Sample Final Exam (finals00)
Covering Chapters 1-9 of *Fundamentals of Signals & Systems*

Problem 1 (20 marks)

Consider the transfer function:

$$H(s) = \frac{s^2 - 2s + 1}{(0.01s^2 + 0.1\sqrt{3}s + 1)(s+2)}.$$

(a) [6 marks] Find the poles and zeros of $H(s)$ (specify how many there are at ∞). Give all possible regions of convergence of the transfer function $H(s)$.

Answer:

Poles are $p_1 = -2$, $p_2 = -5\sqrt{3} + 5j = -8.66 + 5j$, $p_3 = -5\sqrt{3} - 5j = -8.66 - 5j$

Zeros are $z_1 = 1,$ $z_2 = 1$, $z_3 = \infty$

There are 3 possible ROCs:

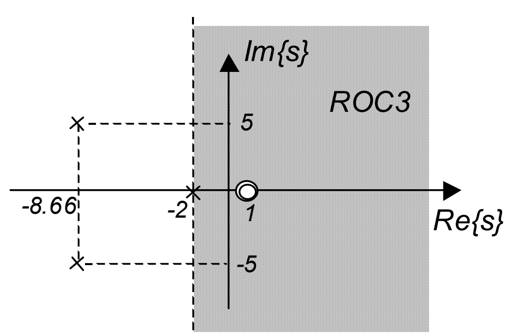
ROC1: $\mathrm{Re}\{s\} < -5\sqrt{3} = -8.66$

ROC2: $-5\sqrt{3} < \mathrm{Re}\{s\} < -2$

ROC3: $\mathrm{Re}\{s\} > -2$

(b) [10 marks] Give the region of convergence of $H(s)$ that corresponds to the impulse response of a stable system, and sketch it on a pole-zero plot. Is the stable system causal? Explain. Compute the impulse response $h(t)$ of this stable system and give its Fourier transform $H(j\omega)$.

Answer:
System is stable for the ROC that contains the $j\omega$-axis: ROC3.
This system is also causal as ROC3 is an open RHP and the transfer function is rational.



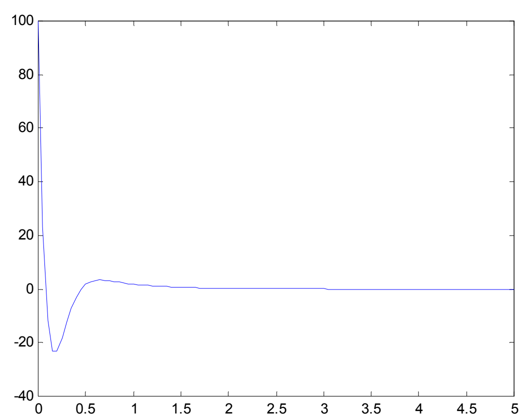
The partial fraction expansion of $H(s)$ yields:

$$H(s)=\frac{s^2-2s+1}{(0.01s^2+0.1\sqrt{3}s+1)(s+2)}=\frac{100(s^2-2s+1)}{(s^2+10\sqrt{3}s+100)(s+2)}$$
$$=\frac{100(s-1)^2}{(s+5\sqrt{3}-j5)(s+5\sqrt{3}+j5)(s+2)}$$
$$=\underbrace{72.58+j225.2}_{A}\frac{1}{s+5\sqrt{3}-j5}+\underbrace{72.58-j225.2}_{B}\frac{1}{s+5\sqrt{3}+j5}+\underbrace{12.976}_{C}\frac{1}{s+2}$$
$$A=\left.\frac{100(s-1)^2}{(s+5\sqrt{3}+j5)(s+2)}\right|_{s=-5\sqrt{3}+j5}=\frac{100(-5\sqrt{3}-1+5j)^2}{(10j)(-5\sqrt{3}+2+5j)}=\frac{10(-5\sqrt{3}-1+5j)^2}{j(-5\sqrt{3}+2)-5}=43.512+j135.24$$
$$B=A^*=43.512-j135.24$$
$$C=\left.\frac{100(s-1)^2}{(s^2+10\sqrt{3}s+100)}\right|_{s=-2}=\frac{900}{(4-20\sqrt{3}+100)}=\frac{900}{69.359}=12.976$$

Using the table and simplifying, we find the following impulse response:

ROC $\mathrm{Re}\{s\}<-1$:

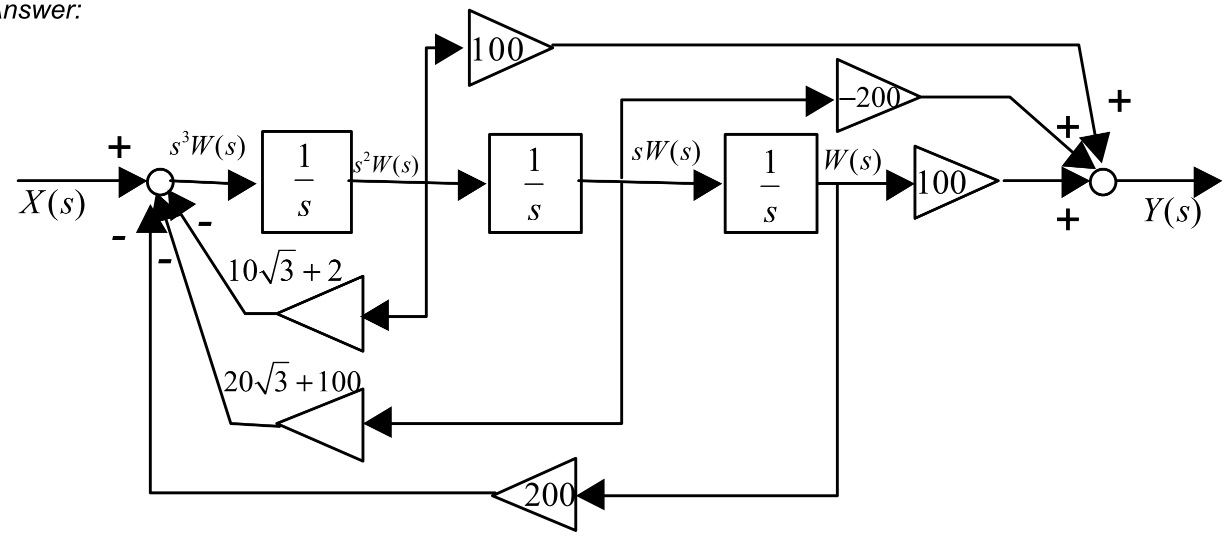
$$h(t)=(43.512+j135.24)e^{(-5\sqrt{3}+5j)t}u(t)+(43.512-j135.24)e^{(-5\sqrt{3}-5j)t}u(t)+12.976e^{-2t}u(t)$$
$$=2e^{-5\sqrt{3}t}\,\mathrm{Re}\left\{(43.512+j135.24)e^{j5t}\right\}u(t)+12.976e^{-2t}u(t)$$
$$=2(142.07)e^{-5\sqrt{3}t}\cos(5t+1.2595)u(t)+12.976e^{-2t}u(t)$$
$$=284.14e^{-5\sqrt{3}t}\cos(5t+1.2595)u(t)+12.976e^{-2t}u(t)$$
$$=2e^{-5\sqrt{3}t}[43.512\cos(5t)-135.24\sin(5t)]u(t)+12.976e^{-2t}u(t)$$



Fourier transform of $h(t)$ is $H(j\omega)$:

$$H(j\omega)=\frac{(j\omega)^2-2j\omega+1}{(0.01(j\omega)^2+0.1\sqrt{3}j\omega+1)(j\omega+2)}$$

(c) [4 marks] Give the direct form realization (block diagram) of $H(s)$.
Answer:



Problem 2 (5 marks)

True or False?

(a) The Fourier transform $X(j\omega)$ of the product of a real signal $x(t)$ and an impulse $\delta(t-1)$ is real.
Answer: False.

(b) The system defined by $y(t) = x(t^2)$ is time-invariant.
Answer: False.

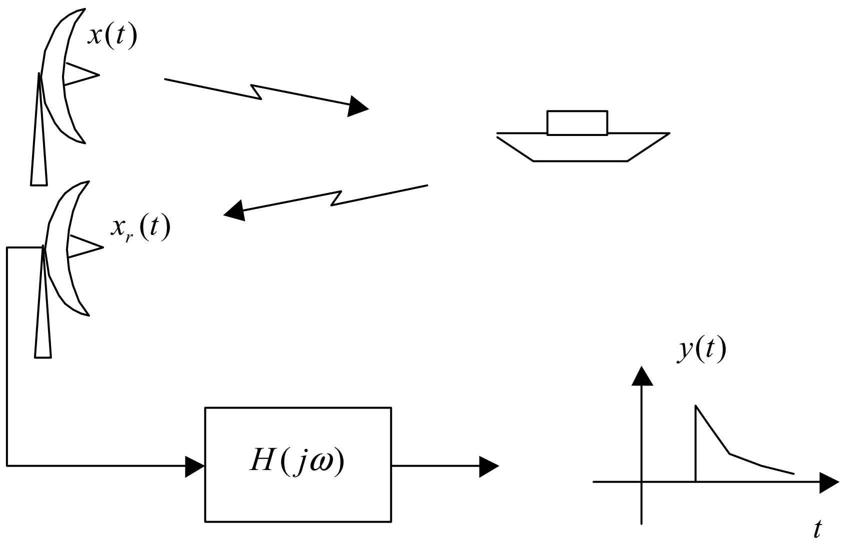
(c) The Fourier series coefficients a_k of a purely imaginary periodic signal $x(t)$ have the following property: $a_k^* = -a_{-k}$.
Answer: True.

(d) The causal linear discrete-time system defined by $y[n-2] + 2y[n-1] + y[n] = x[n]$ is stable.
Answer: False.

(e) The signal $x[n] = e^{j\frac{2}{5}n}$ is not periodic.
Answer: True.

Problem 3 (15 marks)

Consider the radar system depicted below where the radar's emitting antenna emits a pulse $x(t) = e^{-100t}\sin(10^3 t)u(t)$ that gets reflected by a boat and is received by the radar's receiving antenna as $x_r(t) = 0.1e^{100t_0}e^{-100t}\sin(10^3 t - 10^3 t_0)u(t-t_0)$. An LTI filter $H(j\omega)$ processes $x_r(t)$ to generate another pulse $y(t) = \left(e^{-10(t-t_0)} + e^{-20(t-t_0)}\right)u(t-t_0)$ that is used to measure the time of arrival of the received pulse.



