Solutions to Problems in Chapter 8

Problems with Solutions

Problem 8.1

Sketch the pole-zero plots in the s-plane and the Bode plots (magnitude and phase) for the following systems. Specify if the transfer functions have poles or zeros at infinity.

(a) $H(s)=\dfrac{100(s-1)(s+10)}{(s+100)^2},\ \mathrm{Re}\{s\}>-100$.

Answer:

$$H(s)=\frac{100(s-1)(s+10)}{(s+100)^2}=H(s)=\frac{-0.1(-s+1)(s/10+1)}{(0.01s+1)(0.01s+1)},\ \mathrm{Re}\{s\}>-100$$

Break frequencies at $\omega_1=1, \omega_2=10$ (zeros); $\omega_3=100$ (double pole). The pole-zero plot is shown in Figure 8.1, and the Bode plot is in Figure 8.2.

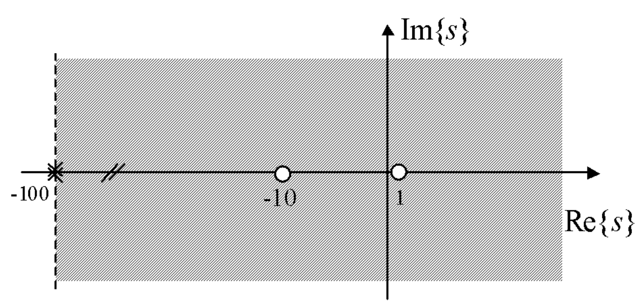


Figure 8.1: Pole-zero plot of transfer function of Problem 8.1(a).

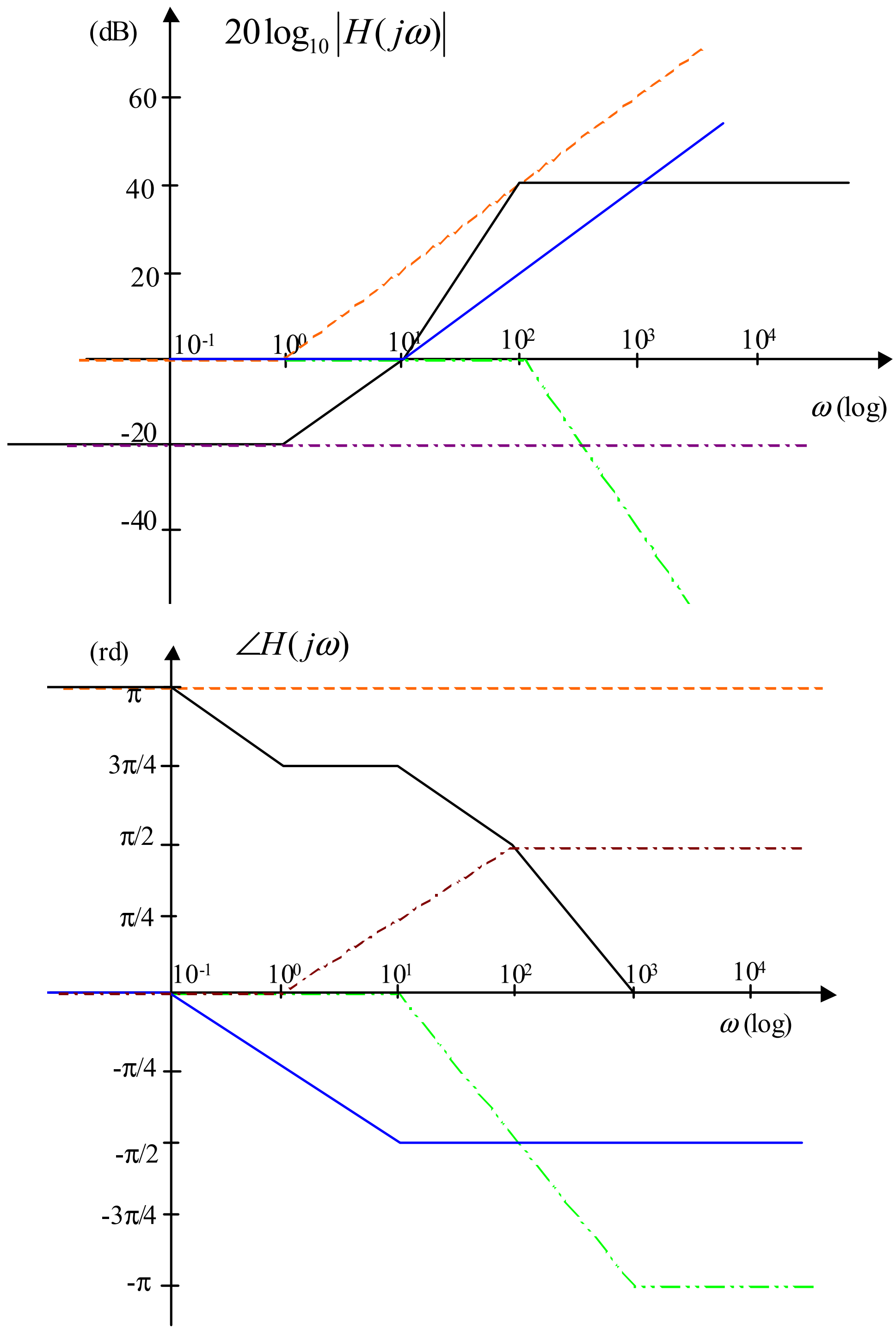


Figure 8.2: Bode plot of transfer function of Problem 8.1(a).

(b) $H(s)=\dfrac{s+10}{s(0.001s+1)},\ \ \mathrm{Re}\{s\}>0$.

Answer:

$$H(s)=\frac{s+10}{s(0.001s+1)}=\frac{10(s/10+1)}{s(s/1000+1)},\ \ \mathrm{Re}\{s\}>0.$$

The pole-zero plot is given in Figure 8.3.

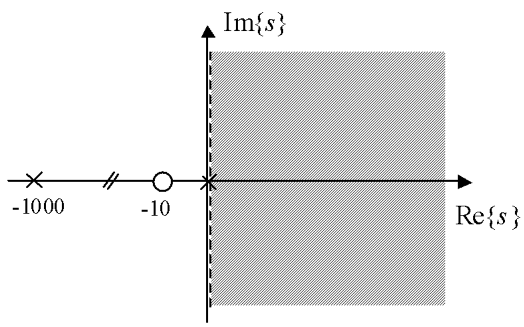


Figure 8.3: Pole-zero plot of transfer function of Problem 8.1(b).

The break frequencies are $\omega_1=10$ (zero); $\omega_2=0, \omega_3=1000$ (poles), and the transfer function has one zero at ∞. The Bode plot is shown in Figure 8.4.

