Solutions to Problems in Chapter 5

Problems with Solutions

Problem 5.1

Sketch the following signals and find their Fourier transforms.

(a) $x(t)=\left(1-e^{-|t|}\right)\left[u(t+1)-u(t-1)\right]$. Show that $X(j\omega)$ is real and even.

Answer:

The signal is sketched in Figure 5.1.

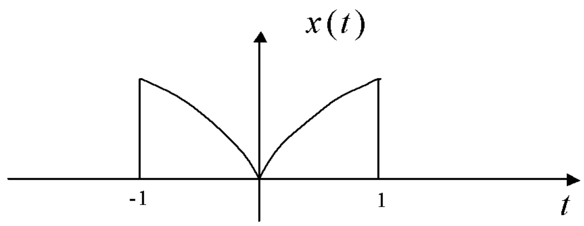

Figure 5.1: Signal of Problem 5.1(a).

$$
\begin{aligned}
X(j\omega) &= \int_{-1}^{1} x(t)e^{-j\omega t}\,dt = \int_{-1}^{1}\left(1-e^{-|t|}\right)e^{-j\omega t}\,dt = \int_{-1}^{1} e^{-j\omega t}\,dt - \int_{-1}^{0} e^{(1-j\omega)t}\,dt - \int_{0}^{1} e^{-(1+j\omega)t}\,dt \\
&= -\frac{1}{j\omega}\left[e^{-j\omega t}\right]_{-1}^{1} - \frac{1}{1-j\omega}\left[e^{(1-j\omega)t}\right]_{-1}^{0} + \frac{1}{1+j\omega}\left[e^{-(1+j\omega)t}\right]_{0}^{1} = \frac{e^{j\omega}-e^{-j\omega}}{j\omega} - \frac{1-e^{-1}e^{j\omega}}{1-j\omega} + \frac{e^{-1}e^{-j\omega}-1}{1+j\omega} \\
&= \frac{2\sin\omega}{\omega} + \frac{(1+j\omega)(-1+e^{-1}e^{j\omega})+(1-j\omega)(-1+e^{-1}e^{-j\omega})}{1+\omega^2} \\
&= \frac{2\sin\omega}{\omega} + \frac{(-1-j\omega+e^{-1}e^{j\omega}+j\omega e^{-1}e^{j\omega})+(-1+j\omega+e^{-1}e^{-j\omega}-j\omega e^{-1}e^{-j\omega})}{1+\omega^2} \\
&= \frac{2\sin\omega}{\omega} + \frac{-2+e^{-1}(e^{j\omega}+e^{-j\omega})+j\omega e^{-1}(e^{j\omega}-e^{-j\omega})}{1+\omega^2} \\
&= \frac{2\sin\omega}{\omega} + \frac{-2+2e^{-1}(\cos\omega-\omega\sin\omega)}{1+\omega^2}
\end{aligned}
$$

This Fourier transform is obviously real. To show that it is even, we consider $X(-j\omega)$:

$$
\begin{aligned}
X(-j\omega) &= \frac{2\sin(-\omega)}{-\omega} + \frac{-2+2e^{-1}(\cos(-\omega)+\omega\sin(-\omega))}{1+(-\omega)^2} \\
&= \frac{-2\sin(\omega)}{-\omega} + \frac{-2+2e^{-1}(\cos(\omega)-\omega\sin(\omega))}{1+\omega^2} = X(j\omega)
\end{aligned}
$$

(b) Periodic signal $x(t)$ in Figure 5.2.

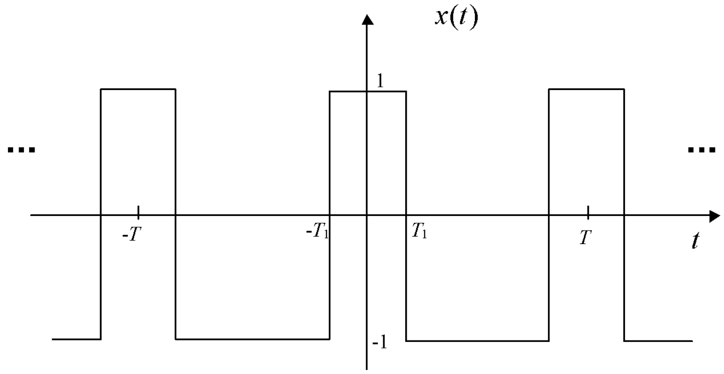

Figure 5.2: Signal of Problem 5.1(b).

