Sample Final Exam (finals00)
Covering Chapters 1-9 of *Fundamentals of Signals & Systems*

Problem 1 (20 marks)

The unit step response of an LTI system was measured to be

$$s(t) = 2e^{-\sqrt{3}t}\sin(t-\frac{\pi}{6})u(t) + u(t) - tu(t).$$

(a) [10 marks] Find the transfer function $H(s)$ of the system. Specify its ROC. Sketch its pole-zero plot.

Answer:

$$H(s) = sS(s) = s\mathcal{L}\left[2e^{-\sqrt{3}t}\sin(t-\frac{\pi}{6})u(t) + u(t) - tu(t)\right]$$

$$= s\mathcal{L}\left[2e^{-\sqrt{3}t}\left(\frac{e^{j(t-\frac{\pi}{6})} - e^{-j(t-\frac{\pi}{6})}}{2j}\right)u(t) + u(t) - tu(t)\right]$$

$$= s\mathcal{L}\left[2e^{-\sqrt{3}t}\left(\frac{(\frac{\sqrt{3}}{2} - j\frac{1}{2})e^{jt} - (\frac{\sqrt{3}}{2} + j\frac{1}{2})e^{-jt}}{2j}\right)u(t) + u(t) - tu(t)\right]$$

$$= s\mathcal{L}\left[\left(\frac{(\frac{\sqrt{3}}{2} - j\frac{1}{2})e^{-\sqrt{3}t+jt} - (\frac{\sqrt{3}}{2} + j\frac{1}{2})e^{-\sqrt{3}t-jt}}{j}\right)u(t) + u(t) - tu(t)\right]$$

$$= s\mathcal{L}\left[\left(\frac{(\frac{\sqrt{3}}{2}e^{-\sqrt{3}t+jt} - \frac{\sqrt{3}}{2}e^{-\sqrt{3}t-jt}) - j\frac{1}{2}e^{-\sqrt{3}t+jt} - j\frac{1}{2}e^{-\sqrt{3}t-jt}}{j}\right)u(t) + u(t) - tu(t)\right]$$

$$= s\mathcal{L}\left[\left(\sqrt{3}e^{-\sqrt{3}t}\sin t - e^{-\sqrt{3}t}\cos t\right)u(t) + u(t) - tu(t)\right]$$

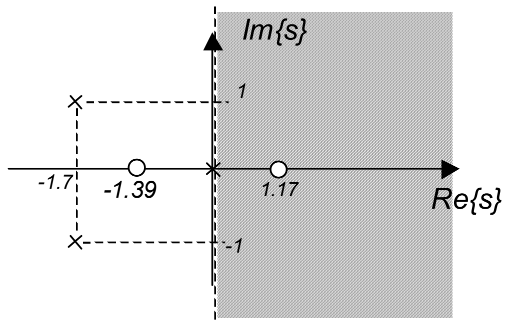
$$= s\left[\frac{\sqrt{3}}{(s+\sqrt{3})^2+1} - \frac{s+\sqrt{3}}{(s+\sqrt{3})^2+1} + \frac{1}{s} - \frac{1}{s^2}\right]$$

$$= \frac{-s^3 + [(s+\sqrt{3})^2+1](s-1)}{s[(s+\sqrt{3})^2+1]} = \frac{-s^3 + [s^2+2\sqrt{3}s+4](s-1)}{s[(s+\sqrt{3})^2+1]}$$

$$= \frac{(2\sqrt{3}-1)s^2 + (4-2\sqrt{3})s - 4}{s[(s+\sqrt{3})^2+1]} = \frac{(2\sqrt{3}-1)\left[s^2 + \frac{(4-2\sqrt{3})}{(2\sqrt{3}-1)}s - \frac{4}{(2\sqrt{3}-1)}\right]}{s[(s+\sqrt{3})^2+1]}$$

ROC: $\mathrm{Re}\{s\} > 0$

Poles are $p_1 = 0$, $p_2 = -\sqrt{3} + j$, $p_3 = -\sqrt{3} - j$, Zeros are zeros of $s^2 + 0.21748s - 1.62331$: $z_1 = -1.3875,$ $z_2 = 1.1700$ $z_3 = \infty$