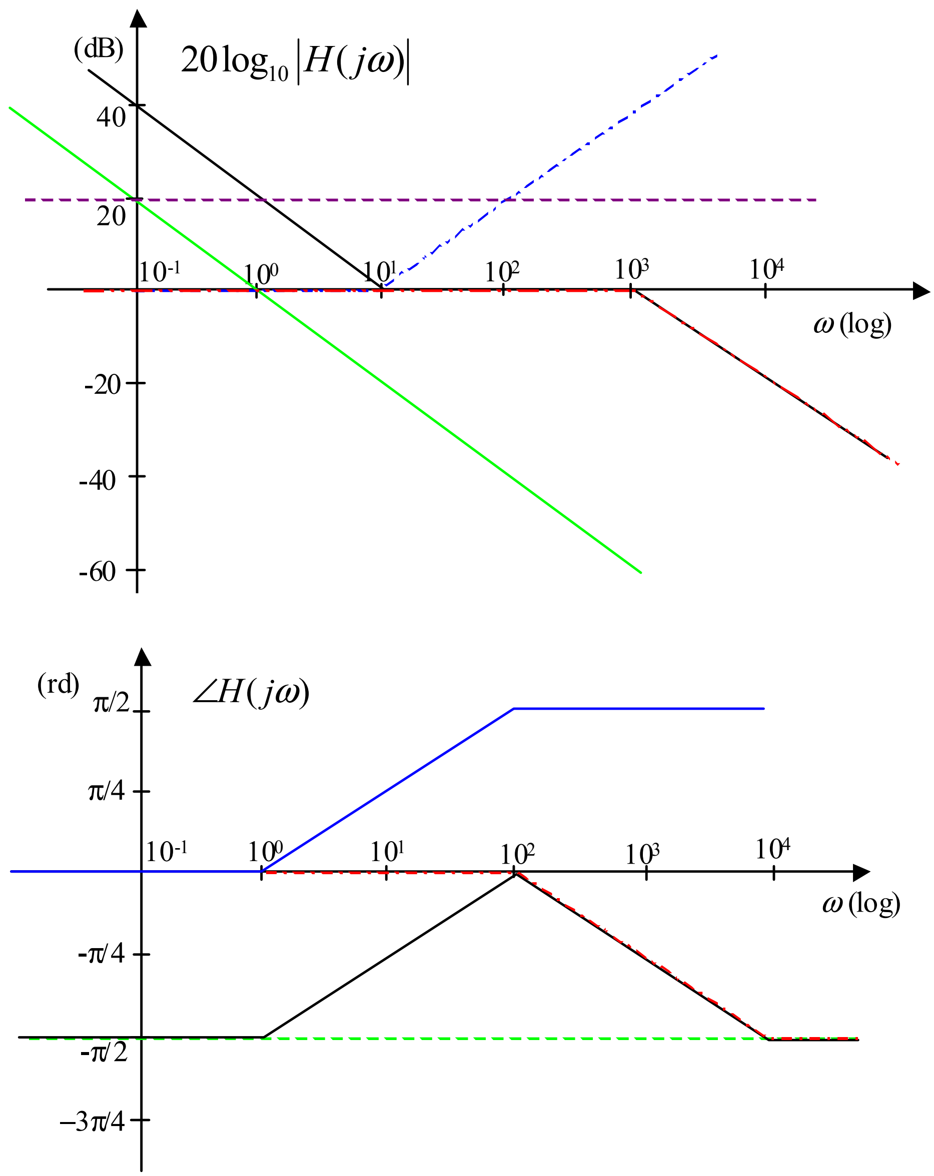


Figure 8.4: Bode plot of transfer function of Problem 8.1(b).

(c) $H(s) = \dfrac{s^2 + 2s + 1}{(s+100)(s^2 + 10s + 100)}, \;\; \mathrm{Re}\{s\} > -5$

Answer:

We can write the transfer function as follows:

$$H(s) = \frac{0.0001(s+1)^2}{(0.01s+1)(0.01s^2 + 0.1s + 1)} = \frac{0.0001(s+1)^2}{(0.01s+1)(s+5-j5\sqrt{3})(s+5+j5\sqrt{3})}, \;\; \mathrm{Re}\{s\} > -5$$

The pole zero plot is shown in Figure 8.5, and the Bode plot is in Figure 8.6.

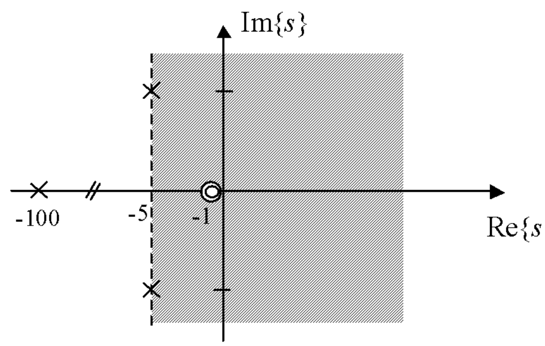


Figure 8.5: Pole-zero plot of transfer function of Problem 8.1(c).

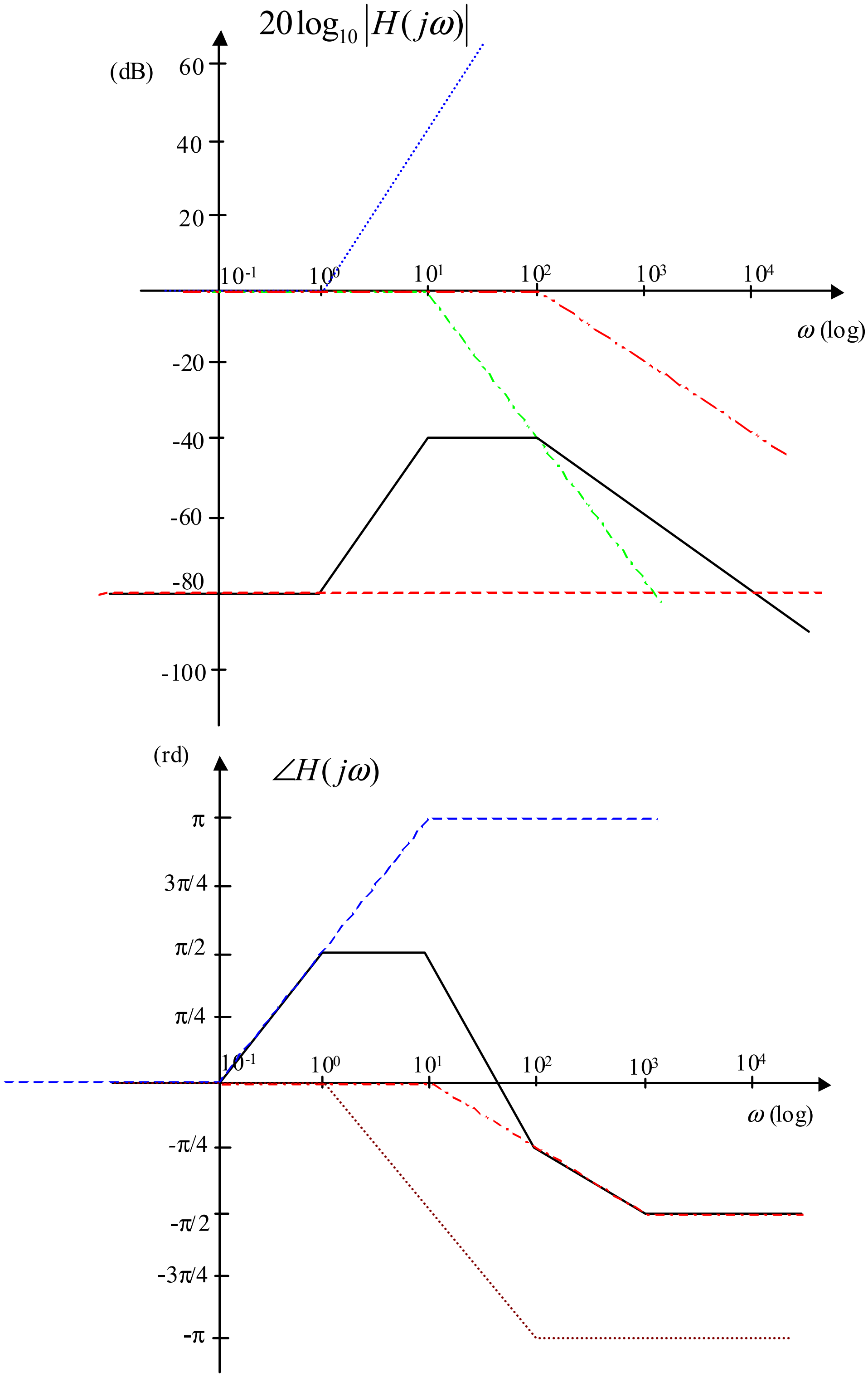


Figure 8.6: Bode plot of transfer function of Problem 8.1(c).

