Solutions to Problems in Chapter 9

Problems with Solutions

Problem 9.1

The circuit in Figure 9.1 has initial conditions on the capacitor $v_C(0^-)$ and inductor $i_L(0^-)$.

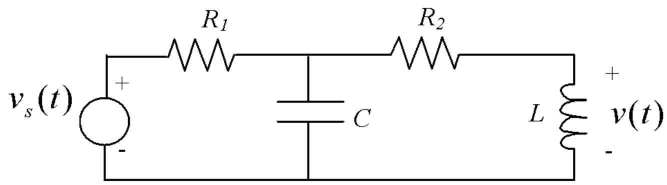


Figure 9.1: Circuit of Problem 9.1.

(a) Transform the circuit using the unilateral Laplace transform.

Answer:

The transform circuit is given in Figure 9.2.

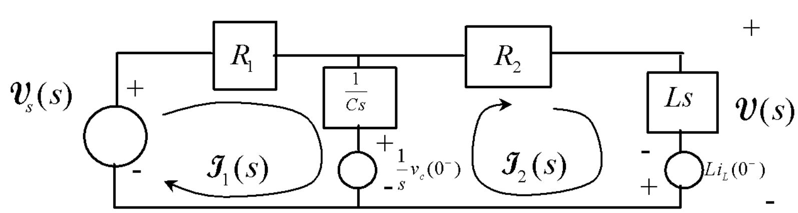


Figure 9.2: Transform circuit of Problem 9.1.

(b) Find the unilateral Laplace transform $\mathcal{V}(s)$ of $v(t)$.

Answer:

Let us use mesh analysis. For mesh 1:

$$\mathcal{V}_s(s) - R_1\mathcal{I}_1(s) - \frac{1}{s}v_c(0^-) - \frac{1}{Cs}[\mathcal{I}_1(s) - \mathcal{I}_2(s)] = 0$$
$$\Rightarrow \quad \mathcal{I}_2(s) = -Cs\mathcal{V}_s(s) + Cv_c(0^-) + (1 + R_1 Cs)\mathcal{I}_1(s)$$

For mesh 2:

$$\frac{1}{Cs}[\mathcal{I}_1(s) - \mathcal{I}_2(s)] + \frac{1}{s}v_C(0^-) + Li_L(0^-) - (R_2 + Ls)\mathcal{I}_2(s) = 0$$
$$\Rightarrow \quad \mathcal{I}_1(s) = (1 + R_2 Cs + LCs^2)\mathcal{I}_2(s) - Cv_C(0^-) - LCsi_L(0^-)$$

Substituting, we obtain:

$$\mathcal{I}_2(s) = -Cs\mathcal{V}_s(s) + Cv_c(0^-) + (1 + R_1 Cs)\left[(1 + R_2 Cs + LCs^2)\mathcal{I}_2(s) - Cv_C(0^-) - LCsi_L(0^-)\right]$$
$$[1 - (1 + R_1 Cs)(1 + R_2 Cs + LCs^2)]\mathcal{I}_2(s) = -Cs\mathcal{V}_s(s) + Cv_c(0^-) - (1 + R_1 Cs)[Cv_C(0^-) + LCsi_L(0^-)]$$
$$-[LR_1C^2s^3 + (R_1R_2C^2 + LC)s^2 + C(R_1 + R_2)s]\mathcal{I}_2(s) = -Cs\mathcal{V}_s(s) + Cv_c(0^-) - (1 + R_1 Cs)[Cv_C(0^-) + LCsi_L(0^-)]$$

Solving for $\mathcal{I}_2(s)$, we get:

$$\mathcal{I}_2(s) = \frac{\mathcal{V}_s(s)}{LR_1Cs^2 + (R_2R_1C + L)s + R_1 + R_2} + \frac{L(1 + R_1 Cs)i_L(0^-) + R_1 Cv_c(0^-)}{LR_1Cs^2 + (R_2R_1C + L)s + R_1 + R_2}$$