(a) [10 marks] Find the frequency response $H(j\omega)$ of the filter. Is the filter BIBO stable? Justify your answer. Write the causal LTI differential equation that would implement this filter.

Answer:

We have $X_r(j\omega)H(j\omega) = Y(j\omega)$, where

$$X_r(j\omega) = 0.1X(j\omega)e^{-j\omega t_0} = \frac{100e^{-j\omega t_0}}{(j\omega+100)^2 + (10^3)^2}$$

$$X(j\omega) = \frac{10^3}{(j\omega+100)^2 + (10^3)^2}$$

and

$$Y(j\omega) = e^{-j\omega t_0}\left(\frac{1}{j\omega+10} + \frac{1}{j\omega+20}\right) = \frac{e^{-j\omega t_0}(2j\omega+30)}{(j\omega+10)(j\omega+20)}$$

Hence

$$H(j\omega) = \frac{Y(j\omega)}{X_r(j\omega)} = \frac{\dfrac{e^{-j\omega t_0}(2j\omega+30)}{(j\omega+10)(j\omega+20)}}{\dfrac{e^{-j\omega t_0}100}{(j\omega+100)^2 + (10^3)^2}} = \frac{(2j\omega+30)\left[(j\omega+100)^2 + (10^3)^2\right]}{100(j\omega+10)(j\omega+20)}$$

This filter is NOT BIBO stable because it is not proper (degree of num>degree of denom.)

Differential equation:

$$100\frac{d^2y(t)}{dt^2}+3000\frac{dy(t)}{dt}+20000y(t)=2\frac{d^3x(t)}{dt^3}+430\frac{d^2x(t)}{dt^2}+2026000\frac{dx(t)}{dt}+30300000x(t)$$

(b) [5 marks] Knowing that electromagnetic waves propagate at the speed of light $c=3\times10^8\text{ m/s}$, and that $t_0=10\mu s$, how far is the boat from the radar?

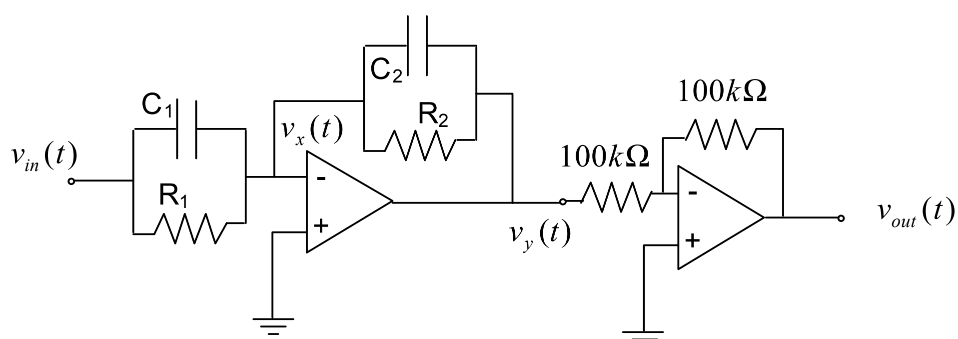
Answer:
The time delay results from the pulse having traveled twice the distance *d* between the boat and the radar, thus

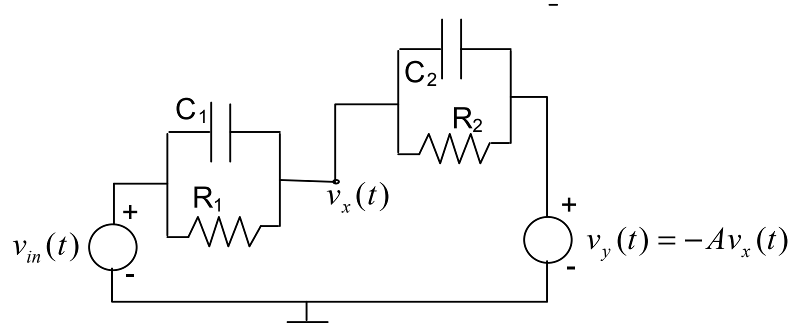
$$d=\frac{ct_0}{2}=\frac{3\times10^8\,\frac{m}{s}\cdot10^{-5}\,s}{2}=1500m$$

Problem 4 (20 marks)

Consider the causal first-order circuit initially at rest depicted below. It can be used to implement a first-order lead or lag, depending on the values of its components. Its ideal circuit model with a voltage-controlled source is also given below.

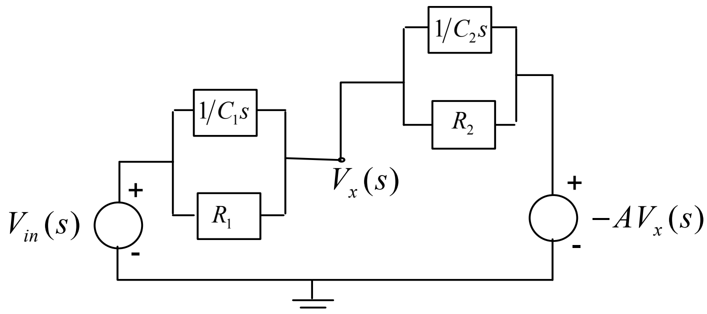
(a) [8 marks] Transform the ideal circuit using the Laplace transform, and use nodal analysis to find the transfer function $H_A(s)=V_{out}(s)/V_{in}(s)$. Then, let $A\to+\infty$ to obtain the ideal transfer function $H(s)=\lim_{A\to+\infty}H_A(s)$. Note that the second op-amp stage is just an inverter such that $v_{out}(t)=-v_y(t)$.





