Solutions to Problems in Chapter 7

Problems with Solutions

Problem 7.1

Consider the system described by:

$$H(s) = \frac{3s^2 - 3s - 6}{s^3 + 12s^2 + 120s + 200}, \ \ \mathrm{Re}\{s\} > -2$$

(a) Find the direct form realization of the transfer function $H(s)$. Is this system BIBO stable? is it causal? Why?

Answer:

The direct form realization is obtained by splitting up the system into two systems as in Figure 7.1.

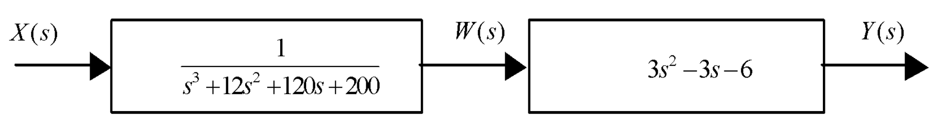


Figure 7.1: Transfer function split up as a cascade of two subsystems, Problem 7.1(a).

The input-output system equation of the first subsystem is:

$$s^3W(s) = -12s^2W(s) - 120sW(s) - 200W(s) + X(s),$$

and we begin the diagram by drawing a cascade of three integrators with their inputs labeled as $s^3W(s),\ s^2W(s),\ sW(s)$. Then, we can draw the feedbacks to a summing junction whose output is the input of the first integrator, labeled $s^3W(s)$. The input signal is also an input to that summing junction. For the second subsystem we have

$$Y(s) = 3s^2W(s) - 3sW(s) - 6W(s),$$

which can be drawn as taps on the integrator inputs, summed up to form the output signal. The resulting direct form realization is shown in Figure 7.2:

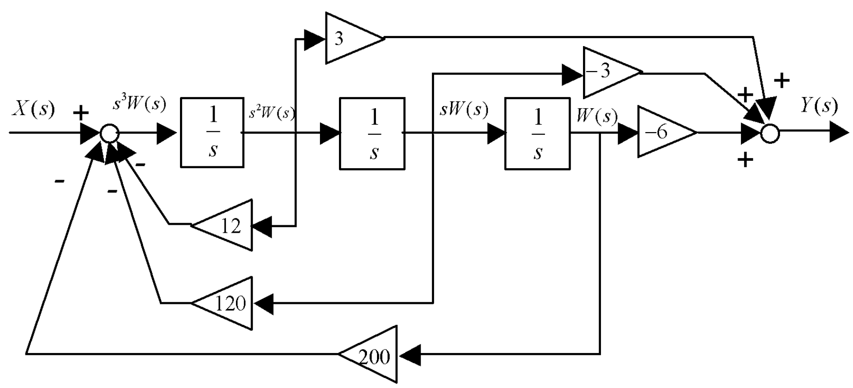


Figure 7.2: Direct form realization of transfer function, Problem 7.1(a).

The system is BIBO stable as its rational transfer function is proper and its ROC contains the $j\omega$-axis. It is also causal because its ROC is an open right-half plane *and* its transfer function is rational.

(b) Give a parallel form realization of $H(s)$ with (possibly complex-rational) first-order blocks.

Answer:

Below is the partial fraction expansion for the parallel realization of the transfer function, shown in :

$$\begin{aligned} H(s) &= \frac{3s^2 - 3s - 6}{s^3 + 12s^2 + 120s + 200}, \ \operatorname{Re}\{s\} > -2 \\ &= \frac{3s^2 - 3s - 6}{(s+2)(s^2 + 10s + 100)}, \ \operatorname{Re}\{s\} > -2 \\ &= \underbrace{\frac{0.14286}{s+2}}_{\operatorname{Re}\{s\}>-2} + \underbrace{\frac{1.4286 + j1.4104}{s + 5 - j5\sqrt{3}}}_{\operatorname{Re}\{s\}>-5} + \underbrace{\frac{1.4286 - j1.4104}{s + 5 + j5\sqrt{3}}}_{\operatorname{Re}\{s\}>-5} \end{aligned}$$