Answer:

This signal is the sum of the constant signal –1 with our familiar rectangular wave of amplitude 2 and duty cycle $\eta = \frac{2T_1}{T}$. Therefore, its Fourier series coefficients are:

$$
\begin{aligned}
a_k &= 2\frac{2T_1}{T}\,\mathrm{sinc}\left(\frac{k2T_1}{T}\right), \quad k \neq 0 \\
a_0 &= \frac{4T_1}{T} - 1
\end{aligned}
$$

Note that since $0 < T_1 < \frac{T}{2}$, then $-1 < a_0 < 1$ depending on the duty cycle. The Fourier transform of the signal is given by:

$$X(j\omega) = 2\pi \sum_{\substack{k=-\infty \\ k\neq 0}}^{+\infty} \frac{4T_1}{T} \mathrm{sinc}(\frac{k2T_1}{T})\delta(\omega - k\frac{2\pi}{T}) + 2\pi(\frac{4T_1}{T} - 1)\delta(\omega)$$

Problem 5.2

Sketch the following signals and compute their Fourier transforms using the integral formula.

(a) $x_1(t) = \begin{cases} \sin\omega_0 t, & -\frac{2\pi}{\omega_0} \leq t \leq \frac{2\pi}{\omega_0} \\ 0, & \text{otherwise} \end{cases}$

Answer:

This real, odd signal is composed of two periods of a sine wave. Its sketch is in Figure 5.3.

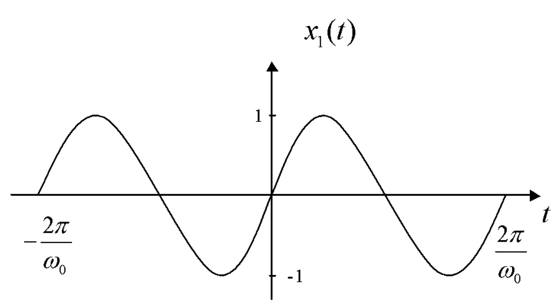

Figure 5.3: Signal composed of two periods of a sine wave in Problem 5.2(a).

Let us compute its Fourier transform:

$$X(j\omega)=\int_{-\frac{2\pi}{\omega_0}}^{\frac{2\pi}{\omega_0}} x(t)e^{-j\omega t}dt=\int_{-\frac{2\pi}{\omega_0}}^{\frac{2\pi}{\omega_0}} \sin(\omega_0 t)e^{-j\omega t}dt=\frac{1}{2j}\int_{-\frac{2\pi}{\omega_0}}^{\frac{2\pi}{\omega_0}} \left(e^{j(\omega_0-\omega)t}-e^{j(-\omega_0-\omega)t}\right)dt$$

$$=-\frac{j}{2j(\omega_0-\omega)}\left[e^{j(\omega_0-\omega)t}\right]_{-\frac{2\pi}{\omega_0}}^{\frac{2\pi}{\omega_0}}+\frac{j}{2j(-\omega_0-\omega)}\left[e^{j(-\omega_0-\omega)t}\right]_{-\frac{2\pi}{\omega_0}}^{\frac{2\pi}{\omega_0}}$$

$$=\frac{1}{2(\omega-\omega_0)}\left[e^{j(\omega_0-\omega)\frac{2\pi}{\omega_0}}-e^{-j(\omega_0-\omega)\frac{2\pi}{\omega_0}}\right]-\frac{1}{2(\omega_0+\omega)}\left[e^{-j(\omega_0+\omega)\frac{2\pi}{\omega_0}}-e^{j(\omega_0+\omega)\frac{2\pi}{\omega_0}}\right]$$

$$=\frac{1}{2(\omega-\omega_0)}\left[e^{-j\omega\frac{2\pi}{\omega_0}}-e^{j\omega\frac{2\pi}{\omega_0}}\right]-\frac{1}{2(\omega_0+\omega)}\left[e^{-j\omega\frac{2\pi}{\omega_0}}-e^{j\omega\frac{2\pi}{\omega_0}}\right]$$

$$=\frac{1}{2(\omega-\omega_0)(\omega_0+\omega)}\left[2\omega_0 e^{-j\omega\frac{2\pi}{\omega_0}}-2\omega_0 e^{j\omega\frac{2\pi}{\omega_0}}\right]$$

$$=\frac{2j\omega_0\sin(\frac{2\pi}{\omega_0}\omega)}{\omega_0{}^2-\omega^2}$$

This Fourier transform is imaginary and odd, as expected.

