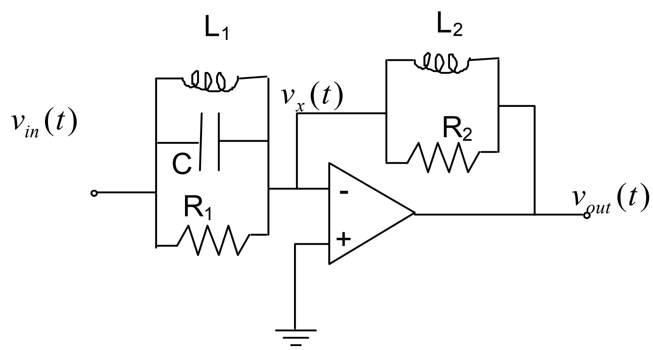
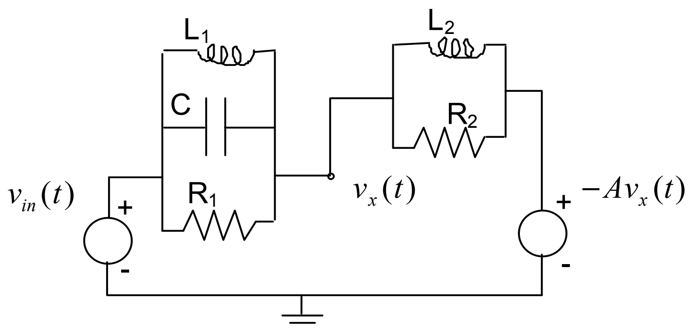
Sample Final Exam (finals01)
Covering Chapters 1-9 of *Fundamentals of Signals & Systems*

Problem 1 (20 marks)

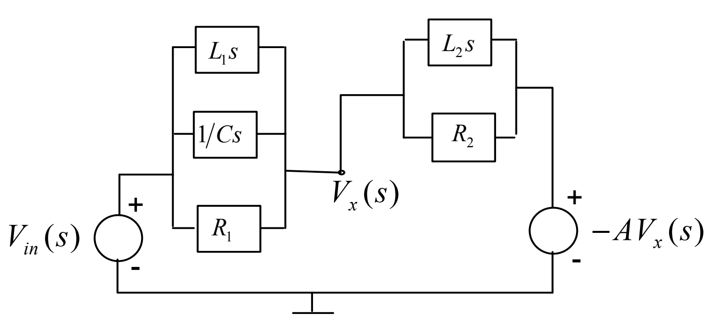
Consider the causal op-amp circuit initially at rest depicted below. Its LTI circuit model with a voltage-controlled source is also given below.

(a) [8 marks] Transform the circuit using the Laplace transform, and find the transfer function $H_A(s) = V_{out}(s)/V_{in}(s)$. Then, let the op-amp gain $A \to +\infty$ to obtain the ideal transfer function $H(s) = \lim_{A\to+\infty} H_A(s)$.





Answer:
The transformed circuit is



There are two supernodes for which the nodal voltages are given by the source voltages. The remaining nodal equation is

$$\frac{V_{in}(s)-V_x(s)}{R_1\left\|\frac{1}{Cs}\right\|L_1 s}+\frac{-AV_x(s)-V_x(s)}{R_2\|L_2 s}=0$$

where $R_1\left\|\frac{1}{Cs}\right\|L_1 s=\frac{1}{Cs+\frac{1}{R_1}+\frac{1}{L_1 s}}=\frac{R_1 L_1 s}{R_1 L_1 Cs^2+L_1 s+R_1}$ and $R_2\|L_2 s=\frac{R_2 L_2 s}{R_2+L_2 s}$. Simplifying the above equation, we get:

$$\frac{R_1 L_1 Cs^2+L_1 s+R_1}{R_1 L_1 s}V_{in}(s)-\left[\frac{(A+1)(R_2+L_2 s)}{R_2 L_2 s}+\frac{R_1 L_1 Cs^2+L_1 s+R_1}{R_1 L_1 s}\right]V_x(s)=0$$

Thus, the transfer function between the input voltage and the node voltage is given by

$$\frac{V_x(s)}{V_{in}(s)}=\frac{\frac{R_1 L_1 Cs^2+L_1 s+R_1}{R_1 L_1 s}}{\frac{(A+1)(R_2+L_2 s)}{R_2 L_2 s}+\frac{R_1 L_1 Cs^2+L_1 s+R_1}{R_1 L_1 s}}.$$

$$=\frac{R_2 L_2 s(R_1 L_1 Cs^2+L_1 s+R_1)}{R_1 L_1 s(A+1)(R_2+L_2 s)+R_2 L_2 s(R_1 L_1 Cs^2+L_1 s+R_1)}$$

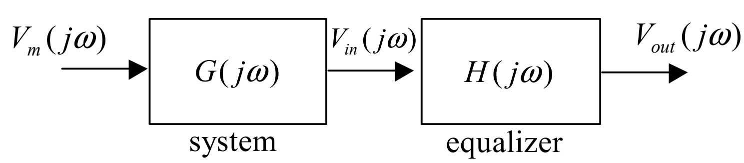
The transfer function between the input voltage and the output voltage is

$$H_A(s)=\frac{V_{out}(s)}{V_{in}(s)}=\frac{-AV_x(s)}{V_{in}(s)}=\frac{-AR_2 L_2 s(R_1 L_1 Cs^2+L_1 s+R_1)}{R_1 L_1 s(A+1)(R_2+L_2 s)+R_2 L_2 s(R_1 L_1 Cs^2+L_1 s+R_1)}$$

The ideal transfer function is the limit as the op-amp gain tends to infinity:

$$H(s)=\lim_{A\to\infty}H_A(s)=-\frac{R_2 L_2(R_1 L_1 Cs^2+L_1 s+R_1)}{R_1 L_1(R_2+L_2 s)}=-\frac{L_2(L_1 Cs^2+\frac{L_1}{R_1}s+1)}{L_1(1+\frac{L_2}{R_2}s)}$$

(b) [5 marks] The circuit is used as a cascade equalizer for the system $G(s)=-50\frac{s+1}{0.01s^2+0.1s+1}$, that is, $|G(j\omega)H(j\omega)|=0dB,\ \forall\omega$. Let $L_1=10H$. Find the values of the remaining circuit components L_2, R_1, R_2, C.