And finally the unilateral Laplace transform of the output voltage is given by

$$\mathcal{V}(s) = Ls\mathcal{I}_2(s) - i_L(0^-) = \frac{Ls\mathcal{V}_s(s)}{LR_1Cs^2 + (R_2R_1C + L)s + R_1 + R_2} + \frac{L^2s(1 + R_1 Cs)i_L(0^-) + R_1 LCsv_c(0^-)}{LR_1Cs^2 + (R_2R_1C + L)s + R_1 + R_2} - i_L(0^-)$$

(c) Give the transfer function $\mathcal{H}(s)$ from the source voltage $\mathcal{V}_s(s)$ to the output voltage $\mathcal{V}(s)$. What type of filter is it? (lowpass, highpass, bandpass?) Assuming that the poles of $\mathcal{H}(s)$ are complex, find expressions for its undamped natural frequency ω_n and damping ratio ζ.

Answer:

Notice that the transfer function from the source voltage to the output voltage is *bandpass*. Its gain at dc and infinite frequencies is 0. The transfer function $\mathcal{H}(s)$ from the source voltage $\mathcal{V}_s(s)$ to the output voltage $\mathcal{V}(s)$ is given by:

$$\mathcal{H}(s) = \frac{\mathcal{V}(s)}{\mathcal{V}_s(s)} = \frac{Ls}{LR_1Cs^2 + (R_2R_1C + L)s + R_1 + R_2}.$$

Its undamped natural frequency is $\omega_n = \sqrt{\frac{R_1 + R_2}{LR_1C}}$. The damping ratio is computed from

$$2\zeta\omega_n = \frac{R_2R_1C + L}{LR_1C} \Leftrightarrow \zeta = \frac{\frac{R_2R_1C + L}{LR_1C}}{2\sqrt{\frac{R_1 + R_2}{LR_1C}}} = \frac{R_2R_1C + L}{2\sqrt{(R_1 + R_2)LR_1C}}$$

(d) Assume that $R_1 = 100\Omega, R_2 = 100\,\Omega, \omega_n = 10$. Find the values of L and C to get Butterworth poles.

Answer:

The Butterworth poles are for:

$$\zeta = \frac{1}{\sqrt{2}} = \frac{R_2R_1C + L}{2\sqrt{(R_1 + R_2)LR_1C}} = \frac{10000C + L}{2\sqrt{20000LC}} \Rightarrow 200\sqrt{LC} = 10000C + L.$$

Furthermore, $\omega_n = 10 = \sqrt{\frac{2}{LC}} \Rightarrow LC = \frac{1}{50}$ and substituting in the previous equation, we get:

$$200\frac{1}{\sqrt{50}} = 10000C + \frac{1}{50C} \Rightarrow 0 = 500000C^2 - 200\sqrt{50}C + 1$$

$$0 = C^2 - \frac{2}{100\sqrt{50}}C + \frac{1}{500000} \Rightarrow C = \frac{1}{100\sqrt{50}}F$$

Finally: $$L = \frac{1}{50C} = \frac{1}{50\frac{1}{100\sqrt{50}}} = \frac{100}{\sqrt{50}}H.$$

Exercises

Problem 9.2

The circuit in Figure 9.3 has initial conditions on its capacitor $v_C(0^-)$ and inductor $i_L(0^-)$.

(a) Transform the circuit using the unilateral Laplace transform.

(b) Find the unilateral Laplace transform $\mathcal{V}(s)$ of $v(t)$.

(c) Give the transfer function $\mathcal{H}(s)$ from the source voltage $\mathcal{V}_s(s)$ to the output voltage $\mathcal{V}(s)$ (it should be second-order). What type of filter is it? (low-pass, high-pass, band-pass?) Assuming that the poles of $\mathcal{H}(s)$ are complex, find expressions for its undamped natural frequency ω_n and damping ratio ζ.

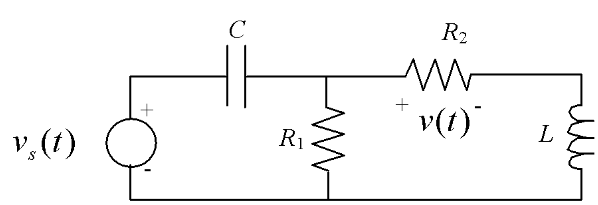


Figure 9.3: Transform circuit of Problem 9.2.