(b) $x_2(t)=x_1(t)*p(t)$ where $x_1(t)$ is as defined in (a) and $p(t)=\sum_{k=-\infty}^{+\infty}\delta(t-k\frac{4\pi}{\omega_0})$ is an impulse train.

Answer:

Note that $x_2(t)=x_1(t)*p(t)$ is just the regular sine wave of frequency ω_0 since $x_2(t)=x_1(t)*p(t)=x_1(t)*\sum_{k=-\infty}^{+\infty}\delta(t-k\frac{4\pi}{\omega_0})=\sum_{k=-\infty}^{+\infty}x_1(t-k\frac{4\pi}{\omega_0})=\sin\left(\omega_0 t\right)$. Thus,

$$X(j\omega)=\int_{-\infty}^{\infty}\sin(\omega_0 t)e^{-j\omega t}dt$$

$$=\frac{1}{2j}\int_{-\infty}^{\infty}e^{j\omega_0 t}e^{-j\omega t}dt-\frac{1}{2j}\int_{-\infty}^{\infty}e^{-j\omega_0 t}e^{-j\omega t}dt$$

We know that the Fourier transform of $e^{j\omega_0 t}$ is given by $2\pi\delta(\omega-\omega_0)$, so

$$X(j\omega) = -\frac{1}{2j}2\pi\delta(\omega+\omega_0) + \frac{1}{2j}2\pi\delta(\omega-\omega_0)$$
$$= j\pi\delta(\omega+\omega_0) - j\pi\delta(\omega-\omega_0)$$

We can obtain the same result by applying the convolution property:

$$x_2(t) = x_1(t) * p(t) \overset{\mathcal{FT}}{\leftrightarrow} X_1(j\omega)P(j\omega) = X(j\omega)$$

Thus,

$$X(j\omega) = X_1(j\omega)P(j\omega) = X_1(j\omega)\frac{\omega_0}{2}\sum_{k=-\infty}^{+\infty}\delta(\omega - k\frac{\omega_0}{2})$$
$$= \frac{\omega_0}{2}\sum_{k=-\infty}^{+\infty} X_1(jk\frac{\omega_0}{2})\delta(\omega - k\frac{\omega_0}{2})$$
$$= \frac{\omega_0}{2}\sum_{k=-\infty}^{+\infty} \frac{2j\omega_0 \sin(k\frac{2\pi}{\omega_0}\frac{\omega_0}{2})}{\omega_0^{\,2} - \frac{1}{4}k^2\omega_0^{\,2}}\delta(\omega - k\frac{\omega_0}{2})$$
$$= \frac{\omega_0}{2}\sum_{k=-\infty}^{+\infty} \underbrace{\frac{2j\omega_0 \sin(k\pi)}{\omega_0^{\,2} - \frac{1}{4}k^2\omega_0^{\,2}}}_{=0 \text{ for } k\neq\pm 2}\delta(\omega - k\frac{\omega_0}{2})$$

The term in the above summation for $X(j\omega)$ is equal to zero for all integers $k \neq \pm 2$. In the case $k = 2$, we have a $0/0$ indeterminacy, and using l'Hopital's rule we find that:

$$\left.\frac{2j\omega_0 \sin(k\pi)}{\omega_0^{\,2} - \frac{1}{4}k^2\omega_0^{\,2}}\right|_{k=2} = \left.\frac{2j\omega_0\pi\cos(k\pi)}{-\frac{1}{2}k\omega_0^{\,2}}\right|_{k=2} = \frac{2j\omega_0\pi}{-\omega_0^{\,2}} = -j\frac{2\pi}{\omega_0}.$$

Similarly, for $k=-2$, we get $\left.\frac{2j\omega_0\sin(k\pi)}{\omega_0^{\,2}-\frac{1}{4}k^2\omega_0^{\,2}}\right|_{k=-2} = j\frac{2\pi}{\omega_0}$, and therefore,

$$
\begin{aligned}
X(j\omega) &= \frac{\omega_0}{2}\sum_{k=-\infty}^{+\infty}\frac{2j\omega_0\sin(k\pi)}{\omega_0^{\,2}-\frac{1}{4}k^2\omega_0^{\,2}}\delta(\omega-k\frac{\omega_0}{2}) \\
&= \frac{\omega_0}{2}\left(j\frac{2\pi}{\omega_0}\right)\delta(\omega+\omega_0)+\frac{\omega_0}{2}\left(-j\frac{2\pi}{\omega_0}\right)\delta(\omega-\omega_0) \\
&= j\pi\delta(\omega+\omega_0)-j\pi\delta(\omega-\omega_0)
\end{aligned}
$$

Problem 5.3

Find the time-domain signals corresponding to the following Fourier transforms.