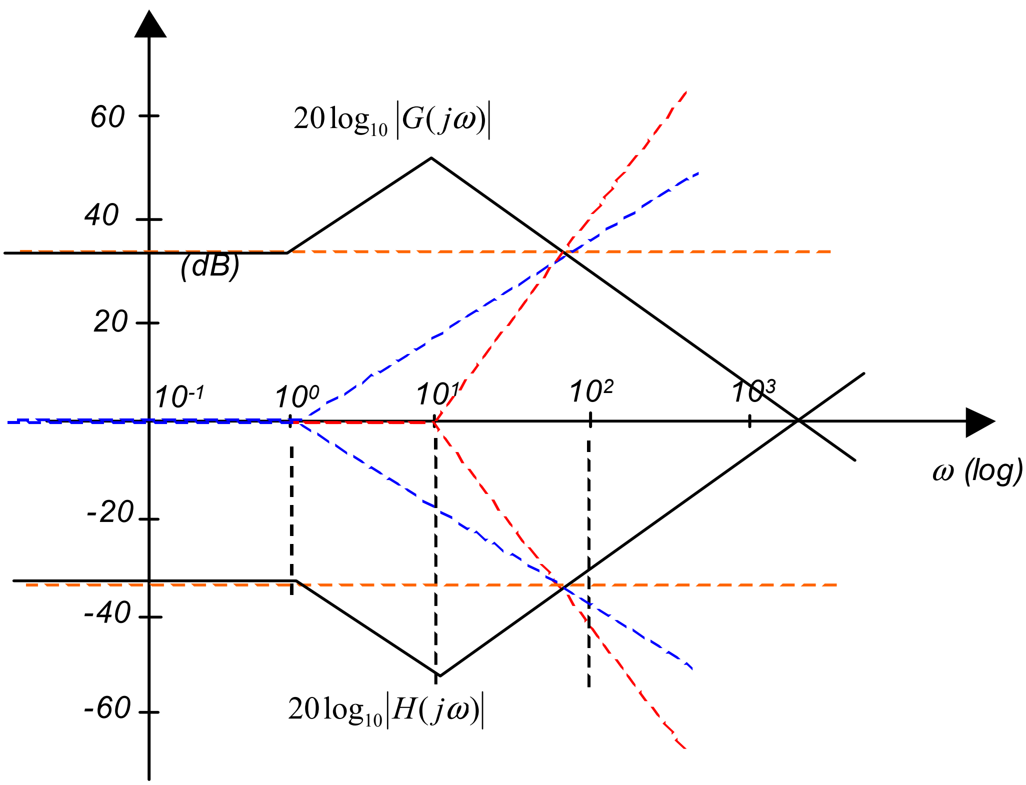
Component values are obtained by setting

$$H(s)=G^{-1}(s)=-0.02\frac{0.01s^2+0.1s+1}{s+1}=-\frac{L_2}{L_1}\frac{(L_1 Cs^2+\frac{L_1}{R_1}s+1)}{(\frac{L_2}{R_2}s+1)}$$

which yields $\Leftrightarrow L_2 = 0.2H,\ R_1 = 100\Omega,\ R_2 = 0.2\Omega,\ C = 0.001F$

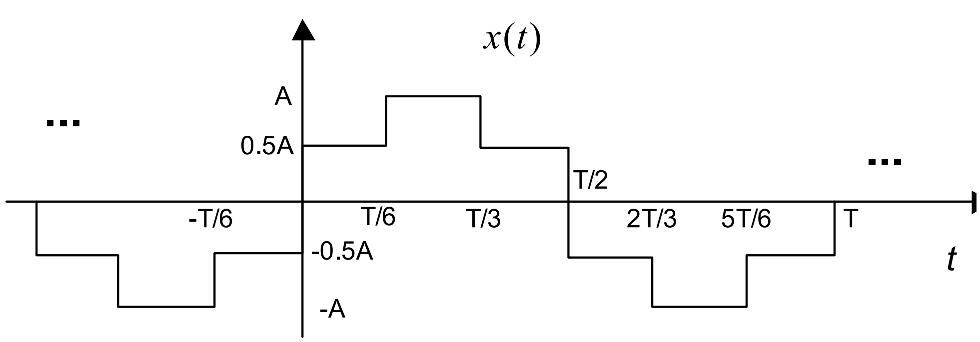
(c) [7 marks] Sketch the Bode plot of $H(s)$ and $G(s)$ (magnitude only, both on the same plot).

Magnitude Bode plot of $G(j\omega) = -50\dfrac{j\omega+1}{0.01(j\omega)^2+0.1(j\omega)+1}$ and desired $H(j\omega) = -0.02\dfrac{0.01(j\omega)^2+0.1(j\omega)+1}{j\omega+1}$:



Problem 2 (20 marks) Digital Signal Generator

A programmable digital signal generator generates a sinusoidal waveform by LTI filtering of a staircase approximation to a sine wave $x(t)$.



(a) [9 marks] Find the Fourier series coefficients a_k of the periodic signal $x(t)$. Show that the even harmonics vanish. Express $x(t)$ as a Fourier series.