Problem 8.2

Consider the mechanical system of Figure 8.7 consisting of a mass m attached to a spring of stiffness k and a viscous damper (dashpot) of damping factor b, both rigidly connected to ground. This basic system model is quite useful to study a number of systems, including a car's suspension, or a flexible robot link.

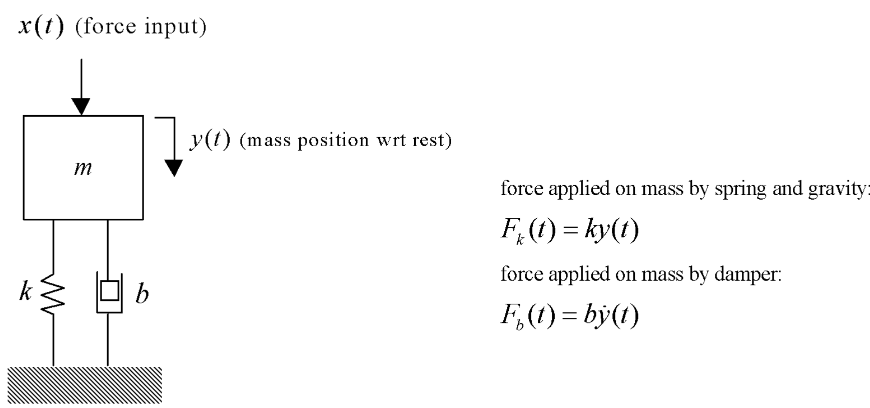


Figure 8.7: Mass-spring-damper system of Problem 8.2.

Assume that the mass-spring-damper system is initially at rest, which means that the spring generates a force equal to the force of gravity to support the mass. The balance of forces on the mass causing motion is the following:

$$x(t) - F_k(t) - F_b(t) = m\ddot{y}(t).$$

(a) Write the differential equation governing the motion of the mass.

Answer:

$$m\ddot{y}(t) + b\dot{y}(t) + ky(t) = x(t)$$

(b) Find the transfer function of the system relating the applied force to the mass position. Express it in the form $H(s)=\dfrac{A\omega_n^2}{s^2+2\varsigma\omega_n s+\omega_n^2}$. What is the damping ratio ζ for this mechanical system? What is its undamped natural frequency ω_n?

Answer:

$$\begin{aligned}H(s)&=\frac{1}{ms^2+bs+k}\\&=\frac{(1/k)(k/m)}{s^2+\frac{b}{m}s+\frac{k}{m}}\end{aligned}$$

The natural frequency is given by $\omega_n^2=\frac{k}{m}\Rightarrow\ \omega_n=\sqrt{\frac{k}{m}}$. Note: the larger the mass, the lower the undamped natural frequency; the stiffer the spring, the higher the undamped natural frequency. The damping ratio of the system is then:

$$\zeta=\frac{\frac{b}{m}}{2\omega_n}=\frac{\frac{b}{m}}{2\sqrt{\frac{k}{m}}}=\frac{b}{2\sqrt{mk}}.$$

For a given dashpot, the larger the mass and/or spring constant, the less damped the system will be.

(c) Let the physical constants have numerical values $m=2\,kg$, $k=8\,N/m$, and $b=4\,N/\tfrac{m}{s}$. Suppose that the applied force is a step $x(t)=3u(t)\,N$. Compute and sketch the resulting mass position for all times. What is the mass position in steady-state? What is the percentage of the first overshoot in the step response? What is the ±5% settling time of the mass? (a numerical answer will suffice.)

Answer:

With the numerical values given, the damping ratio and undamped natural frequencies are:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8\,\frac{\frac{kgm}{s^2}}{m}}{2kg}} = 2\,\tfrac{rd}{s}$$

$$\zeta = \frac{b}{2\sqrt{mk}} = \frac{4\,\frac{N}{m/s}}{2\sqrt{16\,kg\,\frac{N}{m}}} = 0.5$$

The step input force is $x(t) = 3u(t) \leftrightarrow \frac{3}{s},\ \mathrm{Re}\{s\} > 0$. The Laplace transform of the step response is given by:

$$\begin{aligned} Y(s) &= \frac{1.5}{s(s^2 + 2s + 4)} \\ &= \frac{-0.18750 + j0.10826}{s + 1 - j\sqrt{3}} + \frac{-0.18750 - j0.10826}{s + 1 + j\sqrt{3}} + \frac{0.375}{s} \end{aligned}$$

Taking the inverse Laplace transforms of the partial fractions and simplifying, we get

$$\begin{aligned} y(t) &= \left[(-0.18750 + j0.10826)e^{(-1+j\sqrt{3})t} + (-0.18750 - j0.10826)e^{(-1-j\sqrt{3})t} \right] u(t) + 0.375u(t) \\ &= 2e^{-t}\,\mathrm{Re}\left[(-0.18750 + j0.10826)e^{j\sqrt{3}t} \right] u(t) + 0.375u(t) \\ &= 2e^{-t}\left[-0.18750\cos\sqrt{3}t - 0.10826\sin\sqrt{3}t \right] u(t) + 0.375u(t) \quad m \end{aligned}$$

This step response is plotted in Figure 8.8. The mass position in steady state is 0.375m.

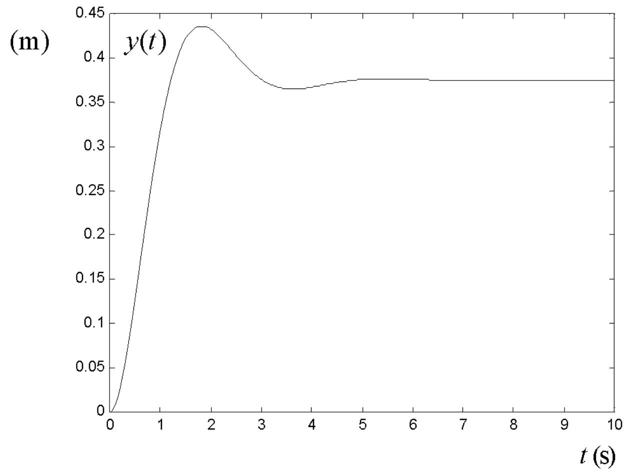


Figure 8.8: Step response of mass-spring-damper system of Problem 8.2.