Problem 9.3

Consider the causal ideal op-amp circuit in Figure 9.4 (initially at rest) which implements a lowpass filter.

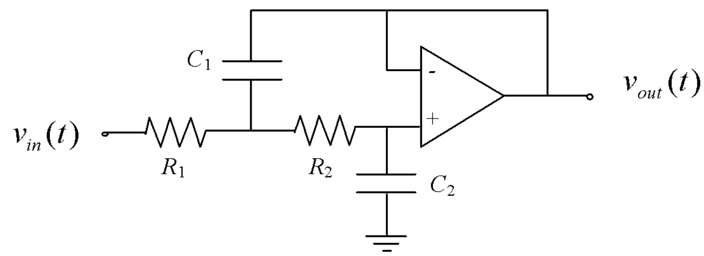


Figure 9.4: Op-amp filter circuit of Problem 9.3.

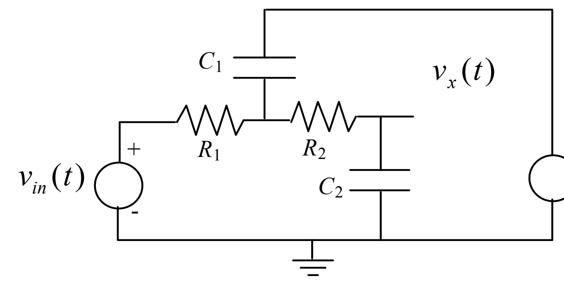
(a) Sketch the LTI model of the circuit with a voltage-controlled source representing the output of the op-amp, and assuming that its input impedance is infinite. Also, assume for this part that the op-amp gain A is finite.

(b) Transform the circuit using the Laplace transform, and find the transfer function $H_A(s) = V_{out}(s)/V_{in}(s)$. Then, let the op-amp gain $A \to +\infty$ to obtain the transfer function $H(s) = \lim_{A\to+\infty} H_A(s)$.

(c) Find expressions for the circuit's undamped natural frequency and damping ratio.

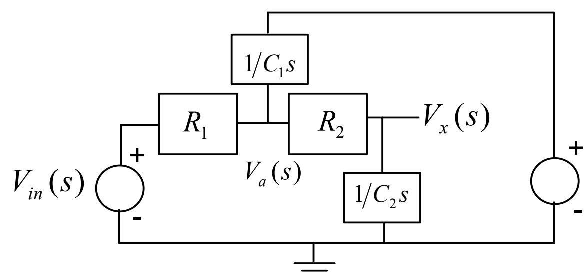
Answer:

(a)



$$v_{out}(t) = A\left[v_x(t) - v_{out}(t)\right]$$
$$= \frac{A}{1+A} v_x(t)$$

(b) The transformed circuit is



$$V_{out}(s) = \frac{A}{1+A} V_x(s)$$

There are two supernodes for which the nodal voltages are given by the source voltages. The remaining two nodal equations are

(1) $$\frac{V_{in}(s) - V_a(s)}{R_1} = \frac{V_a(s) - V_x(s)}{R_2} + \left[V_a(s) - \frac{A}{1+A} V_x(s)\right] C_1 s$$

(2) $$\frac{V_a(s) - V_x(s)}{R_2} = V_x(s) C_2 s$$

From (2): $V_a(s) = (R_2 C_2 s + 1) V_x(s)$. Substituting in (1), we get:

$$R_2 V_{in}(s) - R_2 V_a(s) = R_1 V_a(s) - R_1 V_x(s) + R_1 R_2 C_1 s V_a(s) - \frac{A}{1+A} R_1 R_2 C_1 s V_x(s)$$

$$R_2 V_{in}(s) = (R_1 + R_2 + R_1 R_2 C_1 s) V_a(s) - \left(R_1 + \frac{A}{1+A} R_1 R_2 C_1 s \right) V_x(s)$$

$$R_2 V_{in}(s) = \left((R_1 + R_2 + R_1 R_2 C_1 s)(R_2 C_2 s + 1) - R_1 - \frac{A}{1+A} R_1 R_2 C_1 s \right) V_x(s)$$