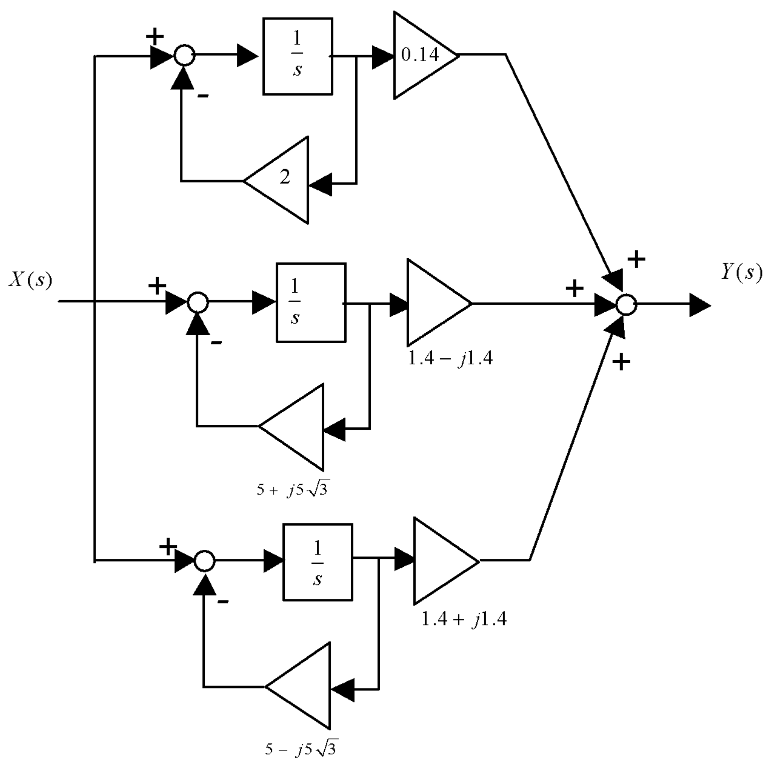


Figure 7.3: Parallel form realization of transfer function, Problem 7.1(b).

(c) Give a parallel form realization of $H(s)$ using a real-rational first-order block and a real-rational second-order block.

Answer:

The transfer function can be expanded as follows:

$$H(s) = \underbrace{\frac{0.14286}{s+2}}_{\mathrm{Re}\{s\}>-2} + \underbrace{\frac{2.857s - 10.14287}{s^2+10s+100}}_{\mathrm{Re}\{s\}>-5},$$

which corresponds to the desired parallel realization in Figure 7.4.

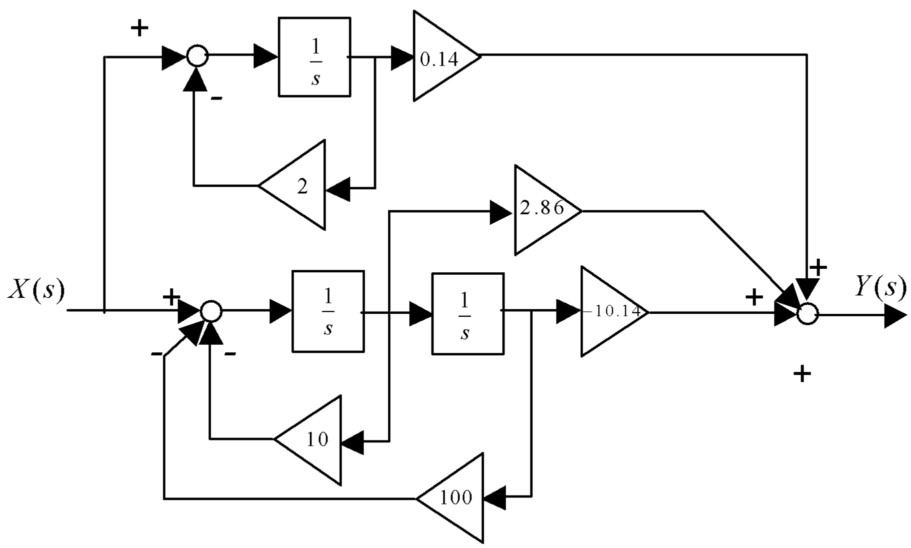


Figure 7.4: Real-rational parallel form realization of transfer function, Problem 7.1(c).

