})}{(2\pi)^2(25+(4\pi)^2)^2} = 0.000256 + 0.0000441 + 0.00000052 = 0.0003
\end{aligned}
$$

Problem 4 (20 marks)

System identification
Suppose that you perform two input-output experiments on an LTI system. In the first experiment, the system has nonzero initial conditions at $t = 0$, but the input is set to zero. You record the output in this case to be:

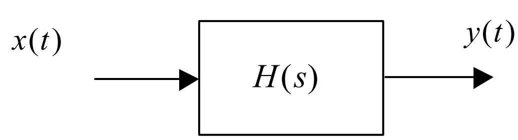
$$y_{zi}(t) = e^{-t}\cos(t - \pi/3)u(t)$$

In the second experiment, the system has the same initial conditions as in the first experiment, but the input is given by:

$$x(t) = e^{-2t}u(t)\ ,$$

and you measured the output to be:

$$y(t) = e^{-t}\left[\cos t + \sin t\right]u(t) - e^{-2t}u(t)\ .$$



(a) [10 marks] Find the transfer function $H(s)$ of the system and its region of convergence. Is the system causal? Is it stable? Justify your answers.

Answer:

First, we have to remove the effect of the initials conditions from the output:

$$y_{zs}(t) = y(t) - y_{zi}(t) = e^{-t}\left[\cos t + \sin t\right]u(t) - e^{-2t}u(t) - e^{-t}\cos(t - \pi/3)u(t)$$
$$= e^{-t}\left[\cos t + \sin t\right]u(t) - e^{-2t}u(t) - e^{-t}\frac{1}{2}(e^{j(t-\pi/3)} + e^{-j(t-\pi/3)})u(t)$$
$$= e^{-t}\left[\cos t + \sin t\right]u(t) - e^{-2t}u(t) - e^{-t}\frac{1}{2}(e^{-j\pi/3}e^{jt} + e^{j\pi/3}e^{-jt})u(t)$$
$$= e^{-t}\left[\cos t + \sin t\right]u(t) - e^{-2t}u(t) - e^{-t}\,\mathrm{Re}(e^{-j\pi/3}e^{jt})u(t)$$
$$= e^{-t}\left[\cos t + \sin t\right]u(t) - e^{-t}[\cos(-\pi/3)\cos t + \sin(\pi/3)\sin t]u(t) - e^{-2t}u(t)$$
$$= e^{-t}\left[\cos t + \sin t\right]u(t) - e^{-t}[0.5\cos t + \frac{\sqrt{3}}{2}\sin t]u(t) - e^{-2t}u(t)$$
$$= e^{-t}\left[0.5\cos t + (1 - \frac{\sqrt{3}}{2})\sin t\right]u(t) - e^{-2t}u(t)$$

Take the Laplace transforms of the input and output signals using the table:

$$X(s) = \frac{1}{(s+2)}, \quad \mathrm{Re}\{s\} > -2$$

$$Y_{zs}(s) = \underbrace{\frac{0.5s + 0.5}{(s+1)^2 + 1^2}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{0.134}{(s+1)^2 + 1^2}}_{\mathrm{Re}\{s\}>-1} - \frac{1}{\underbrace{s+2}_{\mathrm{Re}\{s\}>-2}}$$
$$= \underbrace{\frac{0.5s + 0.634}{(s+1)^2 + 1^2}}_{\mathrm{Re}\{s\}>-1} - \frac{1}{\underbrace{s+2}_{\mathrm{Re}\{s\}>-2}}$$
$$= \frac{(0.5s^2 + 1.634s + 1.268) - (s^2 + 2s + 2)}{(s^2 + 2s + 2)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$
$$= -\frac{0.5s^2 + 0.366s + 0.732}{(s^2 + 2s + 2)(s+2)}, \quad \mathrm{Re}\{s\} > -1$$

Then, the transfer function is obtained as follows:

$$H(s) = \frac{Y_{zs}(s)}{X(s)} = \frac{-\dfrac{0.5s^2 + 0.366s + 0.732}{(s^2 + 2s + 2)(s+2)}}{\dfrac{1}{(s+2)}} = -\frac{0.5s^2 + 0.366s + 0.732}{s^2 + 2s + 2} = -\frac{1}{2}\frac{s^2 + 0.732s + 1.464}{s^2 + 2s + 2}$$