Thus, the transfer function between the input voltage and the node voltage is given by

$$\frac{V_x(s)}{V_{in}(s)} = \frac{R_2}{(R_1 + R_2 + R_1 R_2 C_1 s)(R_2 C_2 s + 1) - R_1 - \frac{A}{1+A} R_1 R_2 C_1 s}.$$

The transfer function between the input voltage and the output voltage is

$$H_A(s) = \frac{V_{out}(s)}{V_{in}(s)} = \frac{\frac{A}{1+A} V_x(s)}{V_{in}(s)} = \frac{\frac{A}{1+A} R_2}{(R_1 + R_2 + R_1 R_2 C_1 s)(R_2 C_2 s + 1) - R_1 - \frac{A}{1+A} R_1 R_2 C_1 s}$$

The ideal transfer function is the limit as the op-amp gain tends to infinity:

$$H(s) = \frac{V_{out}(s)}{V_{in}(s)} = \lim_{A \to \infty} H_A(s) = \frac{R_2}{(R_1 + R_2 + R_1 R_2 C_1 s)(R_2 C_2 s + 1) - R_1 - R_1 R_2 C_1 s}$$

$$= \frac{R_2}{(R_1 R_2 C_2 s + R_2{}^2 C_2 s + R_1 R_2^2 C_1 C_2 s^2 + R_1 + R_2 + R_1 R_2 C_1 s) - R_1 - R_1 R_2 C_1 s}$$

$$= \frac{1}{R_1 C_2 s + R_2 C_2 s + R_1 R_2 C_1 C_2 s^2 + 1} = \frac{1}{R_1 R_2 C_1 C_2 s^2 + (R_1 + R_2) C_2 s + 1}$$

(c) Given a desired TF in the form: $H = \dfrac{1}{\dfrac{s^2}{\omega_n{}^2} + \dfrac{2\zeta}{\omega_n} s + 1}$, we can identify ω_n and ζ:

$$\omega_n = \frac{1}{\sqrt{R_1 R_2 C_1 C_2}}, \quad \zeta = \frac{(R_1 + R_2)C_2}{2\sqrt{R_1 R_2 C_1 C_2}} = \frac{(R_1 + R_2)C_2}{2}\omega_n .$$

Problem 9.4

Consider the causal op-amp circuit initially at rest depicted in Figure 9.5. Its LTI circuit model with a voltage-controlled source is also given in the figure.

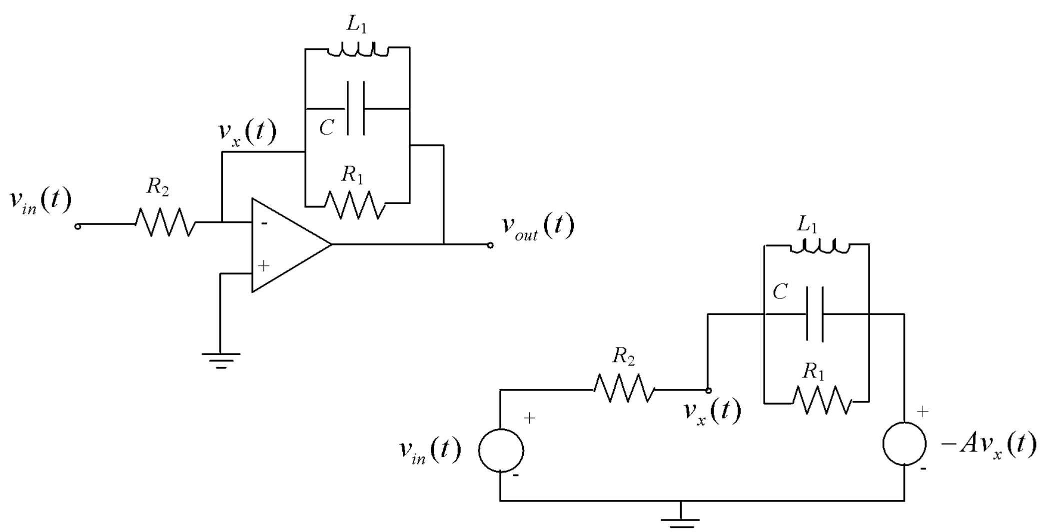


Figure 9.5: Op-amp circuit of Problem 9.4.