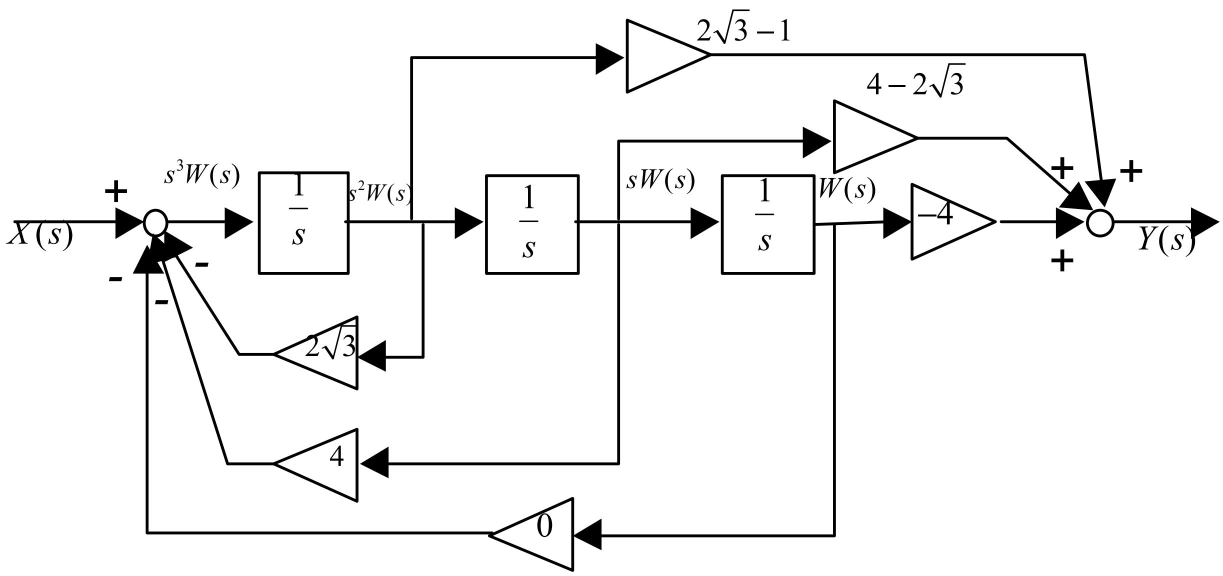
(b) [4 marks] Is the system causal? Is it stable? Justify your answers.

Answer:
System is causal: ROC is an open RHP and transfer function is rational.
This system isn't stable as ROC doesn't include the imaginary axis (or because rightmost pole 0 has a nonnegative real part.)

(c) [6 marks] Give the direct form realization (block diagram) of $H(s)$.

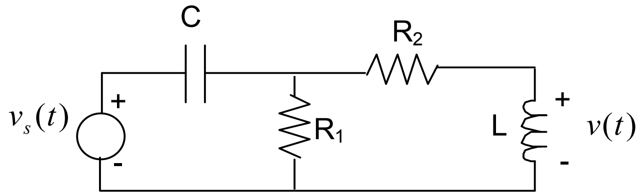
Answer:

$$H(s) = \frac{(2\sqrt{3}-1)s^2 + (4-2\sqrt{3})s - 4}{s^3 + 2\sqrt{3}s^2 + 4s}$$



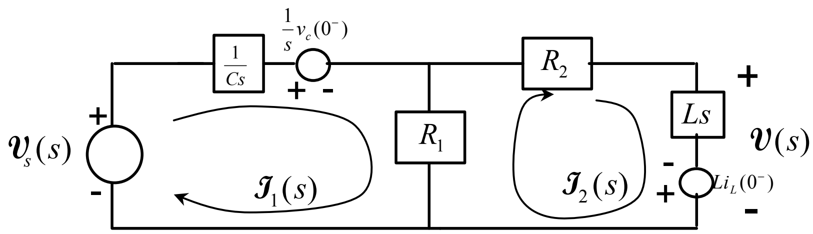
Problem 2 (20 marks)

The following circuit has initial conditions on the capacitor $v_C(0^-)$ and inductor $i_L(0^-)$.



(a) [4 marks] Transform the circuit using the unilateral Laplace transform.

Answer:



(b) [8 marks] Find the unilateral Laplace transform of $v(t)$.