Answer:

The transformed circuit is



There are two supernodes for which the nodal voltages are given by the source voltages. The remaining nodal equation is

$$\frac{V_{in}(s)-V_x(s)}{R_1\left\|\frac{1}{C_1s}\right.}+\frac{-AV_x(s)-V_x(s)}{R_1\left\|\frac{1}{C_1s}\right.}=0$$

where the notation $R_1\left\|\frac{1}{C_1s}\right.=\frac{R_1}{R_1C_1s+1}$ denotes the equivalent impedance of the parallel connection of the resistor and capacitor. Simplifying the above equation, we get:

$$\frac{R_1C_1s+1}{R_1}V_{in}(s)-\left[\frac{(A+1)(R_2C_2s+1)}{R_2}+\frac{R_1C_1s+1}{R_1}\right]V_x(s)=0$$

Thus, the transfer function between the input voltage and the node voltage is given by

$$\frac{V_x(s)}{V_{in}(s)}=\frac{R_1C_1s+1}{R_1[(A+1)C_2+C_1]s+\frac{(A+1)R_1}{R_2}+1}.$$

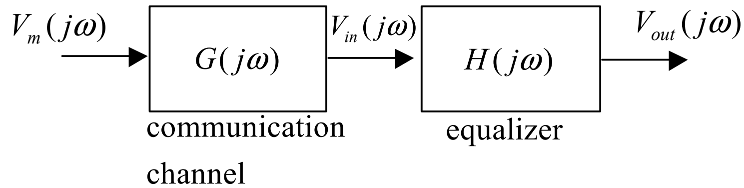
The transfer function between the input voltage and the output voltage is

$$H_A(s)=\frac{V_{out}(s)}{V_{in}(s)}=-\frac{-AV_x(s)}{V_{in}(s)}=\frac{A(R_1C_1s+1)}{R_1[(A+1)C_2+C_1]s+\frac{(A+1)R_1}{R_2}+1}$$

The ideal transfer function is the limit as the op-amp gain tends to infinity:

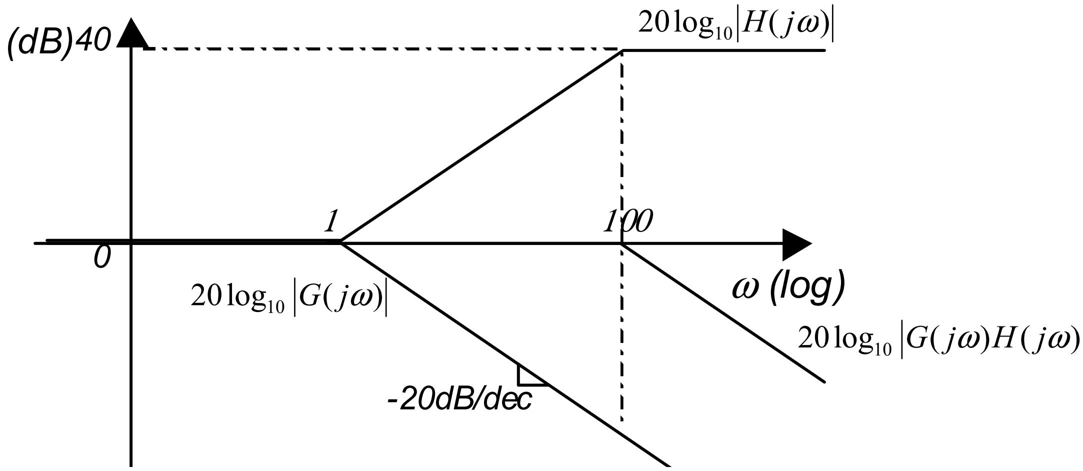
$$H(s)=\lim_{A\to\infty}H_A(s)=\frac{R_2}{R_1}\frac{(R_1C_1s+1)}{(R_2C_2s+1)}$$

(b) [7 marks] Suppose that the circuit is used as an equalizer in the following communication system and $R_1=R_2=1k\Omega$. A rectangular pulse is transmitted over an LTI communication channel with frequency response $G(j\omega)=\frac{1}{j\omega+1}$. The pulse is distorted at the receiving end of the channel, and you are asked to design (i.e., specify the capacitances C_1, C_2 of the capacitors) a first-order LTI equalizer that would reshape the pulse. The specification is that the magnitude of the combined frequency response of the channel and the equalizer $|H(j\omega)G(j\omega)|$ must have a DC gain of 0dB and a -3dB bandwidth of 100 radians/s. Use Bode plots (magnitude only) to guide your designs.