Answer:
First of all, the average over one period is 0, so $a_0 = 0$. For $k \neq 0$,

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-jk\frac{2\pi}{T}t}dt \\
&= -\frac{A}{2T}\int_{-\frac{T}{2}}^{-\frac{T}{3}} e^{-jk\frac{2\pi}{T}t}dt - \frac{A}{T}\int_{-\frac{T}{3}}^{-\frac{T}{6}} e^{-jk\frac{2\pi}{T}t}dt - \frac{A}{2T}\int_{-\frac{T}{6}}^{0} e^{-jk\frac{2\pi}{T}t}dt \\
&\quad + \frac{A}{2T}\int_{\frac{T}{3}}^{\frac{T}{2}} e^{-jk\frac{2\pi}{T}t}dt + \frac{A}{T}\int_{\frac{T}{6}}^{\frac{T}{3}} e^{-jk\frac{2\pi}{T}t}dt + \frac{A}{2T}\int_{0}^{\frac{T}{6}} e^{-jk\frac{2\pi}{T}t}dt \\
&= \frac{A}{2T}\int_{0}^{\frac{T}{6}}\left(e^{-jk\frac{2\pi}{T}t} - e^{jk\frac{2\pi}{T}t}\right)dt + \frac{A}{2T}\int_{\frac{T}{3}}^{\frac{T}{2}}\left(e^{-jk\frac{2\pi}{T}t} - e^{jk\frac{2\pi}{T}t}\right)dt + \frac{A}{T}\int_{\frac{T}{6}}^{\frac{T}{3}}\left(e^{-jk\frac{2\pi}{T}t} - e^{jk\frac{2\pi}{T}t}\right)dt \\
&= \frac{-jA}{T}\int_{0}^{\frac{T}{6}}\sin\left(k\frac{2\pi}{T}t\right)dt - \frac{j2A}{T}\int_{\frac{T}{6}}^{\frac{T}{3}}\sin\left(k\frac{2\pi}{T}t\right)dt - \frac{jA}{T}\int_{\frac{T}{3}}^{\frac{T}{2}}\sin\left(k\frac{2\pi}{T}t\right)dt \\
&= \frac{jA}{T}\left(\frac{T}{2\pi k}\right)\cos\left(k\frac{2\pi}{T}t\right)\Big|_{0}^{\frac{T}{6}} + \frac{j2A}{T}\left(\frac{T}{2\pi k}\right)\cos\left(k\frac{2\pi}{T}t\right)\Big|_{\frac{T}{6}}^{\frac{T}{3}} + \frac{jA}{T}\left(\frac{T}{2\pi k}\right)\cos\left(k\frac{2\pi}{T}t\right)\Big|_{\frac{T}{3}}^{\frac{T}{2}} \\
&= \frac{jA}{2\pi k}\left[\cos\left(k\frac{\pi}{3}\right) - 1 + 2\cos\left(k\frac{2\pi}{3}\right) - 2\cos\left(k\frac{\pi}{3}\right) + \cos(k\pi) - \cos\left(k\frac{2\pi}{3}\right)\right] \\
&= \frac{jA}{2\pi k}\left[-\cos\left(k\frac{\pi}{3}\right) + \cos\left(k\frac{2\pi}{3}\right) - 1 + \cos(k\pi)\right]
\end{aligned}
$$

Note that the coefficients are purely imaginary, which is consistent with our real, odd signal. The even FS coefficients are for $k = 2m,\ m = 1, 2, \ldots$:

$$a_k = a_{2m} = \frac{jA}{2\pi 2m}\left[-\cos\left(m\frac{2\pi}{3}\right)+\cos\left(m\frac{4\pi}{3}\right)-1+\cos(m2\pi)\right]$$
$$= \frac{jA}{2\pi 2m}\left[-\cos\left(-m\frac{\pi}{3}+m\pi\right)+\cos\left(m\frac{\pi}{3}+m\pi\right)\right]$$
$$= \frac{jA}{2\pi 2m}\left[-\cos\left(m\frac{\pi}{3}-m\pi\right)+\cos\left(m\frac{\pi}{3}+m\pi\right)\right]$$
$$= \frac{jA}{2\pi 2m}\left[-\cos\left(m\frac{\pi}{3}+m\pi\right)+\cos\left(m\frac{\pi}{3}+m\pi\right)\right]=0$$

The Fourier series representation of $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\frac{2\pi}{T}t} = \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{jA}{2\pi k}\left[-\cos\left(k\frac{\pi}{3}\right)+\cos\left(k\frac{2\pi}{3}\right)-1+\cos(k\pi)\right]e^{jk\frac{2\pi}{T}t}.$$

(b) [3 marks] Write $x(t)$ using the real form of the Fourier series.

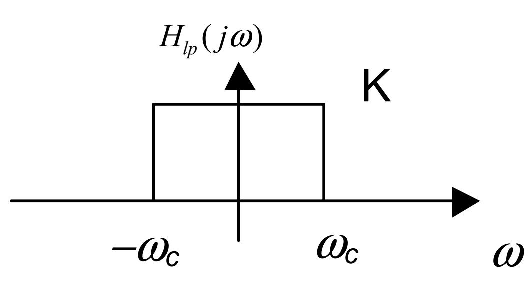
$$x(t) = a_0 + 2\sum_{k=1}^{+\infty}[B_k\cos(k\omega_0 t) - C_k\sin(k\omega_0 t)]$$

Answer:
Recall that the C_k coefficients are the imaginary parts of the a_k's. Hence

$$x(t) = \sum_{k=1}^{+\infty}\frac{-A}{\pi k}\left[-\cos\left(k\frac{\pi}{3}\right)+\cos\left(k\frac{2\pi}{3}\right)-1+\cos(k\pi)\right]\sin(k\omega_0 t).$$

(c) [2 marks] Design an ideal lowpass filter that will produce the perfect sinusoidal waveform $y(t) = \sin\frac{2\pi}{T}t$ at its output with $x(t)$ as its input. Sketch its frequency response and specify its gain K and cutoff frequency ω_c.

Answer:



The cutoff should be between the fundamental and the second harmonic, say $\omega_c = \frac{3\pi}{T}$. The gain should be $K = \frac{-\pi}{A}\left[-\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{2\pi}{3}\right)-1+\cos(\pi)\right]^{-1} = \frac{-\pi}{A}\left[-\frac{1}{2}-\frac{1}{2}-2\right]^{-1} = \frac{\pi}{3A}$.