The ±5% settling time of the mass is found to be $t_s = 2.65$. Percentage of overshoot:

$$OS = 100e^{-\frac{\varsigma\pi}{\sqrt{1-\varsigma^2}}}\% = 100e^{-\frac{0.5\pi}{\sqrt{0.75}}}\% = 16.3\%.$$

Exercises

Problem 8.3

Compute the 95% rise time t_r and the ±5% settling time t_s of the step response of the system

$$H(s) = \frac{0.001s+1}{0.1s+1}, \ \mathrm{Re}\{s\} > -10.$$

Answer:

Here, we have a first-order lag with $\alpha = 0.01$ and time constant $\tau = 0.1$, i.e.,

$$H(s) = \frac{\alpha\tau s+1}{\tau s+1} = \frac{0.01(0.1)s+1}{0.1s+1} = 0.01 + \frac{0.99}{0.1s+1},$$

the step response is: $s(t) = 0.01u(t) + 0.99(1-e^{-10t})u(t)$.

Rise time: $t_r = t_{95\%} - t_{5\%} = 2.9444\tau = 0.29444s$.

±5% Settling time: $t_s = t_{95\%} = 2.9957\tau = 0.29957s$

Problem 8.4

Compute the DC gain in dBs, the peak resonance in dBs, and the quality Q of the second-order causal filter with transfer function: $H(s) = \frac{1000}{s^2 + 2s + 100}$.

Problem 8.5

Compute the actual value of the first overshoot in the step response of the causal LTI system

$$H(s) = \frac{5}{3s^2 + 3s + 6}.$$

Answer:

$H(s) = \frac{5}{3s^2 + 3s + 6} = \frac{5/3}{s^2 + s + 2}$ has a DC gain of $\frac{5}{6} = 0.833$, which is also equal to the settling value of the step response of the system. The damping ratio is found to be $\zeta = \frac{1}{2\sqrt{2}}$, which gives a percentage of overshoot of $OS = 100e^{-\frac{\varsigma\pi}{\sqrt{1-\varsigma^2}}}\% = 100e^{-\frac{\frac{1}{2\sqrt{2}}\pi}{\sqrt{\frac{7}{8}}}}\% = 100e^{-\frac{\pi}{\sqrt{7}}}\% = 30.5\%$. Therefore, the value of the first overshoot in the step response is given by: $1.305 * \frac{5}{6} = 1.088$.

Problem 8.6

Compute the group delay of a communication channel represented by the causal first-order system $H(s)=\dfrac{1}{0.01s+1}$, $\mathrm{Re}\{s\}>-100$. Compute the approximate value of the channel's delay at very low frequencies.

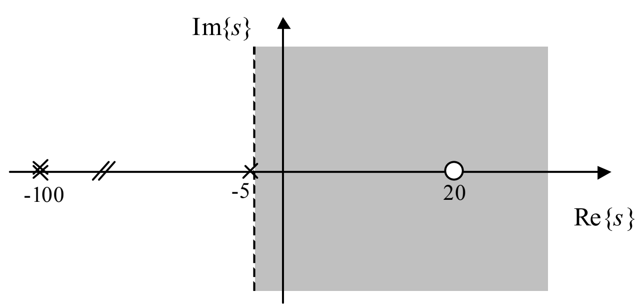
Problem 8.7

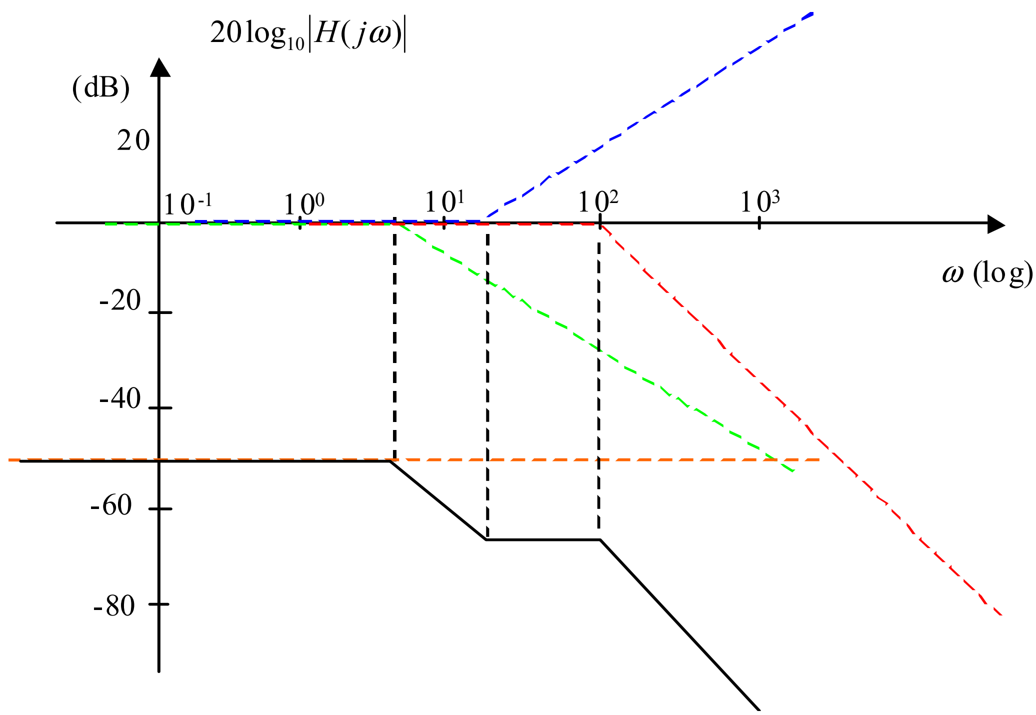
Sketch the pole-zero plots in the s-plane and the Bode plots (magnitude and phase) for the following systems. Specify whether the transfer functions have poles or zeros at infinity.

(a) $H(s)=\dfrac{100(s-20)}{(s+5)(s+100)^2}$, $\mathrm{Re}\{s\}>-5$.

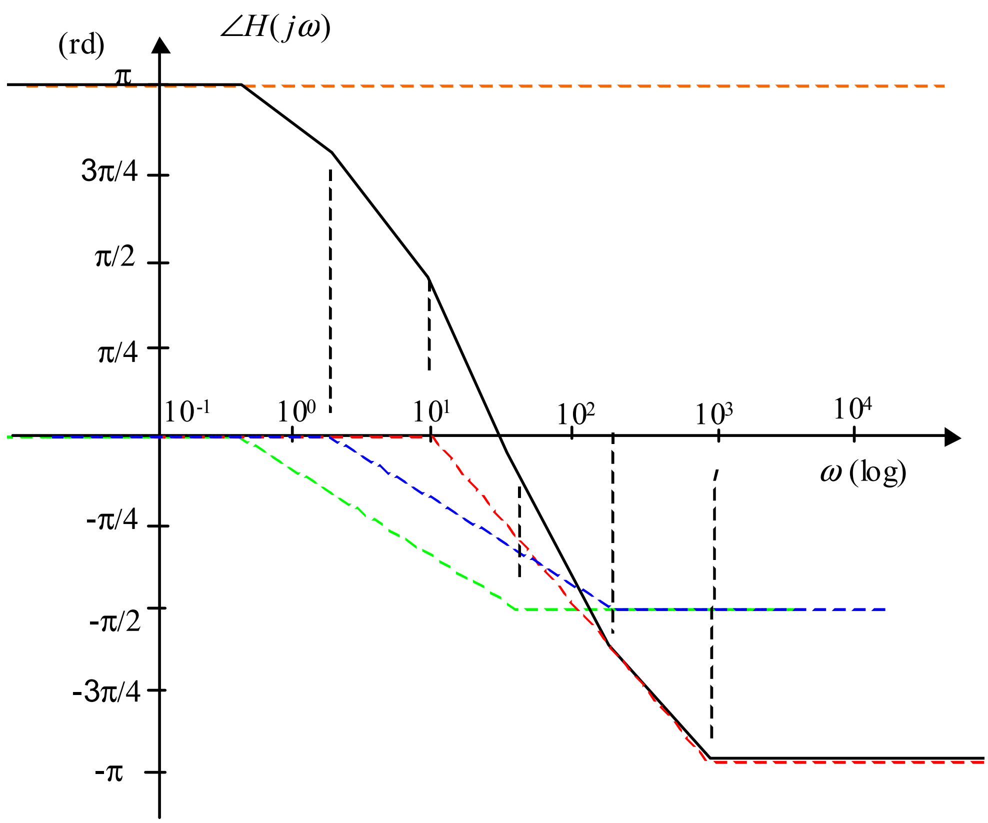
$$H(s)=\frac{100(s-20)}{(s+5)(s+100)^2}=-\frac{1}{250}\frac{(-s/20+1)}{(s/5+1)(s/100+1)^2}$$

Break frequencies at $\omega_1=20$ (zero); $\omega_2=5, \omega_3=100(\times 2)$ (poles), two zeros at ∞.