Answer:
Let's use mesh analysis.
For mesh 1:

$$\mathcal{V}_s(s) - \frac{1}{Cs}\mathcal{I}_1(s) - \frac{1}{s}v_c(0^-) - R_1[\mathcal{I}_1(s) - \mathcal{I}_2(s)] = 0$$

$$\Rightarrow \quad \mathcal{I}_2(s) = -\frac{1}{R_1}\mathcal{V}_s(s) + \frac{1}{R_1 s}v_c(0^-) + (1 + \frac{1}{R_1 Cs})\mathcal{I}_1(s)$$

For mesh 2:

$$R_1[\mathcal{I}_1(s) - \mathcal{I}_2(s)] - (R_2 + Ls)\mathcal{I}_2(s) + Li_L(0^-) = 0$$

$$\Rightarrow \quad \mathcal{I}_1(s) = \frac{1}{R_1}(R_1 + R_2 + Ls)\mathcal{I}_2(s) - \frac{L}{R_1}i_L(0^-)$$

Substituting, we obtain

$$\mathcal{I}_2(s) = -\frac{1}{R_1}\mathcal{V}_s(s) + \frac{1}{R_1 s}v_c(0^-) + (1 + \frac{1}{R_1 Cs})\left[\frac{1}{R_1}(R_1 + R_2 + Ls)\mathcal{I}_2(s) - \frac{L}{R_1}i_L(0^-)\right]$$

$$[R_1^2 Cs - (1 + R_1 Cs)(R_1 + R_2 + Ls)]\mathcal{I}_2(s) = -R_1 Cs\mathcal{V}_s(s) + R_1 Cv_c(0^-) - (1 + R_1 Cs)Li_L(0^-)$$

$$-[LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2]\mathcal{I}_2(s) = -R_1 Cs\mathcal{V}_s(s) + R_1 Cv_c(0^-) - (1 + R_1 Cs)Li_L(0^-)$$

Solving for $\mathcal{I}_2(s)$, we get

$$\mathcal{I}_2(s) = \frac{R_1 C s \mathcal{V}_s(s)}{L R_1 C s^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{(1 + R_1 C s) L i_L(0^-) - R_1 C v_c(0^-)}{L R_1 C s^2 + (L + R_1 R_2 C)s + R_1 + R_2}$$

And finally the output voltage is

$$\mathcal{V}(s) = L s \mathcal{I}_2(s) - L i_L(0^-) = \frac{L R_1 C s^2 \mathcal{V}_s(s)}{L R_1 C s^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{(1 + R_1 C s) L^2 s i_L(0^-) - R_1 C L s v_c(0^-)}{L R_1 C s^2 + (L + R_1 R_2 C)s + R_1 + R_2} - L i_L(0^-)$$

$$= \frac{L R_1 C s^2 \mathcal{V}_s(s)}{L R_1 C s^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{\{(1 + R_1 C s) L^2 s - [L^2 R_1 C s^2 + L(L + R_1 R_2 C)s + L(R_1 + R_2)]\} i_L(0^-) - R_1 C L s v_c(0^-)}{L R_1 C s^2 + (L + R_1 R_2 C)s + R_1 + R_2}$$

$$= \frac{L R_1 C s^2 \mathcal{V}_s(s)}{L R_1 C s^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{-[L R_1 R_2 C s + L(R_1 + R_2)] i_L(0^-) - R_1 C L s v_c(0^-)}{L R_1 C s^2 + (L + R_1 R_2 C)s + R_1 + R_2}$$

(c) [8 marks] Draw the Bode plot (magnitude and phase) of the frequency response from the input voltage $\mathcal{V}_s(j\omega)$ to the output voltage $\mathcal{V}(j\omega)$. Assume that the initial conditions on the capacitor and the inductor are 0. Use the numerical values: $R_1 = 1\,\Omega,\ R_2 = \frac{109}{891}\,\Omega,\ L = \frac{1}{891}\,\mathrm{H},\ C = 1\,\mathrm{F}$.