Problem 7.2

Consider the causal differential system described by

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 25y(t) = \frac{d^2x(t)}{dt^2} - \frac{dx(t)}{dt} + x(t),$$

with initial conditions $\frac{dy(0^-)}{dt} = 3, \quad y(0^-) = -1$. Suppose that this system is subjected to a unit step input signal $x(t) = u(t)$.

Find the system's damping ratio ζ and undamped natural frequency ω_n. Give the transfer function of the system and specify its ROC. Compute the steady-state response $y_{ss}(t)$ and the transient response $y_{tr}(t)$ for $t \geq 0$. Compute the zero-input response $y_{zi}(t)$ and the zero-state response $y_{zs}(t)$.

Answer:

Let us take the unilateral Laplace transform on both sides of the differential equation.

$$\left[s^2\mathcal{Y}(s) - sy(0^-) - \frac{dy(0^-)}{dt}\right] + 3\left[s\mathcal{Y}(s) - y(0^-)\right] + 25\mathcal{Y}(s) = s^2\mathcal{X}(s) - s\mathcal{X}(s) + \mathcal{X}(s)$$

Collecting the terms containing $\mathcal{Y}(s)$ on the left-hand side and putting everything else on the right-hand side, we can solve for $\mathcal{Y}(s)$.

$$\left(s^2 + 3s + 25\right)\mathcal{Y}(s) = s^2\mathcal{X}(s) - s\mathcal{X}(s) + \mathcal{X}(s) + sy(0^-) + 3y(0^-) + \frac{dy(0^-)}{dt}$$

$$\mathcal{Y}(s) = \underbrace{\frac{(s^2 - s + 1)\mathcal{X}(s)}{s^2 + 3s + 25}}_{\text{zero-state resp.}} + \underbrace{\frac{(s+3)y(0^-) + \dfrac{dy(0^-)}{dt}}{s^2 + 3s + 25}}_{\text{zero-input resp.}}$$

The transfer function is $H(s) = \dfrac{s^2 - s + 1}{s^2 + 3s + 25}$, and since the system is causal, the ROC is an open RHP to the right of the rightmost pole. The poles are $p_{1,2} = -1.5 \pm j4.77$. Therefore, the ROC is $\mathrm{Re}\{s\} > -1.5$. The unilateral LT of the input is given by:

$$\mathcal{X}(s) = \frac{1}{s}, \quad \mathrm{Re}\{s\} > 0,$$

thus,

$$\mathcal{Y}(s) = \underbrace{\frac{s^2 - s + 1}{s(s^2 + 3s + 25)}}_{\text{zero-state resp.}} + \underbrace{\frac{-s}{s^2 + 3s + 25}}_{\text{zero-input resp.}}$$
$$= \frac{-s + 1}{s(s^2 + 3s + 25)}$$

Let us compute the zero-state response first:

$$\mathcal{Y}_{zs}(s) = \frac{s^2 - s + 1}{s(s^2 + 3s + 25)}, \quad \mathrm{Re}\{s\} > 0$$
$$= \underbrace{\frac{A(4.77) + B(s + 1.5)}{(s + 1.5)^2 + 22.75}}_{\mathrm{Re}\{s\} > -1.5} + \underbrace{\frac{C}{s}}_{\mathrm{Re}\{s\} > 0}$$
$$= \underbrace{\frac{A(4.77) + B(s + 1.5)}{(s + 1.5)^2 + 22.75}}_{\mathrm{Re}\{s\} > -1.5} + \underbrace{\frac{0.04}{s}}_{\mathrm{Re}\{s\} > 0}$$

Let $s = -1.5$ to compute $\frac{(-1.5)^2 + 1.5 + 1}{(-1.5)(22.75)} = \frac{4.77}{22.75} A - 0.0267 \Rightarrow A = -0.5365$, then multiply both sides by s and let $s \to \infty$ to get $1 = B + 0.04 \quad \Rightarrow B = 0.96$:

$$\mathcal{Y}(s) = \underbrace{\frac{-0.5365(4.77) + 0.96(s + 1.5)}{(s + 1.5)^2 + 22.75}}_{\mathrm{Re}\{s\} > -1.5} + \underbrace{\frac{0.04}{s}}_{\mathrm{Re}\{s\} > 0}$$

Notice that the second term $\frac{0.04}{s}$ is the steady-state response, and thus $y_{ss}(t) = 0.04u(t)$.