(d) [6 marks] Now suppose that the first-order lowpass filter whose differential equation is given below is used to filter $x(t)$.

$$\tau \frac{d}{dt} y(t) + y(t) = Bx(t)$$

where the time constant is chosen to be $\tau = \frac{T}{2\pi}$. Give the Fourier series representation of the output $y(t)$. Compute the total average power in the fundamental components P_{1tot} and in the third harmonic components P_{3tot}. Find the value of the dc gain B such that the output $w(t)$ produced by the fundamental harmonic of the real Fourier series of $x(t)$ has unit amplitude.

Answer:

$$H(s) = \frac{B}{\tau s + 1}$$

$$H(j\omega) = \frac{B}{\tau j\omega + 1}$$

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\frac{2\pi}{T}) e^{jk2\pi t} = \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{B}{jk+1}\frac{jA}{2\pi k}\left[-\cos\left(k\frac{\pi}{3}\right)+\cos\left(k\frac{2\pi}{3}\right)-1+\cos(k\pi)\right] e^{jk\frac{2\pi}{T}t}$$

Power:

$$\begin{aligned}
P_{1tot} &= 2\left|\frac{B}{j+1}\frac{jA}{2\pi}\left[-\cos\left(\frac{\pi}{3}\right)+\cos\left(\frac{2\pi}{3}\right)-1+\cos(\pi)\right]\right|^2 \\
&= 2\left|\frac{B}{j+1}\frac{jA}{2\pi}\left[-\frac{1}{2}-\frac{1}{2}-2\right]\right|^2 = 2\left|\frac{B}{j+1}\frac{j3A}{2\pi}\right|^2 \\
&= B^2\left|\frac{3A}{2\pi}\right|^2 = \frac{9A^2B^2}{4\pi^2} \\
P_{3tot} &= 2\left|\frac{B}{j3+1}\frac{jA}{2\pi}\left[-\cos(\pi)+\cos(2\pi)-1+\cos(\pi)\right]\right|^2 \\
&= 2\left|\frac{B}{j3+1}\frac{jA}{2\pi}[0]\right|^2 = 0
\end{aligned}$$

DC gain *B*:

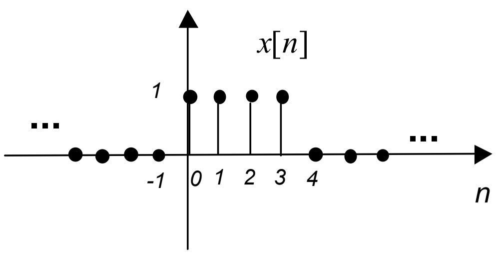
We found that the gain at ω_0 should be

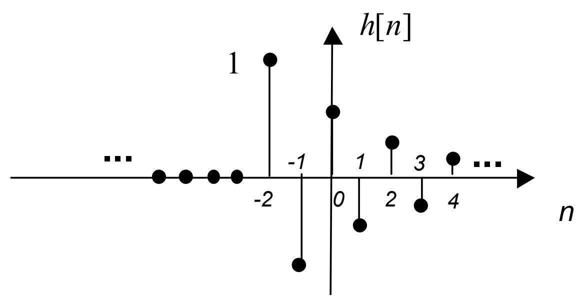
$$\frac{\pi}{3A} = \left|H(j\omega_0)\right| = \frac{B}{\left|\tau j\omega_0 + 1\right|} = \frac{B}{\sqrt{\left(\tau\omega_0\right)^2 + 1}}$$

$$\Leftrightarrow B = \frac{\pi\sqrt{\left(\tau\omega_0\right)^2 + 1}}{3A}$$

Problem 3 (15 marks)

Compute the response $y[n]$ of an LTI system described by its impulse response $h[n]$ shown below and whose magnitude from a time instant to the next decays by 0.9 on its support. The input signal $x[n]$ is the rectangular pulse shown below shown below. Give a mathematical expression for $h[n]$.





Answer:

Mathematical expression for impulse response: $h[n] = (-0.9)^{(n+2)} u[n+2]$

We break down the problem into 3 intervals for n.

<u>For</u> $n < -2$: $h[n-k]$ is zero for k>=0, hence $g[k] = h[k]x[n-k] = 0 \ \forall k$ and $y[n] = 0$.

<u>For</u> $-2 \le n \le 1$: Then $g[k] = h[k]x[n-k] \ne 0$ for $k = 0, \ldots, n+2$. We get

$$\begin{aligned} y[n] &= \sum_{k=0}^{n+2} g[k] = \sum_{k=0}^{n+2} (-0.9)^{n-k+2} = (-0.9)^{n+2} \sum_{k=0}^{n+2} (-0.9)^{-k} \\ &= (-0.9)^{n+2} \sum_{k=0}^{n+2} (-\frac{10}{9})^k = (-0.9)^{n+2} \left(\frac{1-(-0.9)^{-n-3}}{1-(-0.9)^{-1}} \right) \\ &= \frac{(-0.9)^{n+2} - (-0.9)^{-1}}{1-(-0.9)^{-1}} = \frac{(-0.9)^{n+3} - 1}{-1.9} \end{aligned}$$

<u>For</u> $n \ge 2$: Then $g[k] = h[k]x[n-k] \ne 0$ for $k = 0, \ldots, 4$. We get

$$\begin{aligned} y[n] &= \sum_{k=0}^{3} g[k] = \sum_{k=0}^{3} (-0.9)^{n-k+2} = (-0.9)^{n+2} \sum_{k=0}^{3} (-0.9)^{-k} \\ &= (-0.9)^{n+2} \left(\frac{1-(-0.9)^{-4}}{1-(-0.9)^{-1}} \right) \\ &= \frac{(-0.9)^{n+2} - (-0.9)^{n-2}}{1-(-0.9)^{-1}} \end{aligned}$$

In summary, the output signal of the LTI system is

$$y[n] = \begin{cases} 0, & n < -2 \\ \dfrac{(-0.9)^{n+3} - 1}{-1.9}, & -2 \le n < 1 \\ \dfrac{(-0.9)^{n+2} - (-0.9)^{n-2}}{1-(-0.9)^{-1}}, & 2 \le n \end{cases}$$

Problem 4 (10 marks)