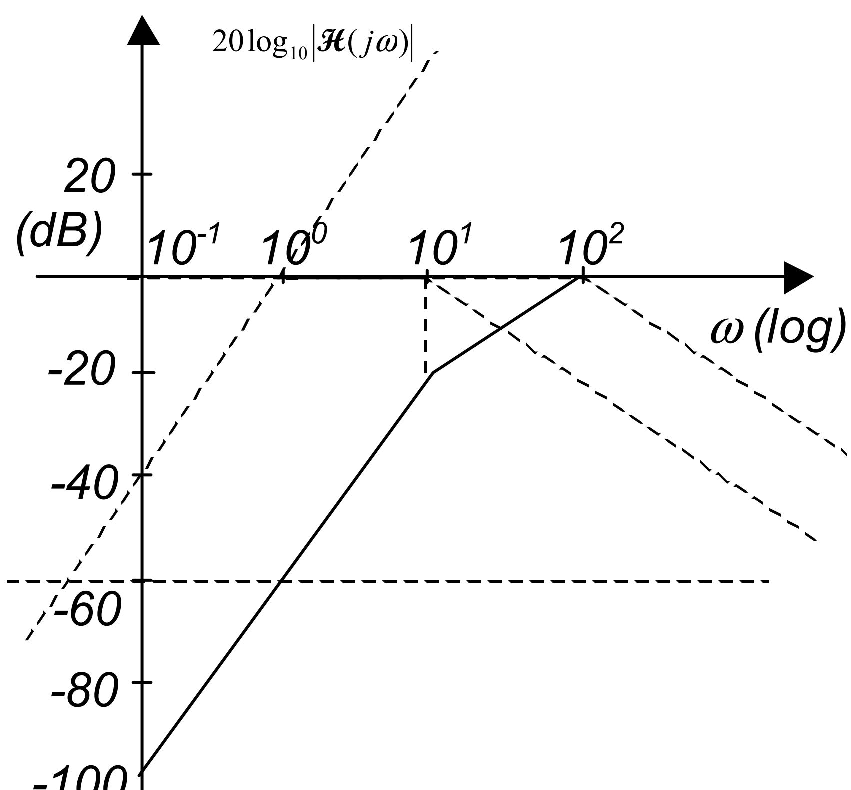
Answer:

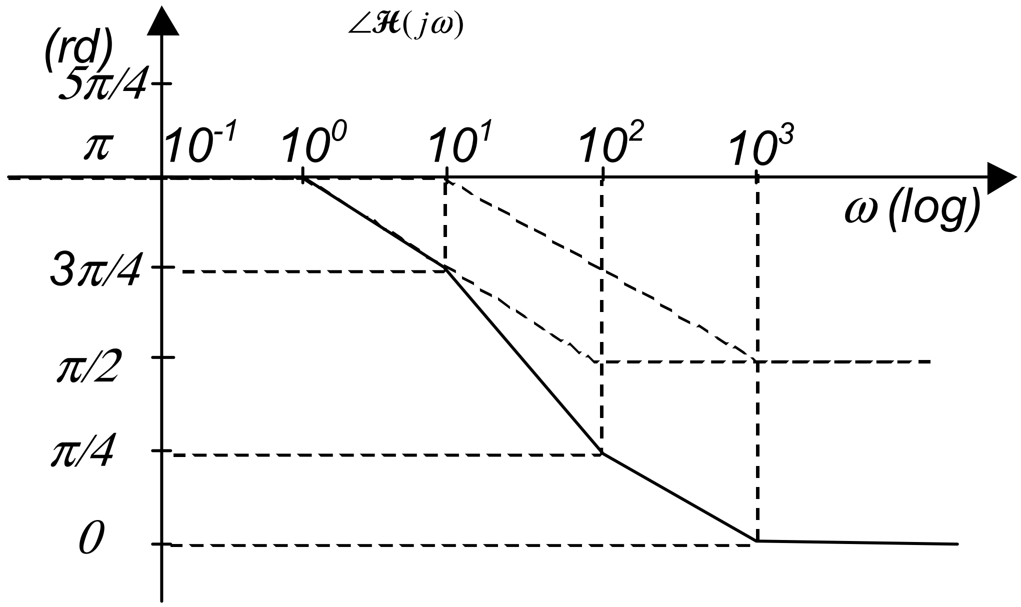
For the values given, the transfer function from the source voltage to the output voltage is