$20\log_{10}|H(j\omega)|$
(dB)
20
-20
-40
-60
-80
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
ω (log)

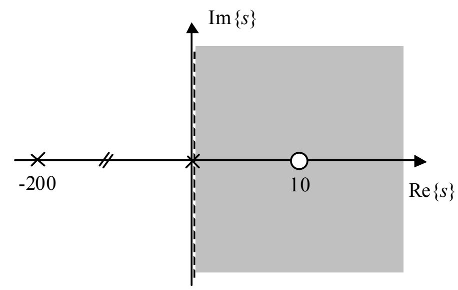


$\angle H(j\omega)$
(rd)
π
3π/4
π/2
π/4
-π/4
-π/2
-3π/4
-π
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
ω (log)

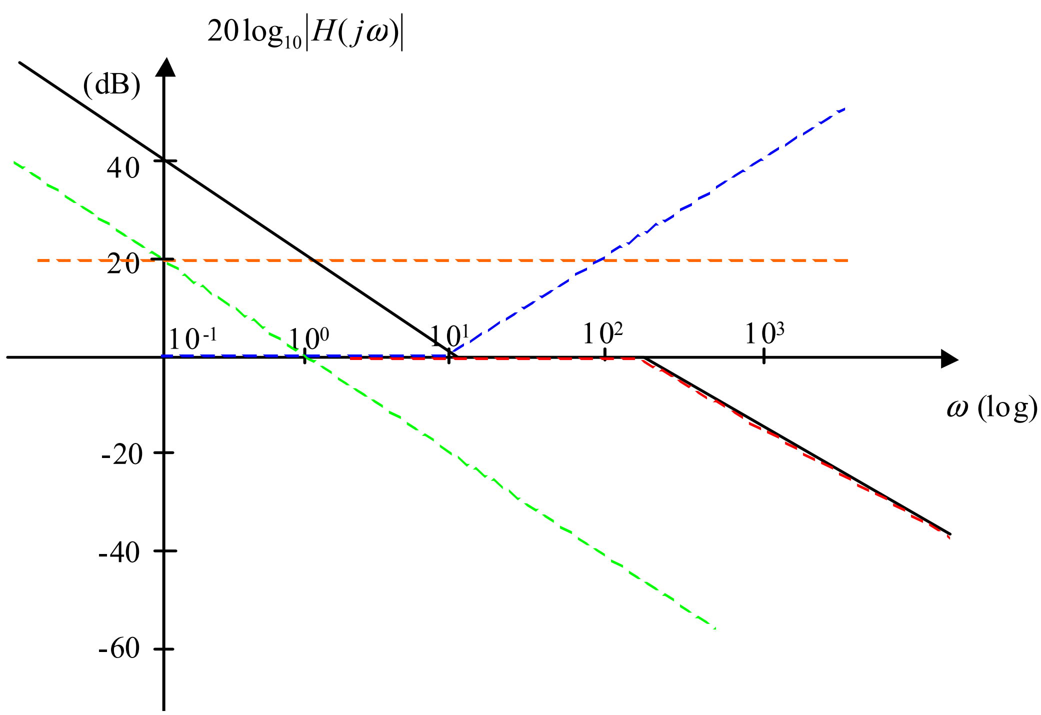
(b) $H(s)=\dfrac{-s+10}{s(0.005s+1)},\;\; \mathrm{Re}\{s\}>0$.

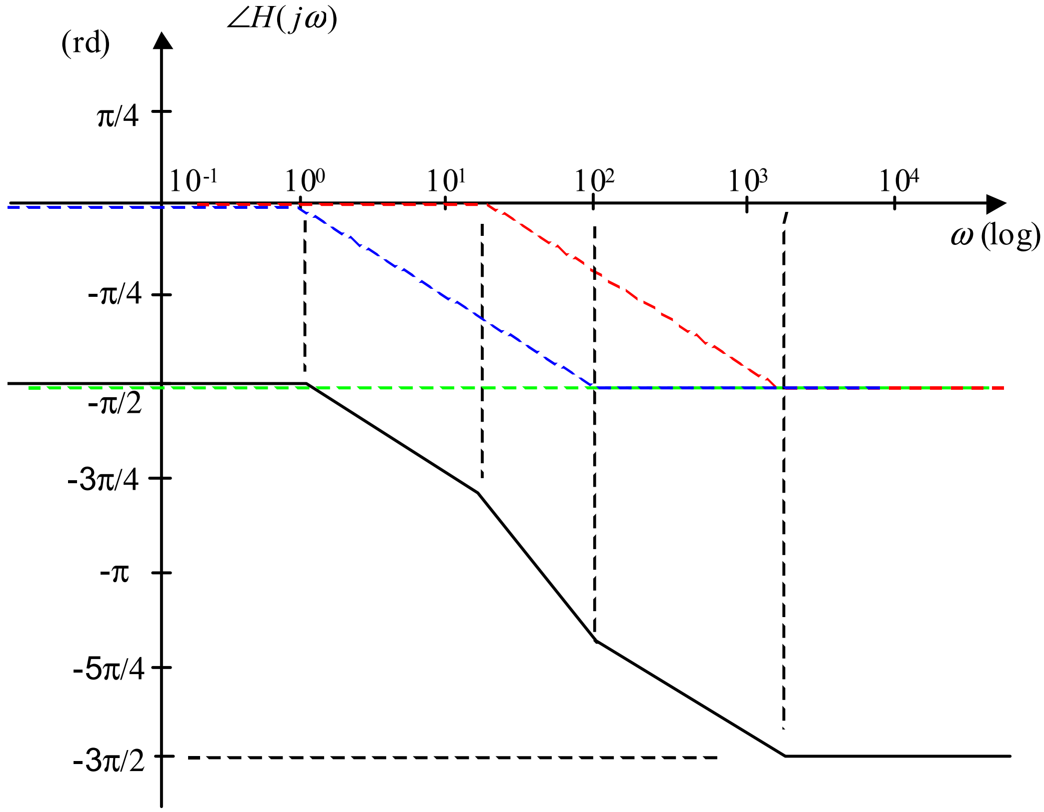
Answer:

$$H(s)=\frac{-s+10}{s(0.005s+1)}=\frac{10(-s/10+1)}{s(0.005s+1)}.$$



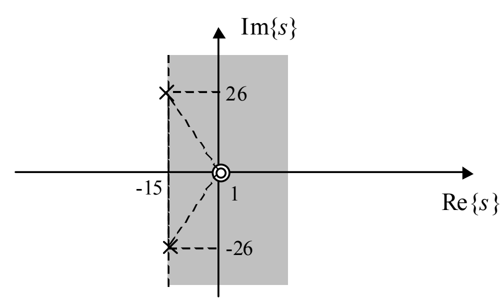
Break frequencies at $\omega_1=10$ (zero); $\omega_2=0,\,\omega_3=200$ (poles), one zero at ∞.