Taking the inverse Laplace transform using the table, we obtain:

$$y_{zs}(t) = \left[-0.5365 e^{-1.5t} \sin(4.77t) + 0.96 e^{-1.5t} \cos(4.77t) \right] u(t) + 0.04u(t).$$

Let us compute the zero-input response:

$$\mathcal{Y}_{zi}(s) = \frac{-s}{s^2+3s+25}, \quad \mathrm{Re}\{s\} > -1.5$$
$$= \frac{\frac{1.5}{4.77}(4.77) - (s+1.5)}{(s+1.5)^2 + 22.75}, \quad \mathrm{Re}\{s\} > -1.5$$

The inverse Laplace transform using the table yields

$$y_{zi}(t) = \left[0.3145e^{-1.5t}\sin(4.77t) - e^{-1.5t}\cos(4.77t)\right]u(t).$$

Finally, the transient response is the sum of the zero-input and zero-state responses minus the steady-state response.

$$y_{tr}(t) = \left[(0.3145 - 0.5365)e^{-1.5t}\sin(4.77t) + (0.96-1)e^{-1.5t}\cos(4.77t)\right]u(t)$$
$$= \left[-0.222e^{-1.5t}\sin(4.77t) - 0.04e^{-1.5t}\cos(4.77t)\right]u(t)$$

Exercises

Problem 7.3

Consider the causal differential system described by: $\frac{1}{4}\frac{d^2y(t)}{dt^2} + \frac{1}{\sqrt{2}}\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} + x(t)$,

with initial conditions $\frac{dy(0^-)}{dt} = 3, \quad y(0^-) = 0$. Suppose that this system is subjected to the input signal: $x(t) = e^{-2t}u(t)$. Find the system's damping ratio ζ and undamped natural frequency ω_n. Compute the output of the system $y(t)$ for $t \geq 0$. Find the steady-state response $y_{ss}(t)$, the transient response $y_{tr}(t)$, the zero-input response $y_{zi}(t)$ and the zero-state response $y_{zs}(t)$ for $t \geq 0$.

Answer:

Let us take the unilateral Laplace transform on both sides of the differential equation:

$$\frac{1}{4}\left[s^2\mathcal{Y}(s) - sy(0^-) - \frac{dy(0^-)}{dt}\right] + \frac{1}{\sqrt{2}}\left[s\mathcal{Y}(s) - y(0^-)\right] + \mathcal{Y}(s) = s\mathcal{X}(s) + \mathcal{X}(s).$$

Collecting the terms containing $\mathcal{Y}(s)$ on the left-hand side and putting everything else on the right-hand side, we can solve for $\mathcal{Y}(s)$:

$$\left(s^2 + 2\sqrt{2}s + 4\right)\mathcal{Y}(s) = 4s\mathcal{X}(s) + 4\mathcal{X}(s) + sy(0^-) + 2\sqrt{2}y(0^-) + \frac{dy(0^-)}{dt}$$

$$\mathcal{Y}(s) = \underbrace{\frac{(4s+4)\mathcal{X}(s)}{s^2 + 2\sqrt{2}s + 4}}_{\text{zero-state resp.}} + \underbrace{\frac{(s+2\sqrt{2})y(0^-) + \frac{dy(0^-)}{dt}}{s^2 + 2\sqrt{2}s + 4}}_{\text{zero-input resp.}}$$