$$\mathcal{H}(s) := \frac{\mathcal{V}(s)}{\mathcal{V}_s(s)} = \frac{s^2}{s^2 + \frac{L + R_1 R_2 C}{L R_1 C}s + \frac{R_1 + R_2}{L R_1 C}} = \frac{s^2}{s^2 + \frac{\frac{1}{891} + \frac{109}{891}}{\frac{1}{891}}s + \frac{1 + \frac{109}{891}}{\frac{1}{891}}}$$

$$= \frac{s^2}{s^2 + 110 s + 1000} = \frac{s^2}{(s + 10)(s + 100)} = \frac{1}{1000}\frac{s^2}{(\frac{1}{10}s + 1)(\frac{1}{100}s + 1)}$$

Bode Plot:





Problem 3 (15 marks)

Consider the causal differential system described by

$$\frac{d^2 y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 4y(t) = 4\frac{dx(t)}{dt} + 4x(t)$$

and with initial conditions $\frac{dy(0^-)}{dt} = -2, \quad y(0^-) = -4$. Suppose that this system is subjected to the input signal

$$x(t) = u(t).$$

Find the system's damping ratio ζ and undamped natural frequency ω_n. Give the transfer function of the system and specify its ROC. Compute the steady-state response $y_{ss}(t)$ and the transient response $y_{tr}(t)$ for $t \geq 0$.

Answer:

Let's take the unilateral Laplace transform on both sides of the differential equation.

$$\left[s^2\mathcal{Y}(s) - sy(0^-) - \frac{dy(0^-)}{dt} \right] + 2\left[s\mathcal{Y}(s) - y(0^-) \right] + 4\mathcal{Y}(s) = 4s\mathcal{X}(s) + 4\mathcal{X}(s)$$

Collecting the terms containing $\mathcal{Y}(s)$ on the left-hand side and putting everything else on the right-hand side, we can solve for $\mathcal{Y}(s)$.

$$\left(s^2+2s+4\right)\mathcal{Y}(s)=4s\mathcal{X}(s)+4\mathcal{X}(s)+sy(0^-)+2y(0^-)+\frac{dy(0^-)}{dt}$$

$$\mathcal{Y}(s)=\underbrace{\frac{4(s+1)\mathcal{X}(s)}{s^2+2s+4}}_{\text{zero-state resp.}}+\underbrace{\frac{(s+2)y(0^-)+\frac{dy(0^-)}{dt}}{s^2+2s+4}}_{\text{zero-input resp.}}$$

The transfer function is $\frac{4(s+1)}{s^2+2s+4}$

and since the system is causal, the ROC is an open RHP to the right of the rightmost pole.

The undamped natural frequency is $\omega_n=2$ and the damping ratio is $\zeta=0.5$. The poles are $p_{1,2}=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}=-1\pm j\sqrt{3}$.

Therefore the ROC is $\mathrm{Re}\{s\}>-1$.

The unilateral LT of the input is given by

$$\mathcal{X}(s)=\frac{1}{s},\quad \mathrm{Re}\{s\}>0,$$

thus

$$\mathcal{Y}(s)=\underbrace{\frac{4\left(s+1\right)}{\left(s^2+2s+4\right)s}}_{\substack{\mathrm{Re}\{s\}>0\\ \text{zero-state resp.}}}+\underbrace{\frac{-4s-10}{s^2+2s+4}}_{\substack{\mathrm{Re}\{s\}>-1\\ \text{zero-input resp.}}}=\frac{-4s^2-6s+4}{\left(s^2+2s+4\right)s}$$

Let's compute the overall response:

$$\begin{aligned}\mathcal{Y}(s)&=\frac{-4s^2-6s+4}{\left(s^2+2s+4\right)s},\quad \mathrm{Re}\{s\}>0\\ &=\underbrace{\frac{A\sqrt{3}+B(s+1)}{\left(s+1\right)^2+3}}_{\mathrm{Re}\{s\}>-1}+\underbrace{\frac{C}{s}}_{\mathrm{Re}\{s\}>0}\\ &=\underbrace{\frac{A\sqrt{3}+B(s+1)}{\left(s+1\right)^2+3}}_{\mathrm{Re}\{s\}>-1}+\underbrace{\frac{1}{s}}_{\mathrm{Re}\{s\}>0}\end{aligned}$$