(a) $X(j\omega)=\dfrac{j\sqrt{2}\omega+1-\omega^2}{(j\omega)(j2\sqrt{2}\omega+4-\omega^2)}+\dfrac{\pi}{4}\delta(\omega)$

Answer:

$$
\begin{aligned}
X(j\omega) &= \frac{j\sqrt{2}\omega+1-\omega^2}{(j\omega)(j2\sqrt{2}\omega+4-\omega^2)}+\frac{\pi}{4}\delta(\omega) \\
&= \frac{(j\omega)^2+\sqrt{2}j\omega+1}{(j\omega)\left[(j\omega)^2+2\sqrt{2}j\omega+4\right]}+\frac{\pi}{4}\delta(\omega) \\
&= \frac{\frac{3}{8}+j\frac{1}{8}}{(j\omega+\sqrt{2}-j\sqrt{2})}+\frac{\frac{3}{8}-j\frac{1}{8}}{(j\omega+\sqrt{2}+j\sqrt{2})}+\frac{1}{4j\omega}+\frac{\pi}{4}\delta(\omega)
\end{aligned}
$$

From Table D.1 of Fourier transform pairs,

$$X(j\omega)\overset{\mathcal{FT}}{\leftrightarrow}$$

$$x(t)=\left(\frac{3}{8}+j\frac{1}{8}\right)e^{(-\sqrt{2}+j\sqrt{2})t}u(t)+\left(\frac{3}{8}-j\frac{1}{8}\right)e^{(-\sqrt{2}-j\sqrt{2})t}u(t)+\frac{1}{4}u(t)$$

$$x(t)=2\,\mathrm{Re}\left\{\left(\frac{3}{8}+j\frac{1}{8}\right)e^{(-\sqrt{2}+j\sqrt{2})t}\right\}u(t)+\frac{1}{4}u(t)$$

$$x(t)=e^{-\sqrt{2}t}\left(\frac{3}{4}\cos(\sqrt{2}t)-\frac{1}{4}\sin(\sqrt{2}t)\right)u(t)+\frac{1}{4}u(t)$$

(b) $X(j\omega)=\dfrac{(j\omega+2)^2(j\omega)}{(j\omega+3)(j\omega+1)}$

Answer:

Let $s=j\omega$. Partial fraction expansion:

$$\begin{aligned}X(s)&=\frac{(s+2)^2 s}{(s+3)(s+1)}\\&=As+B+\frac{C}{s+1}+\frac{D}{s+3}\\&=s+\frac{-0.5}{s+1}+\frac{1.5}{s+3}\\X(j\omega)&=j\omega+\frac{-0.5}{j\omega+1}+\frac{1.5}{j\omega+3}\end{aligned}$$

Thus $X(j\omega)\overset{\mathcal{FT}}{\leftrightarrow}x(t)=\delta'(t)+\left(-\frac{1}{2}e^{-t}+\frac{3}{2}e^{-3t}\right)u(t)$

Problem 5.4

Consider the feedback interconnection in Figure 5.4 of two causal LTI differential systems defined by $S_1:\quad \dfrac{dy(t)}{dt}+y(t)=x(t)$, $S_2:\quad \dfrac{dy(t)}{dt}+2y(t)=\dfrac{dx(t)}{dt}+x(t)$.

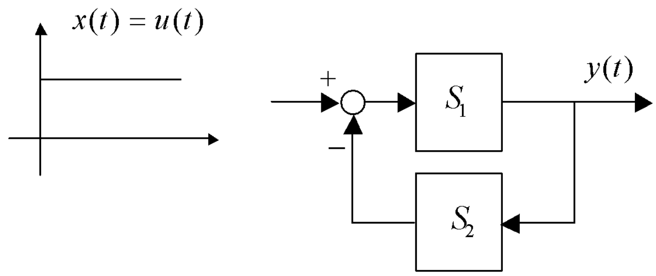

Figure 5.4: Feedback interconnection of two LTI systems in Problem 5.4.

(a) Find the frequency response of the overall system $H(j\omega)$ and plot its magnitude and phase using Matlab.