Since the system is causal, the ROC is an open RHP to the right of the rightmost pole. The undamped natural frequency is $\omega_n = 2$ and the damping ratio is $\zeta = \frac{1}{\sqrt{2}}$. The poles are $p_{1,2} = -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\sqrt{2} \pm j\sqrt{2}$. Therefore the ROC is $\mathrm{Re}\{s\} > -\sqrt{2}$. The unilateral LT of the input is given by $\mathcal{X}(s) = \frac{1}{s+2}, \quad \mathrm{Re}\{s\} > -2$, thus

$$\mathcal{Y}(s) = \underbrace{\frac{4(s+1)}{(s+2)(s^2 + 2\sqrt{2}s + 4)}}_{\substack{\mathrm{Re}\{s\} > -\sqrt{2} \\ \text{zero-state resp.}}} + \underbrace{\frac{3}{s^2 + 2\sqrt{2}s + 4}}_{\substack{\mathrm{Re}\{s\} > -\sqrt{2} \\ \text{zero-input resp.}}}$$

Let us compute the zero-state response first:

$$\mathcal{Y}_{zs}(s) = \frac{4(s+1)}{(s+2)(s^2+2\sqrt{2}s+4)}, \quad \text{Re}\{s\} > -\sqrt{2}$$
$$= \underbrace{\frac{A\sqrt{2}+B(s+\sqrt{2})}{\left(s+\sqrt{2}\right)^2+2}}_{\text{Re}\{s\}>-\sqrt{2}} + \frac{C}{\underbrace{s+2}_{\text{Re}\{s\}>-2}}$$
$$= \underbrace{\frac{A\sqrt{2}+B(s+\sqrt{2})}{\left(s+\sqrt{2}\right)^2+2}}_{\text{Re}\{s\}>-\sqrt{2}} - \frac{1.707}{\underbrace{s+2}_{\text{Re}\{s\}>-2}}$$

Let $s=-\sqrt{2}$ to compute $A=-3\dfrac{1-\sqrt{2}}{2-\sqrt{2}}=2.1213$, then multiply both sides by s and let $s\to\infty$

to get $B=-C=1+1/\sqrt{2}=1.707$:

$$\mathcal{Y}_{zs}(s) = \underbrace{\frac{2.121\sqrt{2}+1.707(s+\sqrt{2})}{\left(s+\sqrt{2}\right)^2+2}}_{\text{Re}\{s\}>-\sqrt{2}} - \frac{1.707}{\underbrace{s+2}_{\text{Re}\{s\}>-2}}$$

Notice that the second term is not a steady-state response, and thus $y_{ss}(t)=0$. Taking the inverse Laplace transform using Table D.4 yields:

$$y_{zs}(t) = \left[-1.707e^{-2t} + 2.121e^{-\sqrt{2}t}\sin(\sqrt{2}t) + 1.707e^{-\sqrt{2}t}\cos(\sqrt{2}t)\right]u(t).$$

The zero-input response is given by:

$$\mathcal{Y}_{zi}(s) = \frac{3}{\underbrace{s^2+2\sqrt{2}s+4}_{\substack{\text{Re}\{s\}>-\sqrt{2} \\ \text{zero-input resp.}}}} = \frac{\frac{3}{\sqrt{2}}\sqrt{2}}{(s+\sqrt{2})^2+2}, \quad \text{Re}\{s\} > -\sqrt{2}$$

which yields: $y_{zi}(t) = \frac{3}{\sqrt{2}} e^{-\sqrt{2}t} \sin(\sqrt{2}t) u(t)$.

The transient response is the sum of $y_{zi}(t)$ and $y_{zs}(t)$ above.

$$y_{tr}(t) = \left[-1.707e^{-2t} + 2.121e^{-\sqrt{2}t} \sin(\sqrt{2}t) + 1.707e^{-\sqrt{2}t} \cos(\sqrt{2}t) + \frac{3}{\sqrt{2}} e^{-\sqrt{2}t} \sin(\sqrt{2}t) \right] u(t)$$

Problem 7.4

Use the unilateral Laplace transform to compute the output response $y(t)$ to the input $x(t) = \cos(10t)u(t)$ of the following causal LTI differential system with initial conditions $y(0^-) = 1$, $\frac{dy(0^-)}{dt} = 1$:

$$\frac{d^2 y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t).$$

Problem 7.5

Compute the steady-state response of the causal LTI differential system $5\frac{dy(t)}{dt} + y(t) = 10x(t)$ to the input $x(t) = \sin(20t)$.