To determine the ROC, first note that the ROC of $Y(s)$ should contain the intersection of the ROC's of $H(s)$ and $X(s)$. There are two possible ROC's for $H(s)$: (a) an open left half-plane to the left of $\mathrm{Re}\{s\} = -1$, (b) an open right half-plane to the right of $\mathrm{Re}\{s\} = -1$. But since the ROC of $X(s)$ is an open right half-plane to the right of $s = -2$, the only possible choice is (b). Hence, the ROC of $H(s)$ is $\mathrm{Re}\{s\} > -1$.

The system is causal as the transfer function is rational and the ROC is a right half-plane. It is also stable as both complex poles $p_{1,2} = -1 \pm j$ are in the open left half-plane.

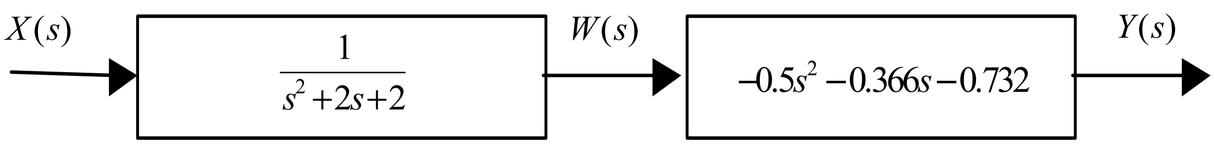
(b) [4 marks] Find an LTI differential equation representing the system.

Answer:
It can be derived from the transfer function obtained in (a):

$$\frac{d^2y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = -0.5\frac{d^2x(t)}{dt^2} - 0.366\frac{dx(t)}{dt} - 0.732x(t)$$

(c) [6 marks] Find the direct form realization of the transfer function $H(s)$.

Answer:
The transfer function can be split up into two systems as follows:



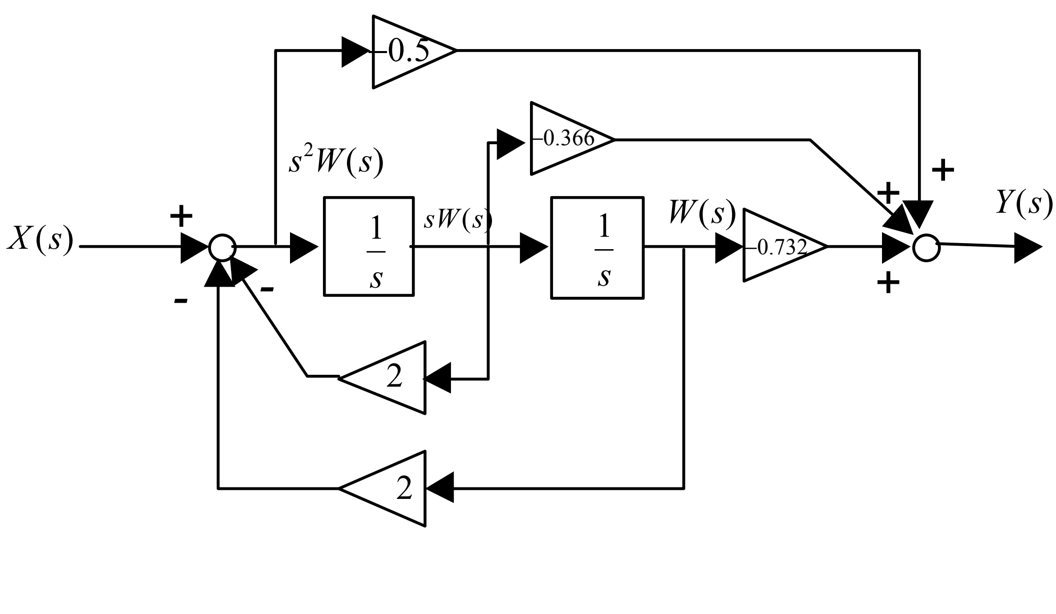
The input-output system equation of the first subsystem is