The unit step response of an LTI system was measured to be

$$s(t) = 2e^{-\sqrt{3}t}\sin(t - \frac{\pi}{6})u(t) + u(t) - tu(t).$$

(a) [6 marks] Find the transfer function $H(s)$ of the system. Specify its ROC. Sketch its pole-zero plot.

$$H(s) = sS(s) = s\mathcal{L}\left[2e^{-\sqrt{3}t}\sin(t-\frac{\pi}{6})u(t) + u(t) - tu(t)\right]$$

$$= s\mathcal{L}\left[2e^{-\sqrt{3}t}\left(\frac{e^{j(t-\frac{\pi}{6})} - e^{-j(t-\frac{\pi}{6})}}{2j}\right)u(t) + u(t) - tu(t)\right]$$

$$= s\mathcal{L}\left[2e^{-\sqrt{3}t}\left(\frac{(\frac{\sqrt{3}}{2} - j\frac{1}{2})e^{jt} - (\frac{\sqrt{3}}{2} + j\frac{1}{2})e^{-jt}}{2j}\right)u(t) + u(t) - tu(t)\right]$$

$$= s\mathcal{L}\left[\left(\frac{(\frac{\sqrt{3}}{2} - j\frac{1}{2})e^{-\sqrt{3}t+jt} - (\frac{\sqrt{3}}{2} + j\frac{1}{2})e^{-\sqrt{3}t-jt}}{j}\right)u(t) + u(t) - tu(t)\right]$$

$$= s\mathcal{L}\left[\left(\frac{(\frac{\sqrt{3}}{2}e^{-\sqrt{3}t+jt} - \frac{\sqrt{3}}{2}e^{-\sqrt{3}t-jt}) - j\frac{1}{2}e^{-\sqrt{3}t+jt} - j\frac{1}{2}e^{-\sqrt{3}t-jt}}{j}\right)u(t) + u(t) - tu(t)\right]$$

$$= s\mathcal{L}\left[\left(\sqrt{3}e^{-\sqrt{3}t}\sin t - e^{-\sqrt{3}t}\cos t\right)u(t) + u(t) - tu(t)\right]$$

$$= s\left[\frac{\sqrt{3}}{(s+\sqrt{3})^2+1} - \frac{s+\sqrt{3}}{(s+\sqrt{3})^2+1} + \frac{1}{s} - \frac{1}{s^2}\right]$$

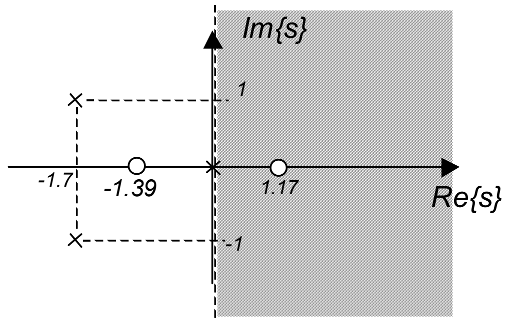
$$= \frac{-s^3 + [(s+\sqrt{3})^2+1](s-1)}{s[(s+\sqrt{3})^2+1]}$$

$$= \frac{-s^3 + [s^2 + 2\sqrt{3}s + 4](s-1)}{s[(s+\sqrt{3})^2+1]}$$

$$= \frac{(2\sqrt{3}-1)s^2 + (4-2\sqrt{3})s - 4}{s[(s+\sqrt{3})^2+1]} = \frac{(2\sqrt{3}-1)\left[s^2 + \frac{(4-2\sqrt{3})}{(2\sqrt{3}-1)}s - \frac{4}{(2\sqrt{3}-1)}\right]}{s[(s+\sqrt{3})^2+1]}$$

ROC: $\mathrm{Re}\{s\} > 0$.

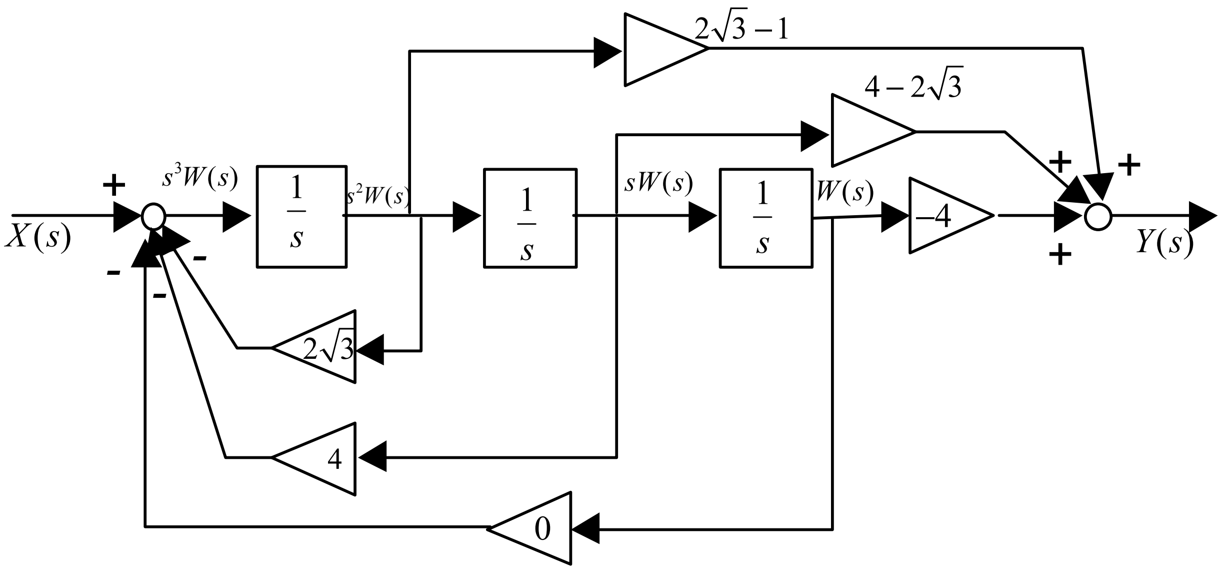
Poles are $p_1 = 0$, $p_2 = -\sqrt{3} + j$, $p_3 = -\sqrt{3} - j$. Zeros are zeros of $s^2 + 0.21748s - 1.62331$: $z_1 = -1.3875$, $z_2 = 1.1700$, $z_3 = \infty$



(b) [2 marks] Is the system causal? Is it stable? Justify your answers.
Answer:
System is causal: ROC is an open RHP and transfer function is rational.
This system isn't stable as ROC doesn't include the imaginary axis (or because rightmost pole 0 has a nonnegative real part.)