Answer:

The frequency response of the system is identified as $H(j\omega) = \frac{10}{5j\omega + 1}$. Its steady-state response is simply:

$$
\begin{aligned}
y_{ss}(t) &= |H(j20)|\sin\left(20t + \angle H(j20)\right) \\
&= \left|\frac{10}{j100+1}\right|\sin\left(20t + \angle\frac{10}{j100+1}\right) \\
&= \frac{10}{\sqrt{10001}}\sin\left(20t + \arctan\left(\frac{-100}{1}\right)\right) \\
&= 0.100\sin\left(20t - 1.5608\right)
\end{aligned}.
$$

Problem 7.6

(a) Find the direct form realization of the transfer function $H(s) = \frac{s^2 - 3s + 12}{3s^2 + 9s + 6}$, $\mathrm{Re}\{s\} > -1$. Is this system BIBO stable? is it causal? Why? Let $y(t)$ be the step response of the system. Compute $y(0^+)$ and $y(+\infty)$.

(b) Give a parallel form realization of $H(s)$ given in (a) with first-order blocks.

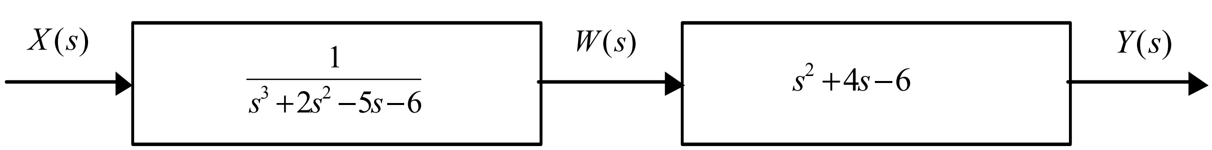
Problem 7.7

(a) Find the direct form realization of the transfer function below. Is this system BIBO stable? is it causal? Why?

$$
H(s) = \frac{s^2 + 4s - 6}{s^3 + 2s^2 - 5s - 6}, \quad -1 < \mathrm{Re}\{s\} < 2
$$

Answer:

The direct form realization is obtained by splitting this system into two subsystems as follows:



The input-output system equation of the first subsystem is

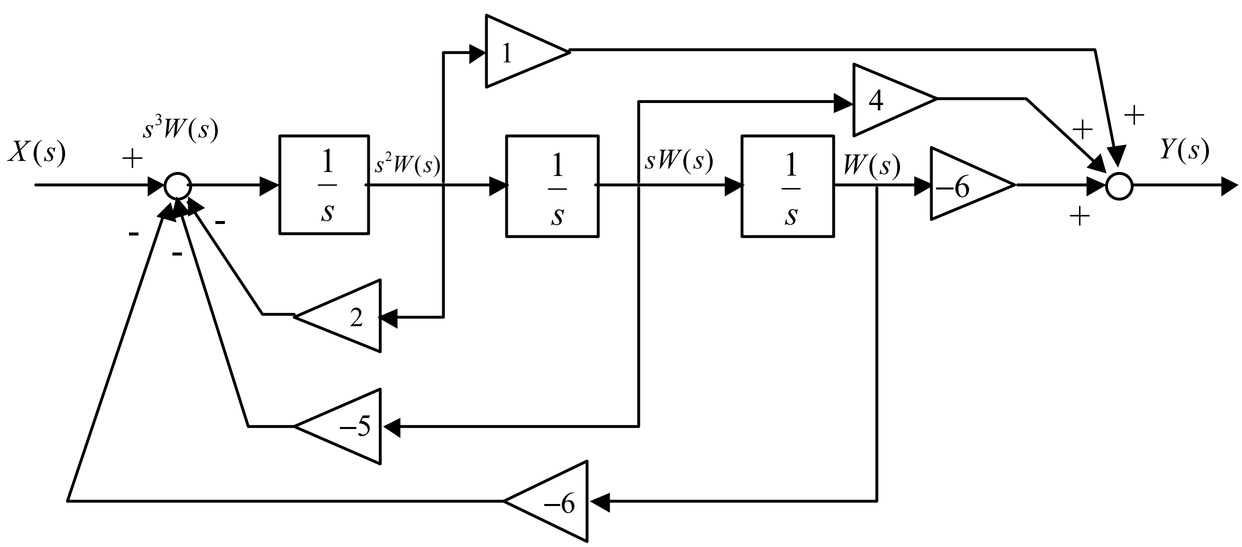
$$s^3W(s) = -2s^2W(s) + 5sW(s) + 6W(s) + X(s),$$