(a) Transform the circuit using the Laplace transform, and find the transfer function $H_A(s) = V_{out}(s)/V_{in}(s)$. Then, let the op-amp gain $A \to +\infty$ to obtain the ideal transfer function $H(s) = \lim_{A \to +\infty} H_A(s)$.

(b) Assume that the transfer function $H_1(s) = \frac{H(s)}{s}$ has a dc gain of -50, and that $H(s)$ has one zero at 0 and two complex conjugate poles with $\omega_n = 10$ rd/s, $\zeta = 0.5$. Let $L_1 = 10H$. Find the values of the remaining circuit components R_1, R_2, C.

(c) Give the frequency response of $H(s)$ and sketch its Bode plot.

Problem 9.5

The circuit in Figure 9.6 has initial conditions on the capacitor $v_C(0^-)$ and inductor $i_L(0^-)$.

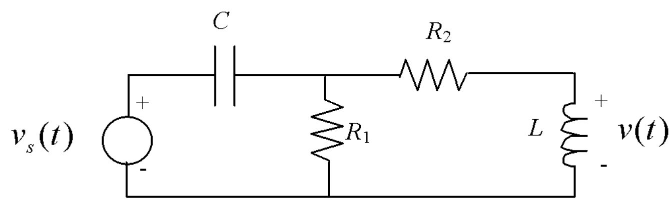
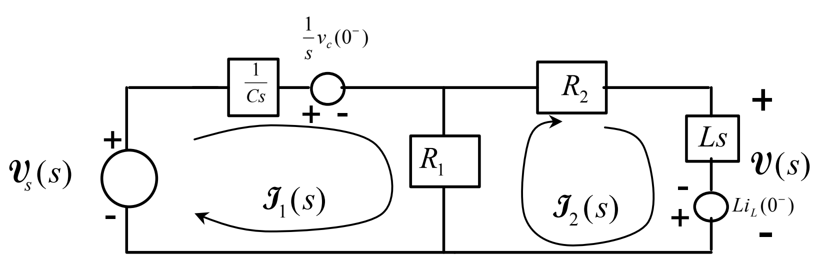


Figure 9.6: Circuit of Problem 9.5.

(a) Transform the circuit using the unilateral Laplace transform.

Answer:



(b) Find the unilateral Laplace transform of $v(t)$.

Answer:

Let's use mesh analysis.

For mesh 1:

$$\mathcal{V}_s(s) - \frac{1}{Cs}\mathcal{I}_1(s) - \frac{1}{s}v_c(0^-) - R_1[\mathcal{I}_1(s) - \mathcal{I}_2(s)] = 0$$
$$\Rightarrow \quad \mathcal{I}_2(s) = -\frac{1}{R_1}\mathcal{V}_s(s) + \frac{1}{R_1 s}v_c(0^-) + (1 + \frac{1}{R_1 Cs})\mathcal{I}_1(s)$$

For mesh 2:

$$R_1[\mathcal{I}_1(s) - \mathcal{I}_2(s)] - (R_2 + Ls)\mathcal{I}_2(s) + Li_L(0^-) = 0$$
$$\Rightarrow \quad \mathcal{I}_1(s) = \frac{1}{R_1}(R_1 + R_2 + Ls)\mathcal{I}_2(s) - \frac{L}{R_1}i_L(0^-)$$