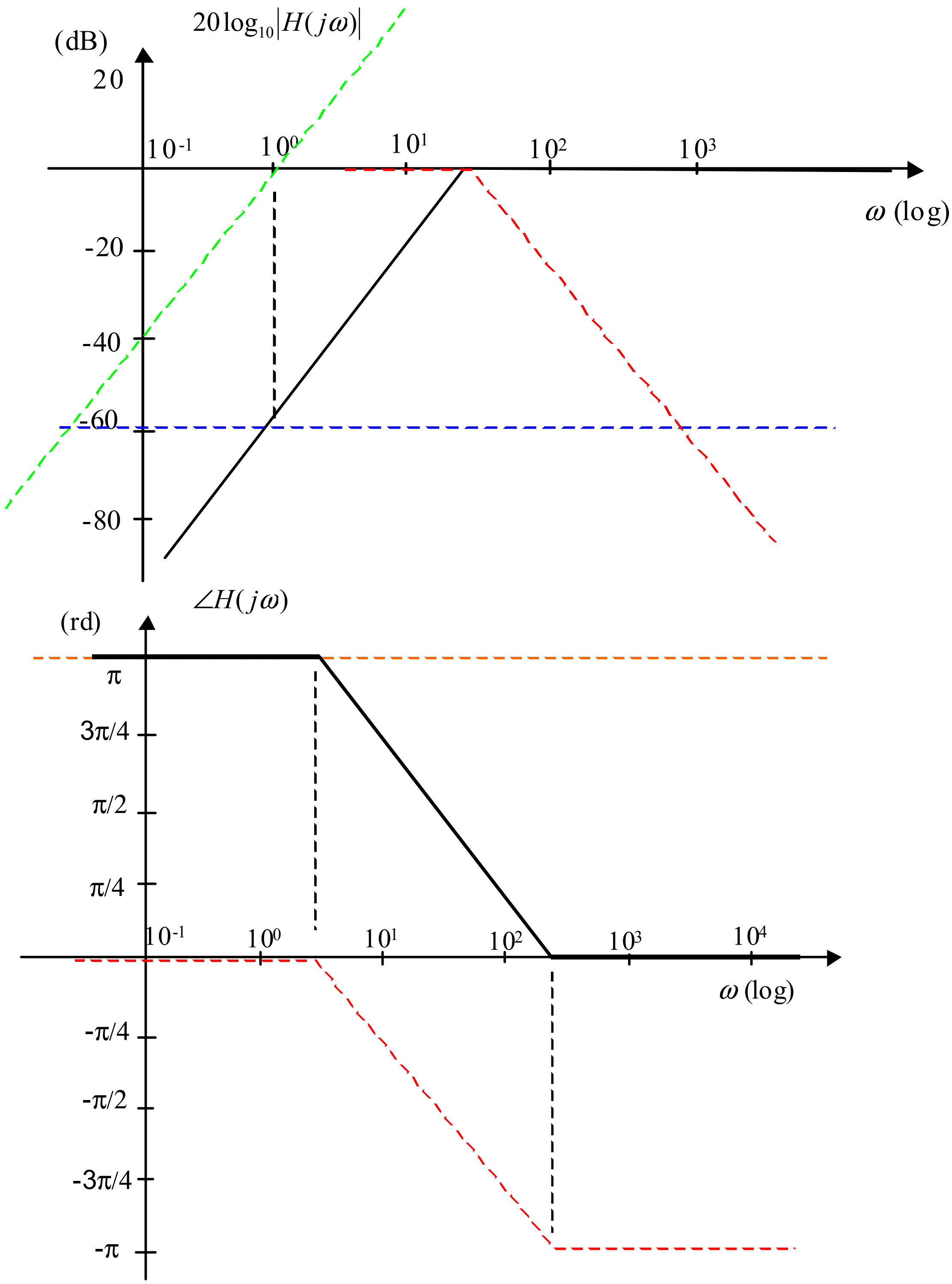


(c) $H(s)=\dfrac{s^2}{s^2+30s+900},\ \ \mathrm{Re}\{s\}>-15$

Answer:



$$H(s)=\frac{s^2}{s^2+30s+900}=\frac{1}{900}\frac{s^2}{\dfrac{s^2}{30^2}+\dfrac{1}{30}s+1},\ \ \omega_n=30,\ \zeta=0.5$$


$20\log_{10}|H(j\omega)|$
(dB)
20
-20
-40
-60
-80
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
ω (log)
$\angle H(j\omega)$
(rd)
π
3π/4
π/2
π/4
-π/4
-π/2
-3π/4
-π
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
10^{4}
ω (log)

Problem 8.8

Sketch the pole-zero plots in the s-plane and the Bode plots (magnitude and phase) for the following systems. Specify if the transfer functions have poles or zeros at infinity.

(a) $H(s) = \dfrac{100(s-10)}{(s+1)(s+10)(s+100)}, \quad \mathrm{Re}\{s\} > -1.$

(b) $H(s) = \dfrac{s+1}{s(0.01s+1)}, \quad \mathrm{Re}\{s\} > 0.$

(c) $H(s) = \dfrac{s(s^2-9)}{(s+100)(s^2+10s+100)}, \quad \mathrm{Re}\{s\} > -5$

Problem 8.9

Consider the causal differential system described by:

$$\frac{1}{4}\frac{d^2 y(t)}{dt^2} + \frac{1}{\sqrt{2}}\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} + x(t),$$

with initial conditions $\dfrac{dy(0^-)}{dt} = 3, \quad y(0^-) = 0$. Suppose that this system is subjected to the input signal $x(t) = e^{-2t}u(t)$.

(a) Find the system's damping ratio ζ and undamped natural frequency ω_n. Compute the output of the system $y(t)$ for $t \geq 0$. Find the steady-state response $y_{ss}(t)$, the transient response $y_{tr}(t)$, the zero-input response $y_{zi}(t)$ and the zero-state response $y_{zs}(t)$ for $t \geq 0$.