and we begin the diagram by drawing a cascade of three integrators with their inputs labeled as $s^3W(s),\ s^2W(s),\ sW(s)$. Then we can draw the feedbacks to a summing junction whose output is the input of the first integrator, labeled $s^3W(s)$. The input signal is also an input to that summing junction.

For the second subsystem we have:

$$Y(s) = s^2W(s) + 4sW(s) - 6W(s),$$

which can be drawn as taps on the integrator inputs, summed up to form the output signal. The direct form realization is then:



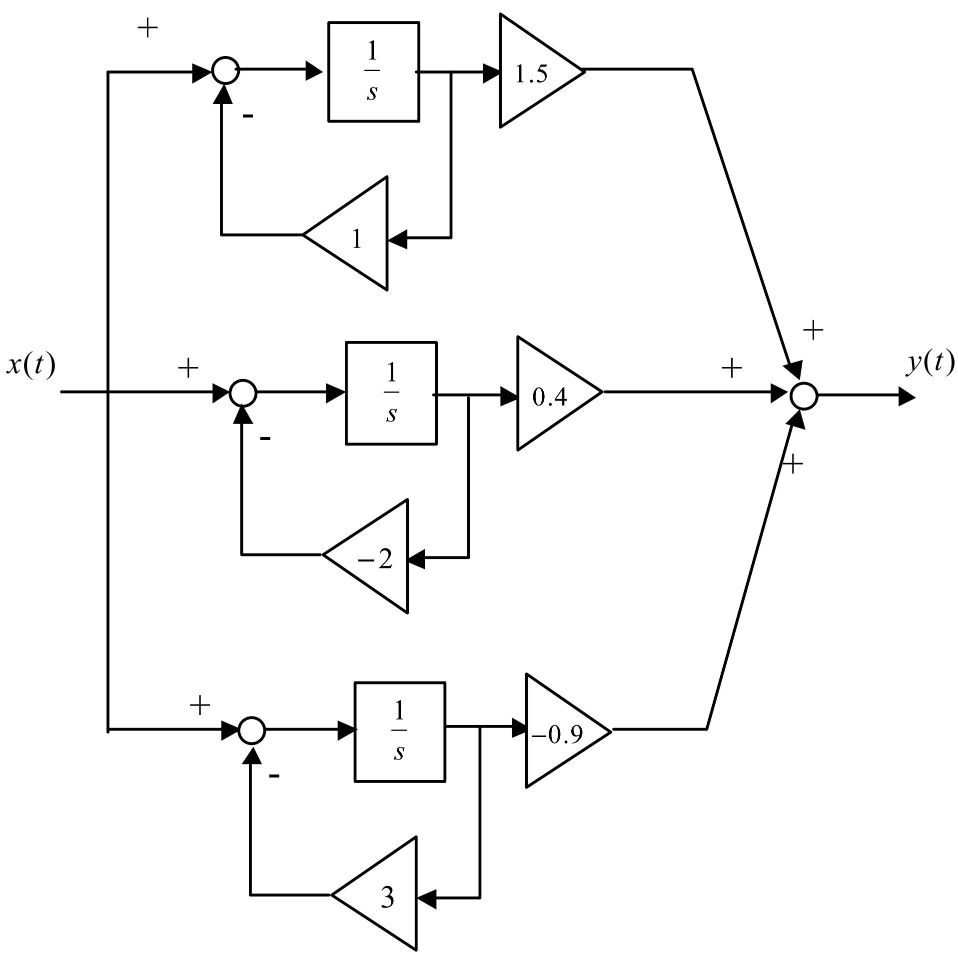
The system is BIBO stable as its rational transfer function is proper and its ROC contains the $j\omega$-axis. It is also causal because its ROC is an open right-half plane and the transfer function is rational.