$$s^2W(s) = -2sW(s) - 2W(s) + X(s),$$

and for the second subsystem we have

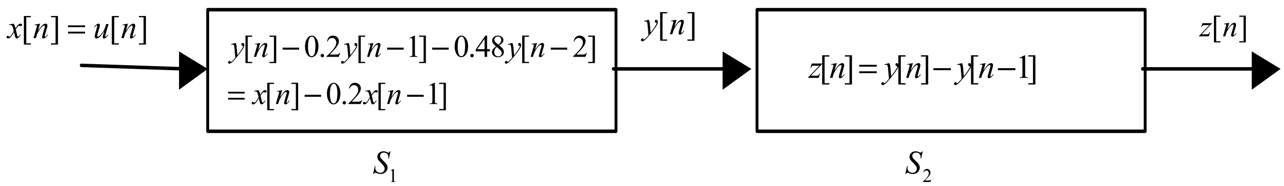
$$Y(s) = -0.5s^2W(s) - 0.366sW(s) - 0.732W(s).$$

The direct form realization of the system is given below:

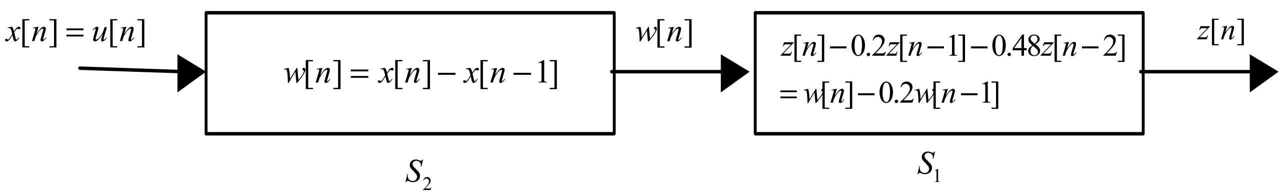


Problem 5 (15 marks)

(a) [12 marks] Compute the step response $z[n]$ of the following causal discrete-time LTI system initially at rest:



Answer:
First, by interchanging the first and second block, we can see that the problem is equivalent to computing the impulse response of S_1:



The intermediate signal is just the unit impulse:

$$w[n] = u[n] - u[n-1] = \delta[n].$$

Hence, the problem reduces to:

$$z[n] - 0.2z[n-1] - 0.48z[n-2] = \delta[n] - 0.2\delta[n-1].$$

We first compute $h_a[n]$:

$$h_a[n] - 0.2h_a[n-1] - 0.48h_a[n-2] = \delta[n]$$

The zeros of the characteristic polynomial are 0.8 and -0.6, so

$$p(z) = (z-0.8)(z+0.6) = z^2 - 0.2z - 0.48$$

homogeneous response for n>0:

$$h_a[n] = A(0.8)^n u[n] + B(-0.6)^n u[n].$$

Initial conditions $h_a[-1] = 0,\ h_a[0] = 1$ lead to

$$h_a[-1] = 0 = 1.250A - 1.667B$$
$$h_a[0] = 1 = A + B$$
$$\Rightarrow A = 0.571,\ B = 0.429$$

Thus,

$$h_a[n] = 0.571(0.8)^n u[n] + 0.429(-0.6)^n u[n],$$

and finally:

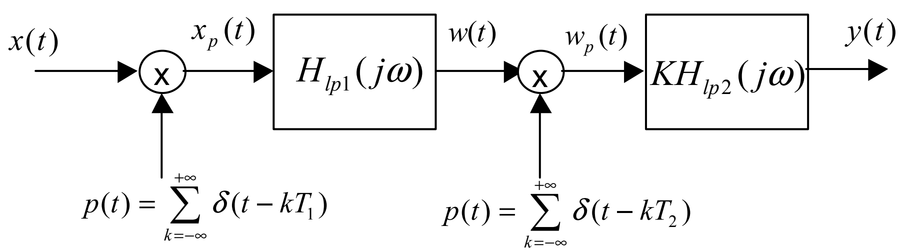
$$
\begin{aligned}
z[n] &= h_a[n] - 0.2h_a[n-1] \\
&= 0.571(0.8)^n u[n] + 0.429(-0.6)^n u[n] - 0.1143(0.8)^{n-1} u[n-1] - 0.0857(-0.6)^{n-1} u[n-1] \\
&= \delta[n] + [0.571 - 0.1143(0.8)^{-1}](0.8)^n u[n-1] + [0.429 - 0.0857(-0.6)^{-1}](-0.6)^n u[n-1] \\
&= \delta[n] + 0.428(0.8)^n u[n-1] + 0.572(-0.6)^n u[n-1] \\
&= 0.428(0.8)^n u[n] + 0.572(-0.6)^n u[n]
\end{aligned}
$$

(b) [3 marks] Compute the impulse response of the system in (a)
Answer:
The impulse response is the first difference of the step response, so:

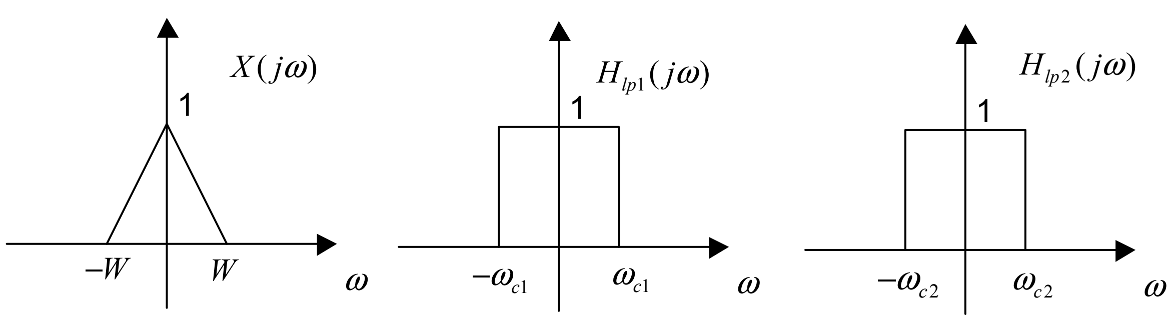
$$
\begin{aligned}
h[n] &= z[n] - z[n-1] \\
&= 0.428(0.8)^n u[n] + 0.572(-0.6)^n u[n] - 0.428(0.8)^{n-1} u[n-1] - 0.572(-0.6)^{n-1} u[n-1]
\end{aligned}
$$

Problem 6 (10 marks)

Consider the following sampling system where the sampling frequencies are $\omega_{s1} = \frac{2\pi}{T_1}$, $\omega_{s2} = \frac{2\pi}{T_2}$.



The spectrum $X(j\omega)$ of the input signal $x(t)$, and the frequency responses of the two ideal lowpass filters, are shown below. The gain of the second lowpass filter is $K > 0$.



(a) [2 marks] For what range of sampling frequencies ω_{s1} is the sampling theorem satisfied for the *first* sampler (from $x(t)$ to $x_p(t)$)?

Answer:

The sampling theorem is satisfied for $\omega_{s1} > 2W$ for the first sampler.

(b) [2 marks] The cutoff frequencies of the lowpass filters are given by $\omega_{c1} = 3W$, $\omega_{c2} = W$. Assume that the lowest admissible sampling frequency is chosen for the first sampler. For what range of sampling frequencies ω_{s2} is the sampling theorem satisfied for the *second* sampler (from $w(t)$ to $w_p(t)$)?