Answer:

Bode plot of $G(j\omega) = \dfrac{1}{j\omega + 1}$ and desired $H(j\omega)$:



The circuit must be a first-order lead with a transfer function of the form

$$H(s) = \frac{\alpha\tau s + 1}{\tau s + 1},$$

where $\tau = 0.01$ and $\alpha = 100$. Frequency response:

$$H(j\omega) = \frac{j\omega + 1}{0.01 j\omega + 1}$$

There are two break frequencies for the Bode plot: $\omega_1 = 1,\ \omega_2 = 100$. The DC gain is 0dB. Identifying with the parameters of $H(s) = \dfrac{(1000C_1 s + 1)}{(1000C_2 s + 1)}$, we find for our design $C_1 = 1000\mu F, C_2 = 10\mu F$.

(c) [5 marks] Compute the response $v_{out}(t)$ of the equalizer circuit to a 1-Volt, 2-second pulse $v_m(t) = u(t) - u(t-2)$ transmitted through the communication channel. Compute the energy in the residual distortion error $e(t) := v_m(t) - v_{out}(t)$.

Answer:

$$V_{out}(s) = H(s)G(s)V_m(s) = \frac{1}{s(0.01s+1)} - \frac{e^{-2s}}{s(0.01s+1)} = \left[\frac{1}{s} - \frac{1}{s+100}\right] - e^{-2s}\left[\frac{1}{s} - \frac{1}{s+100}\right]$$

Taking the inverse transform, we get

$$v_{out}(t) = \left(1 - e^{-100t}\right)u(t) - \left(1 - e^{-100(t-2)}\right)u(t-2)$$

the energy in the residual distortion error is given by

$$\int_{-\infty}^{+\infty} |e(t)|^2\, dt = \int_{0}^{+\infty} \left|-e^{-100t} + e^{-100(t-2)}u(t-2)\right|^2 dt = \int_{0}^{2} e^{-200t} dt + \int_{2}^{+\infty} \left|e^{200} - 1\right|^2 e^{-200t} dt$$

$$= -\frac{1}{200}\left[e^{-200t}\right]_0^2 - \frac{1}{200}\left|e^{200} - 1\right|^2 \left[e^{-200t}\right]_2^{+\infty}$$

$$= -\frac{1}{200}\left[e^{-400} - 1\right] - \frac{1}{200}\left|e^{200} - 1\right|^2 \left[0 - e^{-400}\right]$$

$$= -\frac{1}{200}\left[e^{-400} - 1\right] + \frac{1}{200}\left|1 - e^{-200}\right|^2$$

$$= \frac{1}{100} - \frac{1}{100} e^{-200} \cong 0.01$$

or using Parseval's relation:

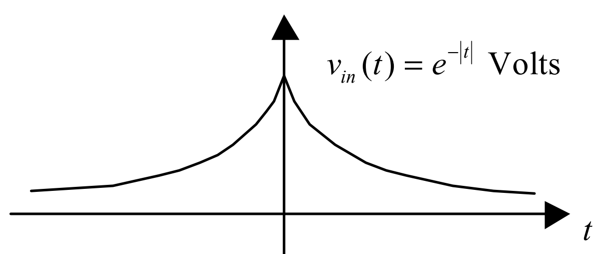
$$\int_{-\infty}^{+\infty} |e(t)|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left|\frac{1 - e^{-j\omega 2}}{0.01 j\omega + 1}\right|^2 d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{(1 - \cos 2\omega)^2 + (\sin 2\omega)^2}{0.0001\omega^2 + 1} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{2 - 2\cos 2\omega}{0.0001\omega^2 + 1} d\omega = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{1 - \cos 2\omega}{0.0001\omega^2 + 1} d\omega$$

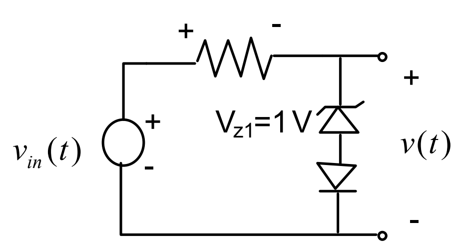
Problem 5 (10 marks)

The following circuit with a Zener diode is an ideal clamping circuit.

The input voltage is shown below

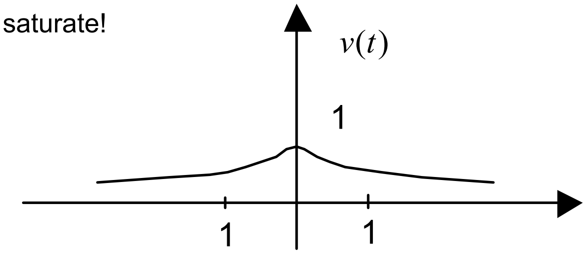


and the output voltage is $v(t) = \begin{cases} v_{in}(t), & v_{in}(t) < 1 \\ 1, & v_{in}(t) \geq 1 \end{cases}$.



[10 marks] Sketch the output voltage $v(t)$ and compute its the Fourier transform $V(j\omega)$.