Let $s=-1$ to compute $\frac{6}{-3}=\frac{1}{\sqrt{3}}A+\frac{1}{-1}\Rightarrow A=-\sqrt{3}$, then multiply both sides by s and let $s\to\infty$ to get $-4=B+1\Rightarrow B=-5$:

$$\mathcal{Y}(s) = \underbrace{\frac{-\sqrt{3}(\sqrt{3}) - 5(s+1)}{(s+1)^2 + 3}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{1}{s}}_{\mathrm{Re}\{s\}>0}$$

Notice that the second term $\frac{1}{s}$ is the steady-state response, and thus $y_{ss}(t) = u(t)$.

Taking the inverse Laplace transform using the table yields

$$y_{tr}(t) = \left[-\sqrt{3}e^{-t}\sin(\sqrt{3}t) - 5e^{-t}\cos(\sqrt{3}t) \right] u(t).$$

Problem 4 (10 marks)

Consider the following second-order, causal difference LTI system S initially at rest:

$$S: \qquad 2y[n] - 1.8y[n-1] + 0.4y[n-2] = x[n] - x[n-2]$$

(a) [4 marks] What is the characteristic polynomial of S? What are its zeros? Is the system stable? Justify your answer.

Answer:
Let's rewrite the difference equation as

$$S: \qquad y[n] - 0.9y[n-1] + 0.2y[n-2] = 0.5x[n] - 0.5x[n-2]$$

$p(z) = z^2 - 0.9z + 0.2 = (z-0.4)(z-0.5)$

The zeros are $z_1 = 0.4,\ z_2 = 0.5$.

(b) [6 marks] Compute the impulse response of S for all *n*.
Answer:
The homogeneous response is given by

$$y[n] = A(0.4)^n + B(0.5)^n,\ \ n > 0.$$

$$h_a[n] - 0.9h_a[n-1] + 0.2h_a[n-2] = \delta[n]$$

The initial conditions for the homogeneous equation for n>0

are $h_a[-1] = 0$ and $h_a[0] = \delta[0] = 1$.
Now we can compute the coefficient A and B:

$$y[-1] = A(0.4)^{-1} + B(0.5)^{-1} = 2.5A + 2B = 0$$
$$y[0] = A + B = 1$$

Hence

$$A = -4, \quad B = 5$$

and the intermediate impulse response is

$$h_a[n] = \left[-4(0.4)^n + 5(0.5)^n \right] u[n]$$

Finally, the impulse response is

$$h[n] = 0.5h_a[n] - 0.5h_a[n-2] = \left[-2(0.4)^n + 2.5(0.5)^n \right] u[n] - \left[-2(0.4)^{n-2} + 2.5(0.5)^{n-2} \right] u[n-2]$$

Problem 5 (5 marks)

True or False?

(a) The Fourier transform $Z(j\omega)$ of the product of a real even signal $x(t)$ and a real odd signal $y(t)$ is real odd.

Answer: False.

(b) The system defined by $y(t) = \int_0^t x(\tau)d\tau$ is time-invariant.

Answer: False.

(c) The Fourier series coefficients a_k of a real and even periodic signal $x(t)$ have the following property: $a_k^* = a_{-k}$.

Answer: True.

(d) The Fourier transform of the convolution of a real even signal with the impulse $\delta(t-1)$ is real.

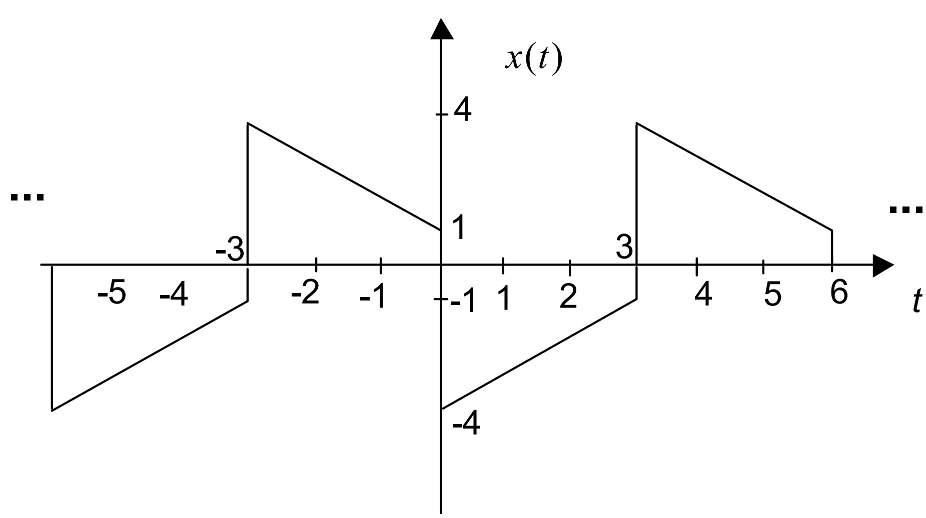
Answer: False.