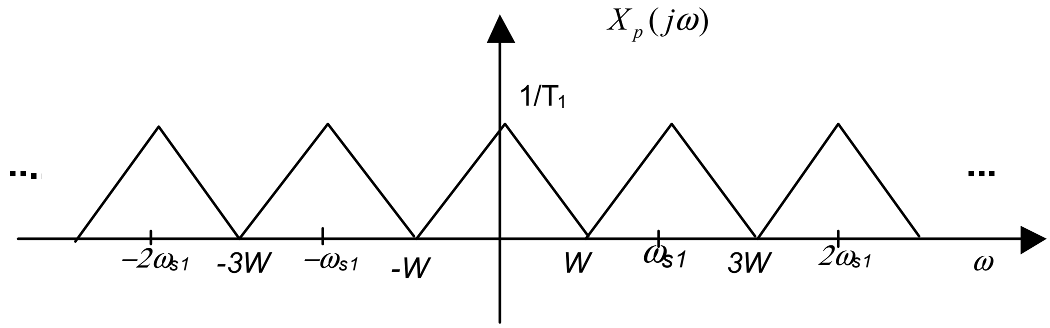
Answer:

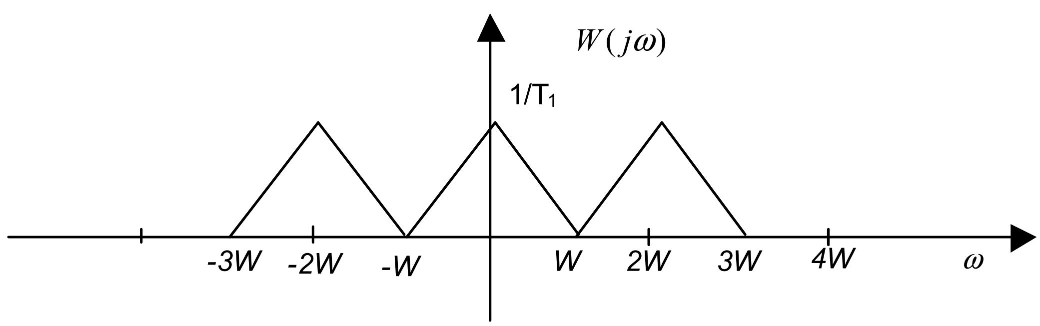
The sampling theorem is satisfied for $\omega_{s2} > 6W$.

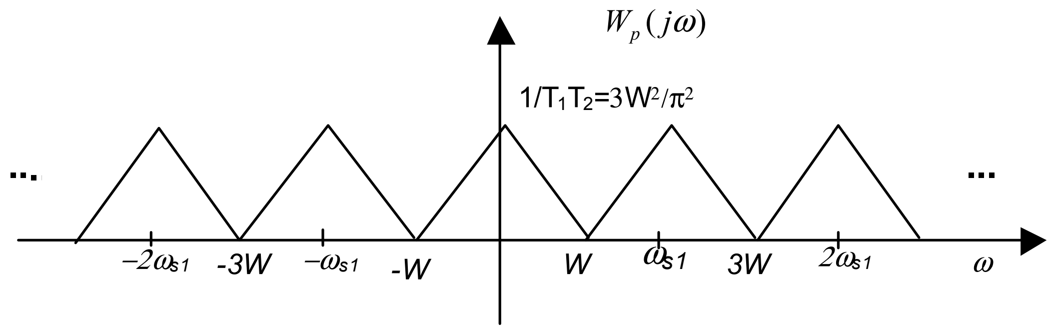
(c) [6 marks] Choosing the lowest sampling frequencies in the ranges that you found for the two samplers, sketch the spectra $X_p(j\omega)$, $W(j\omega)$, $W_p(j\omega)$, and $Y(j\omega)$. Find the gain K of the second filter that leads to $y(t) = x(t)$.

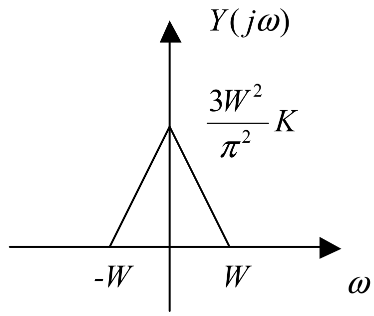
Answer:

For the spectra, we set $\omega_{s1} = 2W$ and $\omega_{s2} = 6W$, so that $T_1 = \frac{\pi}{W}$, $T_2 = \frac{\pi}{3W}$:









Finally, $K = \frac{\pi^2}{3W^2}$.

END OF EXAMINATION