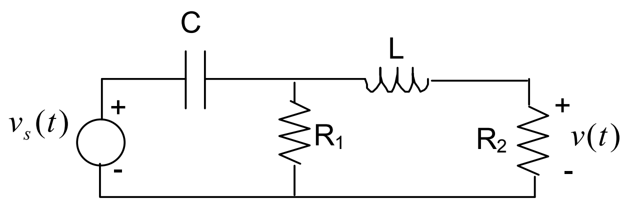
Answer:
The circuit doesn't saturate!



$$
\begin{aligned}
V(j\omega) &= \int_{-\infty}^{\infty} v(t)e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{0} e^{t}e^{-j\omega t}\,dt + \int_{0}^{+\infty} e^{-t}e^{-j\omega t}\,dt \\
&= \int_{-\infty}^{0} e^{(1-j\omega)t}\,dt + \int_{0}^{+\infty} e^{-(1+j\omega)t}\,dt \\
&= \frac{1}{1-j\omega}\left[e^{(1-j\omega)t}\right]_{-\infty}^{0} - \frac{1}{1+j\omega}\left[e^{-(1+j\omega)t}\right]_{0}^{+\infty} \\
&= \frac{1}{1-j\omega} + \frac{1}{1+j\omega} \\
&= \frac{2}{1+\omega^2}
\end{aligned}
$$

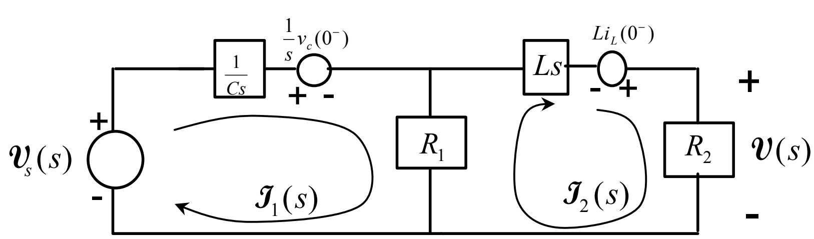
Problem 6 (20 marks)

The following circuit has initial conditions on the capacitor $v_C(0^-)$ and inductor $i_L(0^-)$.



(a) [4 marks] Transform the circuit using the unilateral Laplace transform.

Answer:



(b) [8 marks] Find the unilateral Laplace transform $\mathcal{V}(s)$ of $v(t)$.

Answer:
Let's use mesh analysis.
For mesh 1:

$$\mathcal{V}_s(s) - \frac{1}{Cs}\mathcal{I}_1(s) - \frac{1}{s}v_c(0^-) - R_1[\mathcal{I}_1(s) - \mathcal{I}_2(s)] = 0$$

$$\Rightarrow \quad \mathcal{I}_2(s) = -\frac{1}{R_1}\mathcal{V}_s(s) + \frac{1}{R_1 s}v_c(0^-) + (1 + \frac{1}{R_1 Cs})\mathcal{I}_1(s)$$

For mesh 2:

$$R_1[\mathcal{I}_1(s) - \mathcal{I}_2(s)] - (R_2 + Ls)\mathcal{I}_2(s) + Li_L(0^-) = 0$$

$$\Rightarrow \quad \mathcal{I}_1(s) = \frac{1}{R_1}(R_1 + R_2 + Ls)\mathcal{I}_2(s) - \frac{L}{R_1}i_L(0^-)$$

Substituting, we obtain

$$\mathcal{I}_2(s) = -\frac{1}{R_1}\mathcal{V}_s(s) + \frac{1}{R_1 s}v_c(0^-) + (1 + \frac{1}{R_1 Cs})\left[\frac{1}{R_1}(R_1 + R_2 + Ls)\mathcal{I}_2(s) - \frac{L}{R_1}i_L(0^-)\right]$$

$$[R_1^2 Cs - (1 + R_1 Cs)(R_1 + R_2 + Ls)]\mathcal{I}_2(s) = -R_1 Cs\mathcal{V}_s(s) + R_1 Cv_c(0^-) - (1 + R_1 Cs)Li_L(0^-)$$

$$-[LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2]\mathcal{I}_2(s) = -R_1 Cs\mathcal{V}_s(s) + R_1 Cv_c(0^-) - (1 + R_1 Cs)Li_L(0^-)$$

Solving for $\mathcal{I}_2(s)$, we get

$$\mathcal{I}_2(s) = \frac{R_1 Cs\mathcal{V}_s(s)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{(1 + R_1 Cs)Li_L(0^-) - R_1 Cv_c(0^-)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2}$$

And finally the output voltage is

$$\mathcal{V}(s) = R_2\mathcal{I}_2(s) = \frac{R_1 R_2 Cs\mathcal{V}_s(s)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{(1 + R_1 Cs)LR_2 i_L(0^-) - R_1 R_2 Cv_c(0^-)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2}$$