Substituting, we obtain

$$\mathcal{I}_2(s) = -\frac{1}{R_1}\mathcal{V}_s(s) + \frac{1}{R_1 s}v_c(0^-) + (1 + \frac{1}{R_1 Cs})\left[\frac{1}{R_1}(R_1 + R_2 + Ls)\mathcal{I}_2(s) - \frac{L}{R_1}i_L(0^-)\right]$$
$$[R_1^2 Cs - (1 + R_1 Cs)(R_1 + R_2 + Ls)]\mathcal{I}_2(s) = -R_1 Cs\mathcal{V}_s(s) + R_1 Cv_c(0^-) - (1 + R_1 Cs)Li_L(0^-)$$
$$-[LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2]\mathcal{I}_2(s) = -R_1 Cs\mathcal{V}_s(s) + R_1 Cv_c(0^-) - (1 + R_1 Cs)Li_L(0^-)$$

Solving for $\mathcal{I}_2(s)$, we get

$$\mathcal{I}_2(s) = \frac{R_1 Cs\mathcal{V}_s(s)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{(1 + R_1 Cs)Li_L(0^-) - R_1 Cv_c(0^-)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2}$$

And finally the output voltage is

$$\mathcal{V}(s) = Ls\mathcal{I}_2(s) - Li_L(0^-) = \frac{LR_1 Cs^2\mathcal{V}_s(s)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{(1 + R_1 Cs)L^2 si_L(0^-) - R_1 CLsv_c(0^-)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2} - Li_L(0^-)$$
$$= \frac{LR_1 Cs^2\mathcal{V}_s(s)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{\{(1 + R_1 Cs)L^2 s - [L^2 R_1 Cs^2 + L(L + R_1 R_2 C)s + L(R_1 + R_2)]\}i_L(0^-) - R_1 CLsv_c(0^-)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2}$$
$$= \frac{LR_1 Cs^2\mathcal{V}_s(s)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2} + \frac{-[LR_1 R_2 Cs + L(R_1 + R_2)]i_L(0^-) - R_1 CLsv_c(0^-)}{LR_1 Cs^2 + (L + R_1 R_2 C)s + R_1 + R_2}$$

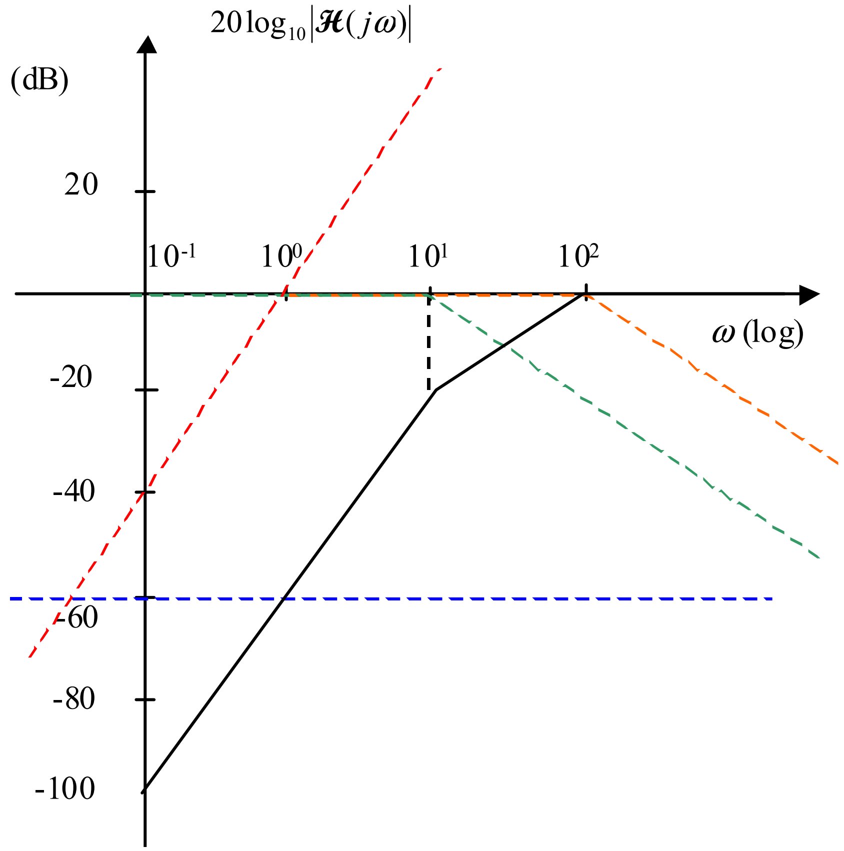
(c) Sketch the Bode plot (magnitude and phase) of the frequency response from the input voltage $\mathcal{V}_s(j\omega)$ to the output voltage $\mathcal{V}(j\omega)$. Assume that the initial conditions on the capacitor and the inductor are 0. Use the numerical values: $R_1 = 1\,\Omega,\ R_2 = \frac{109}{891}\,\Omega,\ L = \frac{1}{891}\,\text{H},\ C = 1\,\text{F}$.