(c) [2 marks] Give the direct form realization (block diagram) of $H(s)$.
Answer:

$$H(s)=\frac{(2\sqrt{3}-1)s^2+(4-2\sqrt{3})s-4}{s^3+2\sqrt{3}s^2+4s}$$



Problem 5 (5 marks)

True or False?

(a) The Fourier transform $Z(j\omega)$ of the convolution of a real even signal $x(t)$ and a real odd signal $y(t)$ is imaginary even.

Answer: False.

(b) The system defined by $y(t) = tx(t)$ is time-invariant.

Answer: False.

(c) The Fourier series coefficients b_k of the periodic signal $y(t) = x(10t)$ are given by $b_k = \frac{1}{10} a_k$, where $x(t) \overset{\mathcal{FS}}{\leftrightarrow} a_k$.

Answer: False.

(d) The Fourier transform $X(j\omega)$ of the product of a real signal $x(t)$ and an impulse $\delta(t)$ is real.

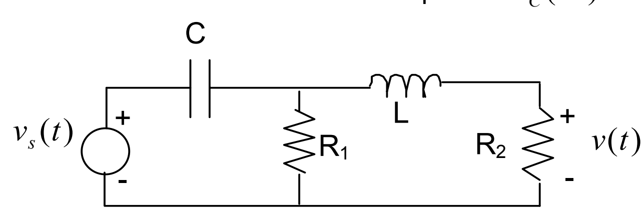
Answer: True.

(e) The fundamental period of the signal $x[n] = \sin(\frac{3\pi}{5} n)\cos(\frac{2\pi}{3} n)$ is 60.

Answer: False. (N=30)

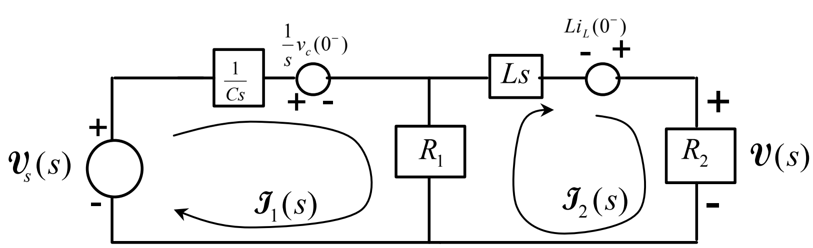
Problem 6 (15 marks)

The following circuit has initial conditions on the capacitor $v_C(0^-)$ and inductor $i_L(0^-)$.



(a) [3 marks] Transform the circuit using the unilateral Laplace transform.

Answer:



(b) [6 marks] Find the unilateral Laplace transform of $v(t)$.

Answer:
Let's use mesh analysis.
For mesh 1:

$$\mathcal{V}_s(s) - \frac{1}{Cs}\mathcal{I}_1(s) - \frac{1}{s}v_c(0^-) - R_1[\mathcal{I}_1(s) - \mathcal{I}_2(s)] = 0$$

$$\Rightarrow \quad \mathcal{I}_2(s) = -\frac{1}{R_1}\mathcal{V}_s(s) + \frac{1}{R_1 s}v_c(0^-) + (1 + \frac{1}{R_1 Cs})\mathcal{I}_1(s)$$

For mesh 2:

$$R_1[\mathcal{I}_1(s) - \mathcal{I}_2(s)] - (R_2 + Ls)\mathcal{I}_2(s) + Li_L(0^-) = 0$$

$$\Rightarrow \quad \mathcal{I}_1(s) = \frac{1}{R_1}(R_1 + R_2 + Ls)\mathcal{I}_2(s) - \frac{L}{R_1}i_L(0^-)$$

Substituting, we obtain

$$\mathcal{I}_2(s) = -\frac{1}{R_1}\mathcal{V}_s(s) + \frac{1}{R_1 s}v_c(0^-) + (1 + \frac{1}{R_1 Cs})\left[\frac{1}{R_1}(R_1 + R_2 + Ls)\mathcal{I}_2(s) - \frac{L}{R_1}i_L(0^-)\right]$$

$$[R_1^2 Cs - (1 + R_1 Cs)(R_1 + R_2 + Ls)]\mathcal{I}_2(s) = -R_1 Cs\mathcal{V}_s(s) + R_1 Cv_c(0^-) - (1 + R_1 Cs)Li_L(0^-)$$

$$-[LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2]\mathcal{I}_2(s) = -R_1 Cs\mathcal{V}_s(s) + R_1 Cv_c(0^-) - (1 + R_1 Cs)Li_L(0^-)$$

Solving for $\mathcal{I}_2(s)$, we get

$$\mathcal{I}_2(s) = \frac{R_1 Cs\mathcal{V}_s(s)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{(1 + R_1 Cs)Li_L(0^-) - R_1 Cv_c(0^-)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2}$$

And finally the output voltage is

$$\mathcal{V}(s) = R_2\mathcal{I}_2(s) = \frac{R_1 R_2 Cs\mathcal{V}_s(s)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{R_2(1 + R_1 Cs)Li_L(0^-) - R_1 R_2 Cv_c(0^-)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2}$$

(c) [6 marks] Draw the Bode plot (magnitude and phase) of the frequency response from the input voltage $\mathcal{V}_s(j\omega)$ to the output voltage $\mathcal{V}(j\omega)$. Assume that the initial conditions on the capacitor and the inductor are 0. Use the numerical values: $R_1 = 1\,\Omega,\ R_2 = \frac{109}{891}\,\Omega,\ L = \frac{1}{891}\,\mathrm{H},\ C = 1\,\mathrm{F}$.