Answer:

Let's take the unilateral Laplace transform on both sides of the differential equation.

$$\frac{1}{4}\left[s^2\mathcal{Y}(s)-sy(0^-)-\frac{dy(0^-)}{dt}\right]+\frac{1}{\sqrt{2}}\left[s\mathcal{Y}(s)-y(0^-)\right]+\mathcal{Y}(s)=s\mathcal{X}(s)+\mathcal{X}(s)$$

Collecting the terms containing $\mathcal{Y}(s)$ on the left-hand side and putting everything else on the right-hand side, we can solve for $\mathcal{Y}(s)$.

$$\left(s^2+2\sqrt{2}s+4\right)\mathcal{Y}(s)=4s\mathcal{X}(s)+4\mathcal{X}(s)+sy(0^-)+2\sqrt{2}y(0^-)+\frac{dy(0^-)}{dt}$$

$$\mathcal{Y}(s)=\underbrace{\frac{(4s+4)\mathcal{X}(s)}{s^2+2\sqrt{2}s+4}}_{\text{zero-state resp.}}+\underbrace{\frac{(s+2\sqrt{2})y(0^-)+\frac{dy(0^-)}{dt}}{s^2+2\sqrt{2}s+4}}_{\text{zero-input resp.}}$$

Since the system is causal, the ROC is an open RHP to the right of the rightmost pole. The undamped natural frequency is $\omega_n=2$ and the damping ratio is $\zeta=\frac{1}{\sqrt{2}}$. The poles are $p_{1,2}=-\zeta\omega_n\pm j\omega_n\sqrt{1-\zeta^2}=-\sqrt{2}\pm j\sqrt{2}$. Therefore the ROC is $\mathrm{Re}\{s\}>-\sqrt{2}$. The unilateral LT of the input is given by

$$\mathcal{X}(s)=\frac{1}{s+2},\quad \mathrm{Re}\{s\}>-2,$$

thus,

$$\mathcal{Y}(s)=\underbrace{\frac{4(s+1)}{(s+2)(s^2+2\sqrt{2}s+4)}}_{\substack{\mathrm{Re}\{s\}>-\sqrt{2}\\ \text{zero-state resp.}}}+\underbrace{\frac{3}{s^2+2\sqrt{2}s+4}}_{\substack{\mathrm{Re}\{s\}>-\sqrt{2}\\ \text{zero-input resp.}}}.$$

Let's compute the zero-state response first:

$$\mathcal{Y}_{zs}(s) = \frac{4(s+1)}{(s+2)(s^2+2\sqrt{2}s+4)}, \quad \mathrm{Re}\{s\} > -\sqrt{2}$$

$$= \underbrace{\frac{A\sqrt{2}+B(s+\sqrt{2})}{\left(s+\sqrt{2}\right)^2+2}}_{\mathrm{Re}\{s\}>-\sqrt{2}} + \frac{C}{\underbrace{s+2}_{\mathrm{Re}\{s\}>-2}}$$

$$= \underbrace{\frac{A\sqrt{2}+B(s+\sqrt{2})}{\left(s+\sqrt{2}\right)^2+2}}_{\mathrm{Re}\{s\}>-\sqrt{2}} - \frac{1.707}{\underbrace{s+2}_{\mathrm{Re}\{s\}>-2}}$$

Let $s=-\sqrt{2}$ to compute $A = -3\dfrac{1-\sqrt{2}}{2-\sqrt{2}} = 2.1213$, then multiply both sides by s and let $s \to \infty$ to get $B = -C = 1+1/\sqrt{2} = 1.707$:

$$\mathcal{Y}_{zs}(s) = \underbrace{\frac{2.121\sqrt{2}+1.707(s+\sqrt{2})}{\left(s+\sqrt{2}\right)^2+2}}_{\mathrm{Re}\{s\}>-\sqrt{2}} - \frac{1.707}{\underbrace{s+2}_{\mathrm{Re}\{s\}>-2}}$$

Notice that the second term is not a steady-state response, and thus $y_{ss}(t)=0$. Taking the inverse Laplace transform using the table yields

$$y_{zs}(t) = \left[-1.707e^{-2t} + 2.121e^{-\sqrt{2}t}\sin(\sqrt{2}t) + 1.707e^{-\sqrt{2}t}\cos(\sqrt{2}t)\right]u(t).$$

The zero-input response is given by:

$$\mathcal{Y}_{zi}(s) = \frac{3}{\underbrace{s^2+2\sqrt{2}s+4}_{\substack{\mathrm{Re}\{s\}>-\sqrt{2} \\ \text{zero-input resp.}}}}$$

$$= \frac{\frac{3}{\sqrt{2}}\sqrt{2}}{(s+\sqrt{2})^2+2}, \quad \mathrm{Re}\{s\} > -\sqrt{2}$$

which yields:

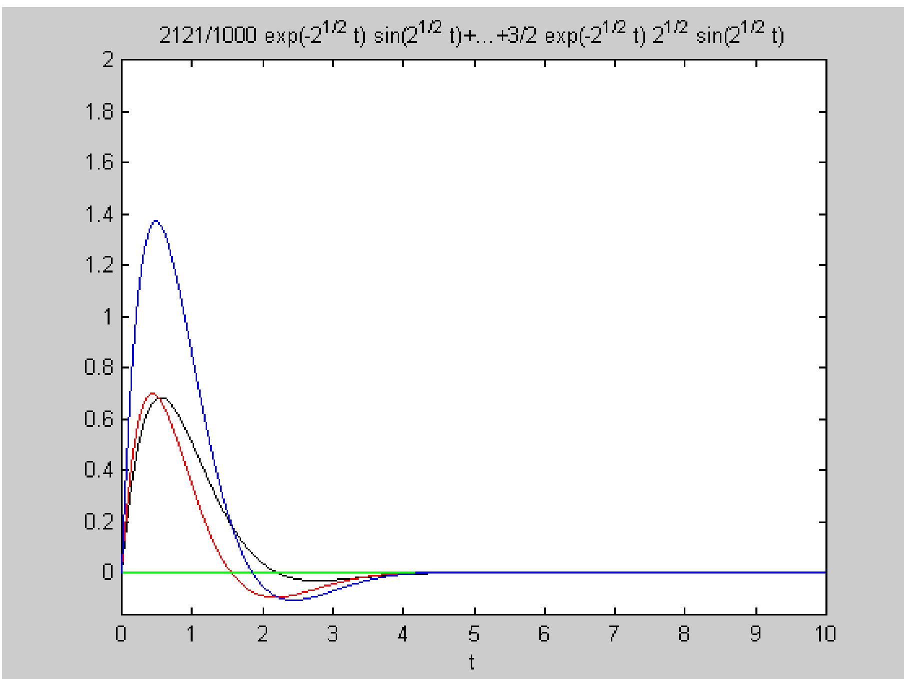
$$y_{zi}(t) = \frac{3}{\sqrt{2}} e^{-\sqrt{2}t} \sin(\sqrt{2}t) u(t).$$

The transient response is the sum of $y_{zi}(t)$ and $y_{zs}(t)$ above.

$$y_{tr}(t) = \left[-1.707e^{-2t} + 2.121e^{-\sqrt{2}t} \sin(\sqrt{2}t) + 1.707e^{-\sqrt{2}t} \cos(\sqrt{2}t) + \frac{3}{\sqrt{2}} e^{-\sqrt{2}t} \sin(\sqrt{2}t) \right] u(t)$$

(b) Plot $y_{ss}(t)$, $y_{tr}(t)$, $y_{zi}(t)$, $y_{zs}(t)$ for $t \geq 0$, all on the same figure.