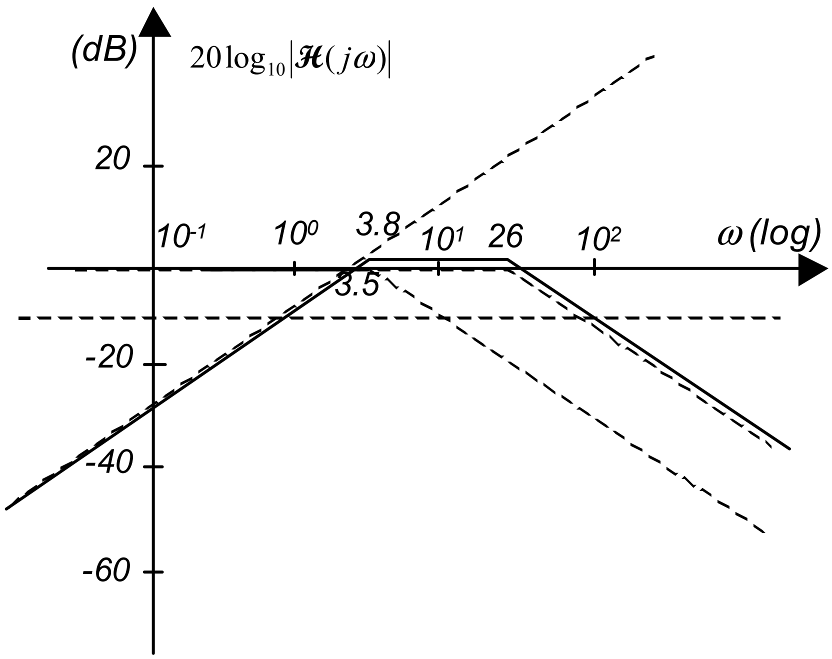
(c) [8 marks] Draw the Bode plot (magnitude and phase) of the frequency response from the input voltage $\mathcal{V}_s(j\omega)$ to the output voltage $\mathcal{V}(j\omega)$. Assume that the initial conditions on the capacitor and the inductor are 0. Use the numerical values: $R_1 = 1\,\Omega,\ R_2 = 1\,\Omega,\ L = 0.035425\text{ H},\ C = 0.564576\text{ F}$. What type of filter is it (lowpass, bandpass or highpass)?

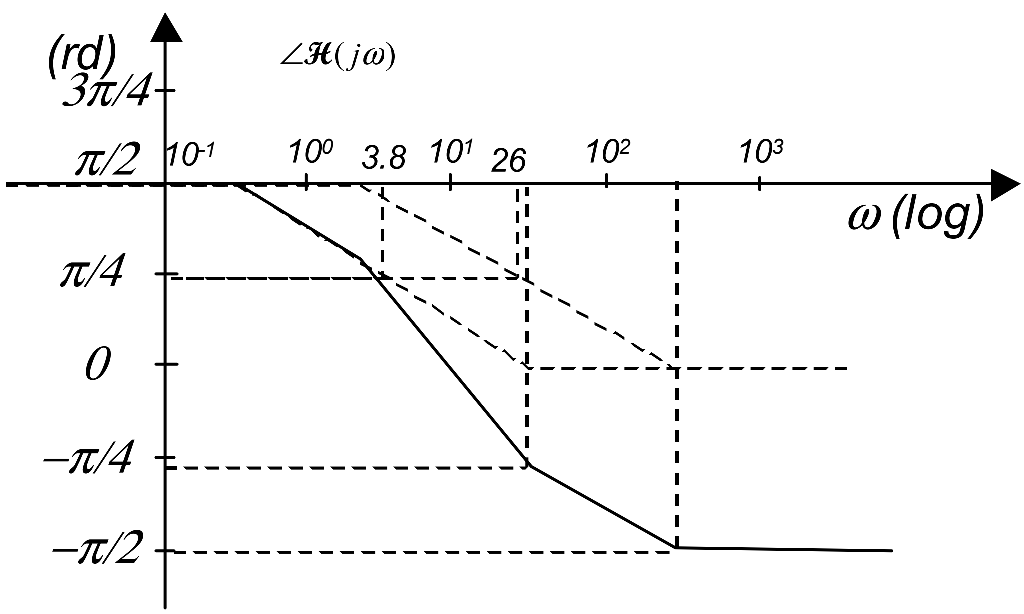
$$\mathcal{H}(s) = \frac{\mathcal{V}(s)}{\mathcal{V}_s(s)} = \frac{R_1 R_2 Cs}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2} = \frac{\dfrac{R_1 R_2 C}{R_1 + R_2}s}{\dfrac{LR_1 C}{R_1 + R_2}s^2 + \dfrac{(L + R_1 R_2 C)}{R_1 + R_2}s + 1}$$

Frequency response:

$$\mathcal{H}(j\omega) = \frac{\frac{R_1R_2C}{R_1+R_2}j\omega}{\frac{LR_1C}{R_1+R_2}(j\omega)^2 + \frac{(L+R_1R_2C)}{R_1+R_2}(j\omega)+1} = \frac{0.2823j\omega}{0.01(j\omega)^2+0.3(j\omega)+1}$$

from which we find $\omega_n = 10\text{rd/s}, \zeta = 1.5$ and the poles are $p_1 = -15+10\sqrt{(1.5)^2-1} = -3.82$ and $p_1 = -15-10\sqrt{(1.5)^2-1} = -26.18$. Gain is -11dB. Bode plot:

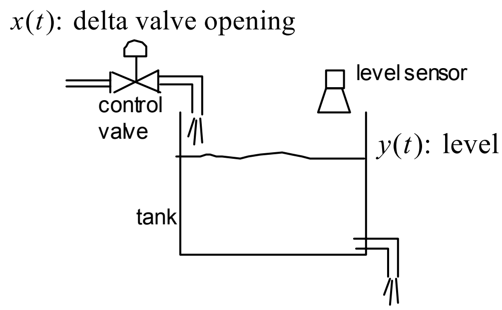




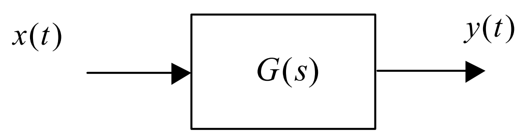
This is a bandpass filter.