Answer:

For the values given, the transfer function from the source voltage to the output voltage is

$$\mathcal{H}(s) := \frac{\mathcal{V}(s)}{\mathcal{V}_s(s)} = \frac{\frac{R_2}{L}s}{s^2 + \frac{L + R_1 R_2 C}{L R_1 C}s + \frac{R_1 + R_2}{L R_1 C}}$$

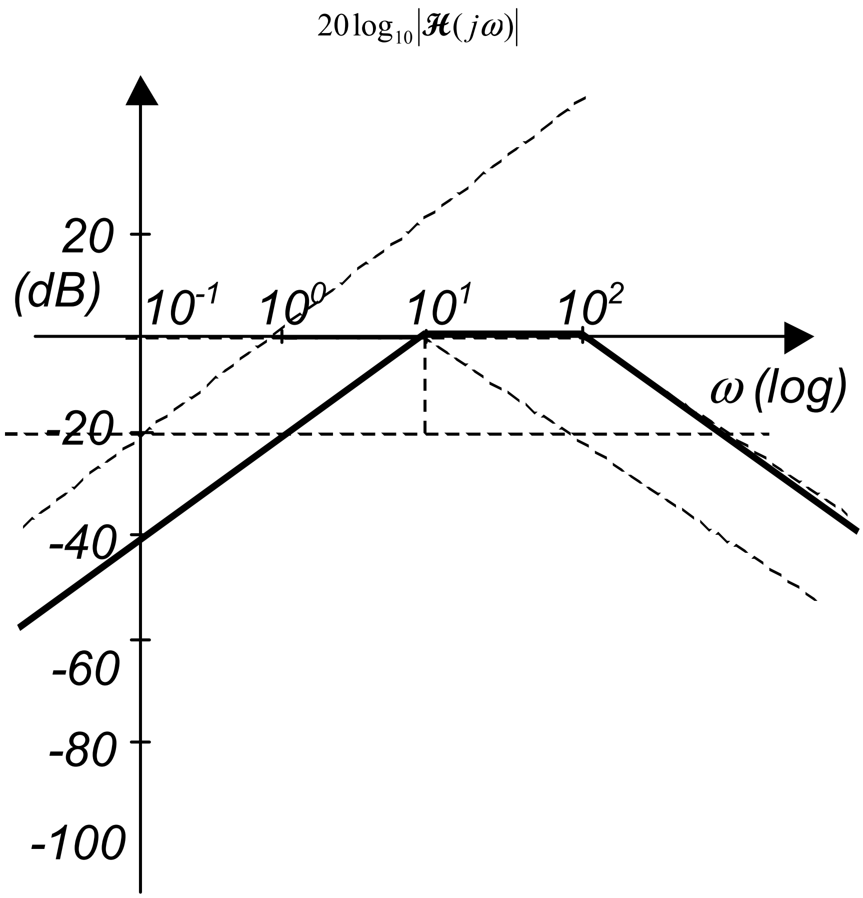
$$= \frac{109s}{s^2 + \frac{\frac{1}{891} + \frac{109}{891}}{\frac{1}{891}}s + \frac{1 + \frac{109}{891}}{\frac{1}{891}}}$$

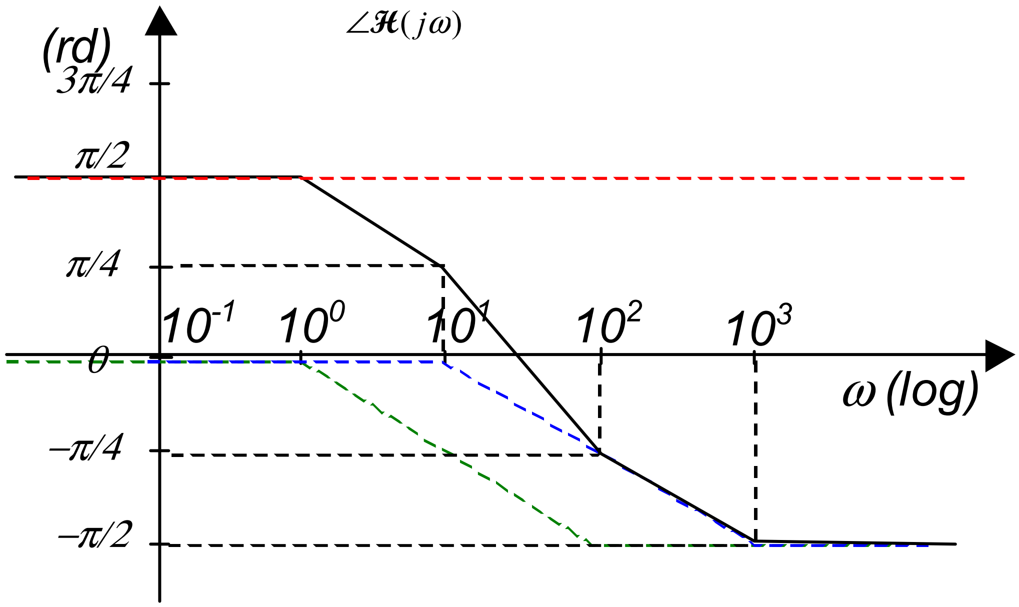
$$= \frac{109s}{s^2 + 110s + 1000}$$

$$= \frac{109s}{(s+10)(s+100)}$$

$$= 0.109\frac{s}{(\frac{1}{10}s+1)(\frac{1}{100}s+1)}$$

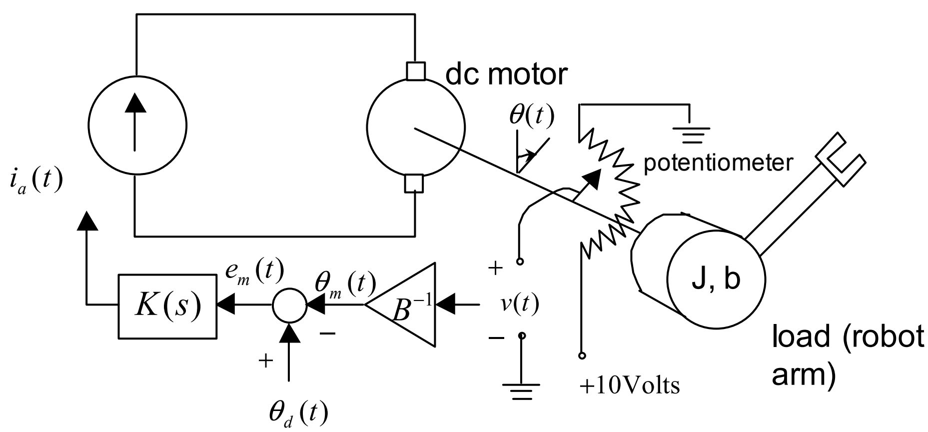
Bode Plot:





Problem 7 (15 marks)

A classical technique to control the position of an inertial load (e.g., a robot arm) driven by a permanent-magnet DC motor is to vary the armature current based on a potentiometer measurement of the robot's joint angle $\theta(t)$, as shown in the diagram below.



Before closing the feedback loop, we first look at the open-loop dynamics of this system in (a) and (b). The torque $\tau(t)$ in Newton-metres applied to the load by the motor is proportional to the armature current in Amps:

$$\tau(t) = Ai_a(t).$$

The robot arm is modeled as an inertia J with viscous friction represented by the coefficient b. The equation of movement for the load is

$$J\frac{d^2\theta(t)}{dt^2}+b\frac{d\theta(t)}{dt}=\tau(t).$$

(a) [2 marks] Assume that the system is initially at rest. Find the open-loop transfer function $G(s):=\Theta(s)/I_a(s)$ relating the joint angle $\Theta(s)$ to the motor's armature current input $I_a(s)$.

Answer:

$$G(s)=\frac{\Theta(s)}{I_a(s)}=\frac{A}{s(Js+b)}$$

(b) [6 marks] Assume that $A=1\ \mathrm{Nm/A}$, $J=1\ \mathrm{Nm}/{}^{\mathrm{rd}}\!/_{\mathrm{s}^2}$ and $b=1\ \mathrm{Nm}/{}^{\mathrm{rd}}\!/_{\mathrm{s}}$. Compute the open-loop unit step response $s_\omega(t)$ of the load's *angular velocity* $\omega(t)=\dfrac{d\theta(t)}{dt}$ for a unit step in armature current. What is the ±5% settling time t_s (an approximation is OK)?

Answer:

$$G_\omega(s)=\frac{s\Theta(s)}{I_a(s)}=\frac{1}{s+1}$$

The unit step response $s_\omega(t)$ is given by

$$S_\omega(s)=\frac{1}{s}G_\omega(s)=\frac{\Theta(s)}{I_a(s)}=\frac{1}{s(s+1)}=\frac{1}{s(s+1)}=\frac{1}{s}-\frac{1}{s+1}$$

Taking the inverse Laplace transform, we get

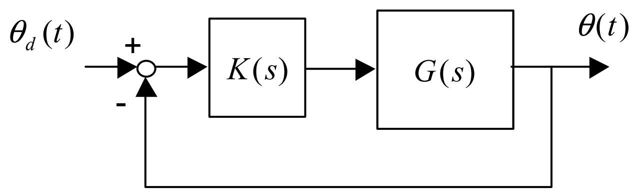
$$s_\omega(t)=(1-e^{-t})u(t)$$

The ±5% settling time is approximately equal to three time constants: $t_s\approx 3\ s$.

(c) [7 marks] Based on the above figure, find the gain B of the joint angle sensor (potentiometer) in Volts/radian. Assume, as shown in the figure above, that there is a gain B^{-1} in the feedback path canceling out the sensor gain, and thus the link angle $\theta(t)$ is measured perfectly, i.e., $\theta_m(t)=\theta(t)$. We want to analyze the feedback controller $K(s)=\lambda(s+\alpha)$ to control the robot's joint angle. That is, we want $\theta(t)$ to track the desired angle $\theta_d(t)$.

Answer:

Sensor gain is $B=\dfrac{10}{\pi}V/rd$



Find the closed-loop transfer function $H(s) := \Theta(s)/\Theta_d(s)$. Find expressions for the closed-loop damping ratio ζ and undamped natural frequency ω_n in terms of the controller parameters α and λ. For $\alpha = 4$, find a value of λ that will result in a closed-loop damping ratio of $\zeta = \dfrac{1}{\sqrt{2}}$.

Closed-loop transfer function:

$$\begin{aligned} H(s) &= \frac{K(s)G(s)}{1+K(s)G(s)} \\ &= \frac{\dfrac{\lambda(s+\alpha)}{s(s+1)}}{1+\dfrac{\lambda(s+\alpha)}{s(s+1)}} \\ &= \frac{\lambda(s+\alpha)}{s(s+1)+\lambda(s+\alpha)} \\ &= \frac{\lambda(s+\alpha)}{s^2+(1+\lambda)s+\alpha\lambda} \end{aligned}$$

We get $\omega_n = \sqrt{\alpha\lambda}$ and $\zeta = \dfrac{1+\lambda}{2\sqrt{\alpha\lambda}}$.

For $\alpha = 4$ and given $\zeta = \dfrac{1}{\sqrt{2}}$, we have:

$$\begin{aligned} &\zeta = \frac{1}{\sqrt{2}} = \frac{1+\lambda}{2\sqrt{4\lambda}} \\ &\Rightarrow \frac{1}{2} = \frac{(1+\lambda)^2}{16\lambda} \Rightarrow 8\lambda = 1+2\lambda+\lambda^2 \\ &\Rightarrow 1-6\lambda+\lambda^2 = 0 \\ &\Rightarrow \lambda_{1,2} = \frac{6\pm\sqrt{36-4}}{2} = 3\pm 2\sqrt{2} \end{aligned}$$

both values are positive (for stability) and hence acceptable.

END OF EXAMINATION