Answer:

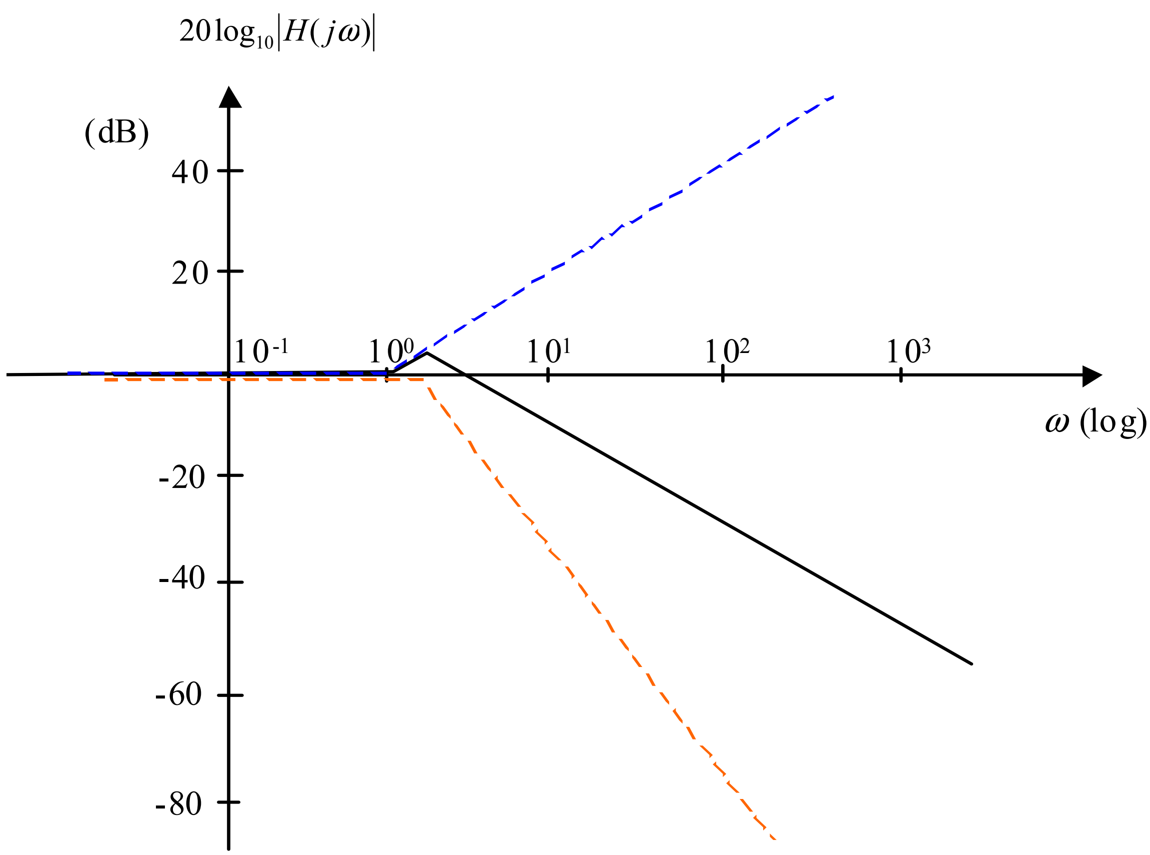


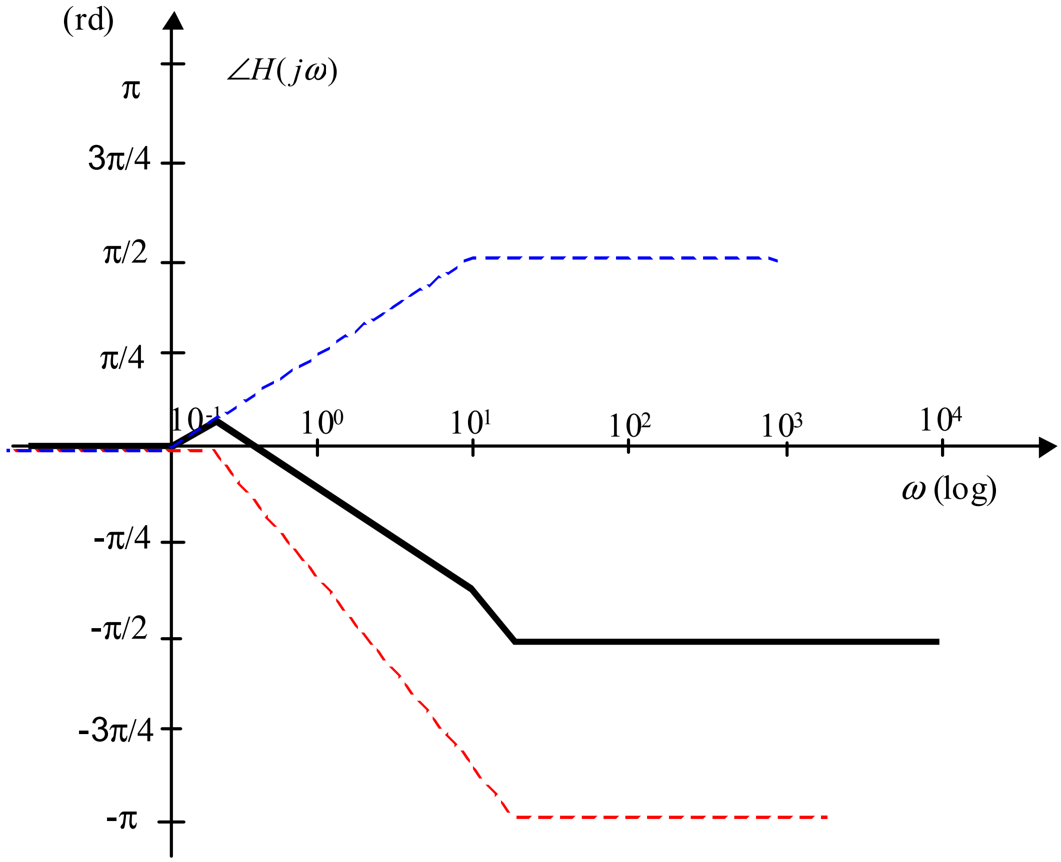
(c) Find the frequency response of the system and sketch its Bode plot.

Answer:

The transfer function of the system is $H(s)=\dfrac{4s+4}{s^2+2\sqrt{2}s+4}$, and the frequency response is given by:

$$H(j\omega)=\frac{4j\omega+4}{(j\omega)^2+2\sqrt{2}j\omega+4}=\frac{(j\omega+1)}{\frac{(j\omega)^2}{4}+\frac{\sqrt{2}}{2}j\omega+1}$$





Problem 8.10

Consider the causal differential system described by its direct form realization shown in Figure 8.9.

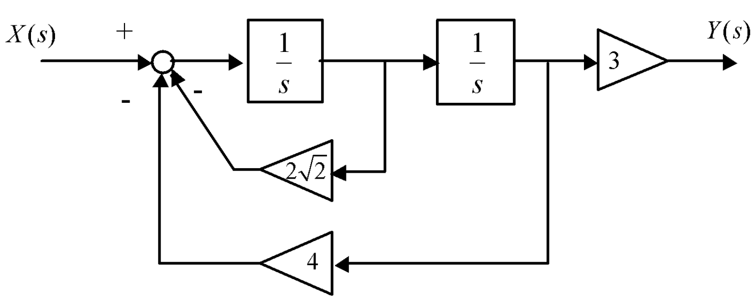


Figure 8.9: System of Problem 8.10.

This system has initial conditions $\frac{dy(0^-)}{dt} = -1, \quad y(0^-) = 2$. Suppose that the system is subjected to the unit step input signal $x(t) = u(t)$.

(a) Write the differential equation of the system. Find the system's damping ratio ζ and undamped natural frequency ω_n. Give the transfer function of the system and specify its ROC. Sketch its pole-zero plot. Is the system stable? Justify.

(b) Compute the step response of the system (including the effect of initial conditions), its steady-state response $y_{ss}(t)$ and its transient response $y_{tr}(t)$ for $t \geq 0$. Identify the zero-state response and the zero-input response in the Laplace domain.

(c) Compute the percentage of first overshoot in the step response of the system assumed this time to be initially at rest.