(e) The fundamental period of the signal $x[n] = \sin(\frac{3\pi}{7}n)e^{j\pi n}$ is 14.

Answer: True.

Problem 6 (15 marks)

(a) [12 marks] Consider the periodic signal $x(t)$ depicted below. Give a mathematical expression for $x(t)$. Find its fundamental frequency ω_0. Compute its Fourier series coefficients a_k. Express $x(t)$ as a Fourier series.



Answer:

This signal can be written as:

$$x(t)=\sum_{m=-\infty}^{+\infty}-((t-6m)-1)\left[u(t+3-6m)-u(t-6m)\right]+((t-6m)-4)\left[u(t-6m)-u(t-3-6m)\right]$$

Its fundamental period and frequency are $T=6,\ \omega_0=\frac{2\pi}{6}=\frac{\pi}{3}$. The average value over one period is given by:

$$a_0=\frac{1}{6}\int_0^6 x(t)dt=0$$

The FS coefficients a_k for $k\neq 0$ are given by

$$\begin{aligned}
a_k&=\frac{1}{T}\int_T x(t)e^{-jk\frac{\pi}{3}t}dt\\
&=\frac{1}{6}\int_{-3}^{0}(1-t)e^{-jk\frac{\pi}{3}t}dt+\frac{1}{6}\int_{0}^{3}(t-4)e^{-jk\frac{\pi}{3}t}dt\\
&=\frac{1}{-6jk\frac{\pi}{3}}\left[(1-t)e^{-jk\frac{\pi}{3}t}\right]_{-3}^{0}-\frac{1}{6jk\frac{\pi}{3}}\int_{-3}^{0}e^{-jk\frac{\pi}{3}t}dt+\frac{1}{-6jk\frac{\pi}{3}}\left[(t-4)e^{-jk\frac{\pi}{3}t}\right]_{0}^{3}+\frac{1}{6jk\frac{\pi}{3}}\int_{0}^{3}e^{-jk\frac{\pi}{3}t}dt\\
&=\frac{1}{-jk2\pi}\left[1-4e^{jk\pi}\right]+\frac{1}{6\left(jk\frac{\pi}{3}\right)^2}\left[e^{-jk\frac{\pi}{3}t}\right]_{-3}^{0}+\frac{1}{-jk2\pi}\left[-e^{-jk\pi}+4\right]+\frac{1}{-6\left(jk\frac{\pi}{3}\right)^2}\left[e^{-jk\frac{\pi}{3}t}\right]_{0}^{3}\\
&=\frac{1}{-jk2\pi}\left[5-5e^{jk\pi}\right]-\frac{3}{2k^2\pi^2}\left[1-e^{jk\pi}\right]-\frac{3}{2k^2\pi^2}\left[1-e^{-jk\pi}\right]\\
&=\frac{j5}{k2\pi}\left[1-e^{jk\pi}\right]-\frac{3}{k^2\pi^2}\left[1-e^{jk\pi}\right]\\
&=\left(-\frac{3}{k^2\pi^2}+j\frac{5}{k2\pi}\right)\left[1-(-1)^k\right]
\end{aligned}$$

The Fourier series representation of $x(t)$ is

$$x(t)=\sum_{k=-\infty}^{+\infty}a_k e^{jk\frac{\pi}{3}t}=\sum_{\substack{k=-\infty\\k\neq 0}}^{+\infty}\left(-\frac{3}{k^2\pi^2}+j\frac{5}{k2\pi}\right)\left[1-(-1)^k\right]e^{jk\frac{\pi}{3}t}.$$

(b) [3 marks] Compute the Fourier transform of $x(t)$.