(b) Give a parallel form realization of $H(s)$ with first-order blocks.

Answer:

Parallel realization:

$$\begin{aligned}
H(s) &= \frac{s^2+4s-6}{s^3+2s^2-5s-6}, \quad -1<\mathrm{Re}\{s\}<2 \\
&= \frac{s^2+4s-6}{(s+1)(s-2)(s+3)}, \quad -1<\mathrm{Re}\{s\}<2 \\
&= \underbrace{\frac{1.5}{s+1}}_{\mathrm{Re}\{s\}>-1} + \underbrace{\frac{0.4}{s-2}}_{\mathrm{Re}\{s\}<2} - \underbrace{\frac{0.9}{s+3}}_{\mathrm{Re}\{s\}>-3}
\end{aligned}$$

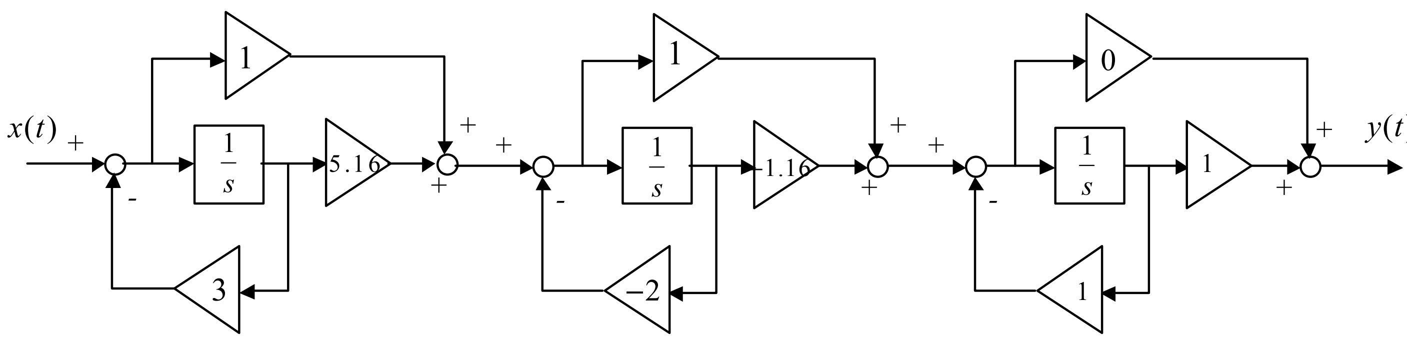


(c) Give a cascade form realization of $H(s)$ with first-order blocks.

Answer:

Cascade realization (this is one possibility, other possibilities when grouping poles and zeros in a different way):

$$\begin{aligned} H(s) &= \frac{s^2+4s-6}{s^3+2s^2-5s-6}, \quad -1<\mathrm{Re}\{s\}<2 \\ &= \frac{(s-1.1623)(s+5.1623)}{(s+1)(s-2)(s+3)}, \quad -1<\mathrm{Re}\{s\}<2 \\ &= \left(\frac{1}{s+1}\right)\left(\frac{s-1.1623}{s-2}\right)\left(\frac{s+5.1623}{s+3}\right), \quad -1<\mathrm{Re}\{s\}<2 \end{aligned}$$



Problem 7.8

Consider the causal differential system described by:

$$\frac{1}{2}\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+2y(t)=-\frac{dx(t)}{dt}-x(t),$$

with initial conditions $\frac{dy(0^-)}{dt} = 1, \quad y(0^-) = 2$. Suppose that this system is subjected to the input signal $x(t) = u(t)$. Give the transfer function of the system and specify its ROC. Compute the steady-state response $y_{ss}(t)$ and the transient response $y_{tr}(t)$ for $t \geq 0$.