Answer:

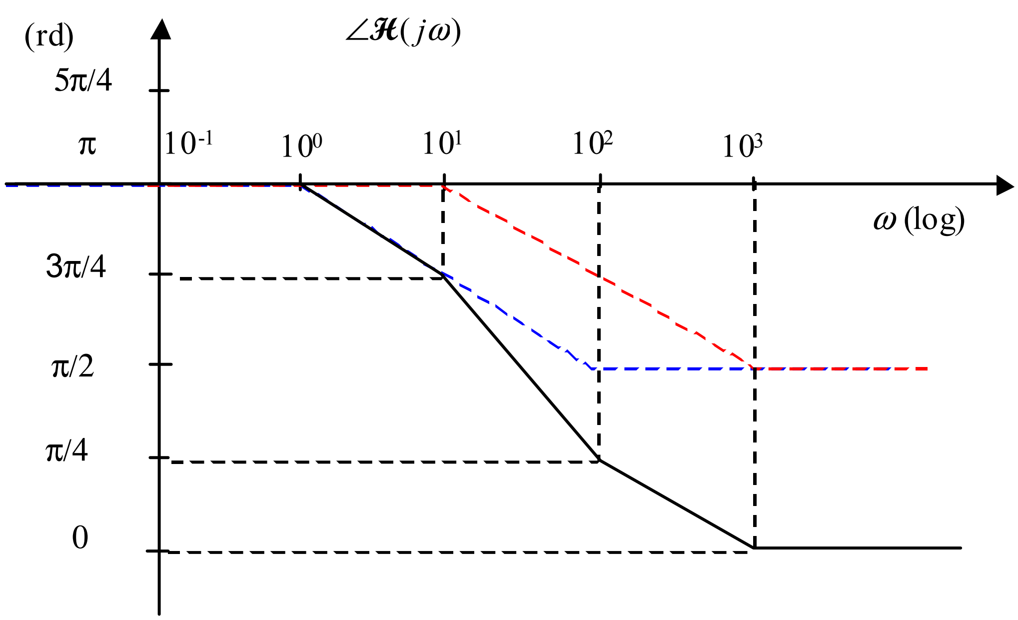
For the values given, the transfer function from the source voltage to the output voltage is

$$\begin{aligned}
\mathcal{H}(s) := \frac{\mathcal{V}(s)}{\mathcal{V}_s(s)} &= \frac{s^2}{s^2 + \frac{L + R_1 R_2 C}{L R_1 C} s + \frac{R_1 + R_2}{L R_1 C}} \\
&= \frac{s^2}{s^2 + \frac{\frac{1}{891} + \frac{109}{891}}{\frac{1}{891}} s + \frac{1 + \frac{109}{891}}{\frac{1}{891}}} \\
&= \frac{s^2}{s^2 + 110s + 1000} \\
&= \frac{s^2}{(s+10)(s+100)} \\
&= \frac{1}{1000} \frac{s^2}{(\frac{1}{10}s + 1)(\frac{1}{100}s + 1)}
\end{aligned}$$

Bode Plot:



20log10|ℋ(jω)|
(dB)
20
-20
-40
-60
-80
-100
10-1
100
101
102
ω (log)



(rd)
∠ℋ(jω)
5π/4
π
3π/4
π/2
π/4
0
10-1
100
101
102
103
ω (log)