Answer:

$$X(j\omega) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$
$$= \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \left(-\frac{6}{k^2\pi} + j\frac{5}{k}\right)\left[1-(-1)^k\right]\delta(\omega - k\frac{\pi}{3})$$

Problem 7 (15 marks)

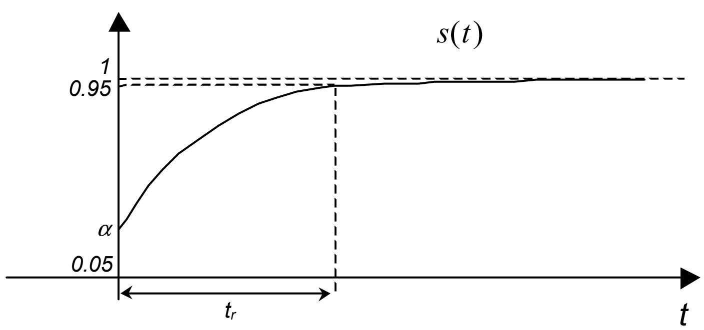
(a) [8 marks] Compute the 95% rise time of the unit step response $s(t)$ of $H(s) = \frac{0.01s+1}{0.1s+1},\ \operatorname{Re}\{s\} > -10$. Sketch $s(t)$, indicating the important features.

Answer:

$$H(s) = \frac{0.01s+1}{0.1s+1} = \frac{\alpha\tau s+1}{\tau s+1},\ \operatorname{Re}\{s\} > -10, \alpha = 0.1, \tau = 0.1$$

Notice that $\alpha > 0.05$. The step response is

$s(t) = \alpha u(t) + (1-\alpha)(1-e^{-\frac{t}{\tau}})u(t)$.



The 95% rise time is given by the difference between the times when the response reaches the value 0.95 and the value 0.05. Since $\alpha > 0.05$,the output is already larger than 0.05 at $t = 0^+$.

$$0.95 = \alpha + (1-\alpha)(1-e^{-\frac{t_{95\%}}{\tau}})$$
$$\Rightarrow \quad t_{95\%} = -\tau(\ln 0.05 - \ln(1-\alpha)) = [2.9957 + \ln(1-\alpha)]\tau = [2.9957 + \ln(0.9)]0.1 = 0.289$$
$$\Rightarrow \quad t_s = t_{95\%} = 0.289$$

(b) [7 marks] For the stable, causal second-order system $H(s) = \frac{3}{0.02s^2 + as + 2}$, find the value of the real parameter a that will cause the system to have a 10% first overshoot in its step response.

Answer:

$$H(s) = \frac{3}{0.02s^2 + as + 2} = \frac{1.5}{0.01s^2 + 0.5as + 1} = \frac{1.5}{\omega_n^{-2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

$$\Rightarrow \omega_n = 10, \ \ a = \frac{4\zeta}{\omega_n} = 0.4\zeta$$

For a 10% first overshoot:

$$10 = 100e^{-\frac{\varsigma\pi}{\sqrt{1-\varsigma^2}}} \Rightarrow e^{-\frac{\varsigma\pi}{\sqrt{1-\varsigma^2}}} = 0.1$$

$$-\frac{\varsigma\pi}{\sqrt{1-\varsigma^2}} = \ln 0.1 = -2.3026$$

$$\varsigma^2 = \frac{(2.3026)^2}{\pi^2}(1-\varsigma^2)$$

$$\Rightarrow \varsigma^2\left(1 + \frac{(2.3026)^2}{\pi^2}\right) = \frac{(2.3026)^2}{\pi^2}$$

$$\Rightarrow \varsigma^2 = \frac{\frac{(2.3026)^2}{\pi^2}}{1 + \frac{(2.3026)^2}{\pi^2}} = \frac{(2.3026)^2}{\pi^2 + (2.3026)^2} = 0.3495$$

$$\Rightarrow \varsigma = 0.59$$

Therefore $a = 0.4\zeta = 0.4(0.59) = 0.24$.

END OF EXAMINATION