Problem 7 (10 marks)

Suppose we want to control the level of liquid in a tank that is part of a continuous chemical process. We can do this by varying the opening of the control valve based on a feedback measurement of the tank level. The control valve input signal $x(t)$ is taken to be a delta variation of valve opening from its nominal opening. The output signal $y(t)$ is the tank level.



The open-loop tank dynamics are modeled by the following nonminimum phase transfer function



$$G(s) = \frac{-(s-2)}{s(s+1)}$$

(a) [5 marks] Compute and sketch the unit step response $y(t)$ of the tank level for a unit step in delta valve opening $x(t)$.

$$Y(s) = \frac{-(s-2)}{s^2(s+1)} = \frac{A}{(s+1)} + \frac{B}{s} + \frac{C}{s^2}$$

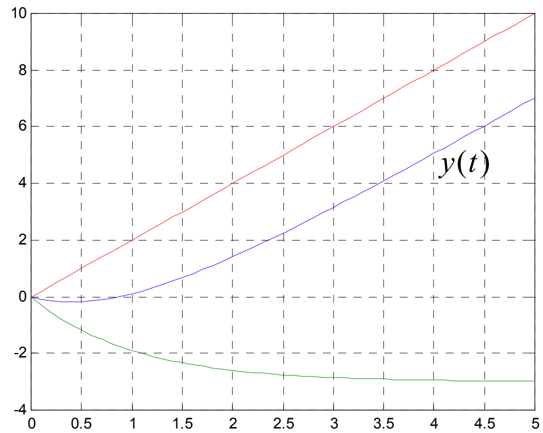
we find

$$Y(s) = \frac{3}{(s+1)} + \frac{-3}{s} + \frac{2}{s^2}$$

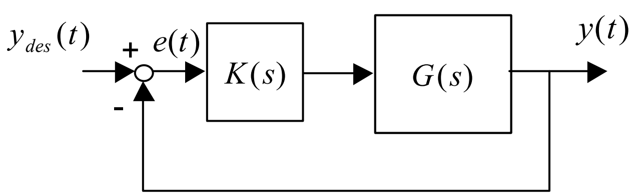
inverse transform:

$$y(t) = 3e^{-t}u(t) - 3u(t) + 2tu(t)$$

Plot:



(b) [5 marks] Assume that we can measure the tank level $y(t)$ perfectly. We want to test the feedback controller $K(s) = \lambda(s+\alpha)$ to control the level. That is, we want the tank level $y(t)$ to track the desired level $y_{des}(t)$.



Find the closed-loop transfer function $H(s) := Y(s)/Y_{des}(s)$. What is the final value $y(+\infty)$ of the step response?

Answer:

Closed-loop transfer function:

$$H(s) = \frac{K(s)G(s)}{1+K(s)G(s)}$$
$$= \frac{\dfrac{-\lambda(s+\alpha)(s-2)}{s(s+1)}}{1+\dfrac{-\lambda(s+\alpha)(s-2)}{s(s+1)}}$$
$$= \frac{-\lambda(s+\alpha)(s-2)}{s(s+1)-\lambda(s+\alpha)(s-2)}$$
$$= \frac{-\lambda(s+\alpha)(s-2)}{s^2+s-\lambda(s^2+(\alpha-2)s-2\alpha)}$$
$$= \frac{-\lambda(s+\alpha)(s-2)}{(1-\lambda)s^2+\left[1-\lambda(\alpha-2)\right]s+2\alpha\lambda}$$

With controller parameters $\lambda = 0.5$ and $\alpha = 3$:

$$H(s) = \frac{-0.5(s+3)(s-2)}{0.5s^2+0.5s+3} = \frac{-(s+3)(s-2)}{s^2+s+6}$$

Unit step response: NOT REQUIRED!!!

$$Y(s) = \frac{-(s+3)(s-2)}{s(s^2+s+6)}$$
$$= \frac{A}{s} + \frac{B(s+0.5)+C\sqrt{6}}{s^2+s+6}$$
$$= \frac{A}{s} + \frac{B(s+0.5)+C\sqrt{6}}{(s+0.5)^2+(\frac{\sqrt{23}}{2})^2} \quad \frac{\sqrt{23}}{2} = 2.398, \quad A=1$$

$$\Leftrightarrow$$

$$-s^2+s+6 = (s^2+s+6)+Bs(s+0.5)+Cs\sqrt{6}$$

$$\Leftrightarrow$$

$$B = -2,$$

$$0.5(-2)+C\sqrt{6} = 1 \Leftrightarrow C = \frac{2}{\sqrt{6}} = \sqrt{\frac{2}{3}}$$

Hence,

$$y(t) = u(t) - 2e^{-0.5t}\cos(\frac{\sqrt{23}}{2}t)u(t) + \sqrt{\frac{2}{3}}e^{-0.5t}\sin(\frac{\sqrt{23}}{2}t)u(t)$$

and the final value is

$y(+\infty) = H(0) = 1$.

END OF EXAMINATION