Answer:

$$
\begin{aligned}
&S_1: \quad (j\omega)Y(j\omega)+Y(j\omega)=X(j\omega)\\
&\Rightarrow H_1(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{1}{j\omega+1}\\
&S_2: \quad (j\omega)Y(j\omega)+2Y(j\omega)=(j\omega)X(j\omega)+X(j\omega)\\
&\Rightarrow H_2(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{j\omega+1}{j\omega+2}
\end{aligned}
$$

The overall closed-loop frequency response is obtained by first writing the loop equations for the error signal $e(t)$ (output of the summing junction), and the output.

$$
\begin{aligned}
E(j\omega) &= X(j\omega)+H_2(j\omega)H_1(j\omega)E(j\omega)\\
Y(j\omega) &= H_1(j\omega)E(j\omega)
\end{aligned}
$$

Solving the first equation for $E(j\omega)$, we obtain:

$$E(j\omega)=\frac{1}{1+H_2(j\omega)H_1(j\omega)}X(j\omega)$$

$$Y(j\omega)=\underbrace{\frac{H_1(j\omega)}{1+H_2(j\omega)H_1(j\omega)}}_{H(j\omega)}X(j\omega)$$

Thus,

$$H(j\omega)=\frac{H_1(j\omega)}{1+H_2(j\omega)H_1(j\omega)}=\frac{\frac{1}{j\omega+1}}{1+\frac{1}{j\omega+1}\frac{j\omega+1}{j\omega+2}}.$$

$$=\frac{\frac{1}{j\omega+1}}{1+\frac{1}{j\omega+2}}=\frac{j\omega+2}{(j\omega+3)(j\omega+1)}$$

Magnitude and phase:

$$\left|H(j\omega)\right|=\frac{\sqrt{\omega^2+4}}{\sqrt{\omega^2+9}\sqrt{\omega^2+1}}$$

$$\angle H(j\omega)=\arctan\left(\frac{\omega}{2}\right)+\arctan\left(\frac{-\omega}{3}\right)+\arctan\left(\frac{-\omega}{1}\right)$$

Using Matlab, we obtain the frequency response plots of Figure 5.5. These so-called Bode plots have a logarithmic frequency axis, and a logarithmic scale for the magnitude as well (more on Bode plots in Chapter 8.)

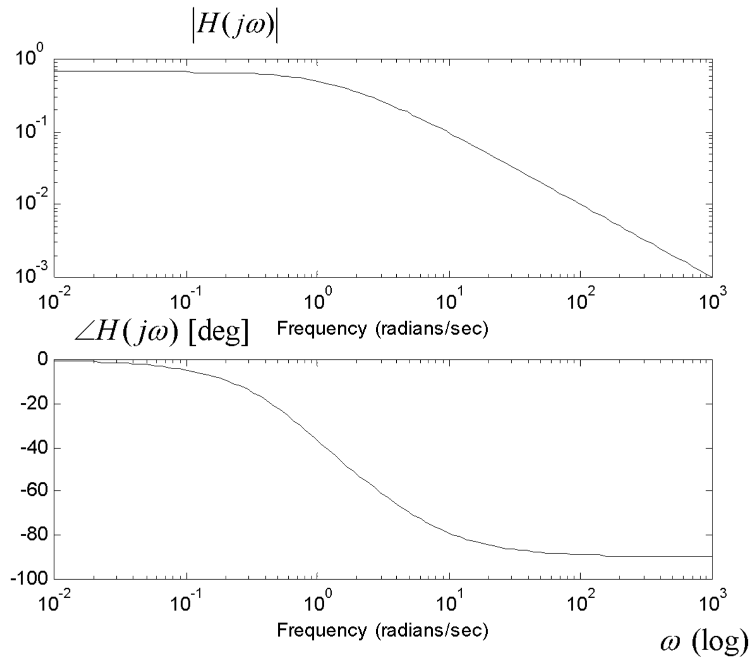

Figure 5.5: Frequency response of feedback system in Problem 5.4(a).

(b) Find the output signal $y(t)$ (the step response) using the Fourier transform technique, and sketch it.

Answer:

$$
\begin{aligned}
Y(j\omega) = H(j\omega)X(j\omega) &= \frac{j\omega+2}{(j\omega+3)(j\omega+1)}\left(\frac{1}{j\omega}+\pi\delta(\omega)\right) \\
&= \frac{j\omega+2}{(j\omega)(j\omega+3)(j\omega+1)}+\frac{2}{3}\pi\delta(\omega) \\
&= \frac{-\frac{1}{6}}{j\omega+3}+\frac{-\frac{1}{2}}{j\omega+1}+\frac{\frac{2}{3}}{j\omega}+\frac{2}{3}\pi\delta(\omega)
\end{aligned}
$$

Taking the inverse transform, we obtain the step response shown in Figure 5.6.

$$y(t) = \frac{2}{3}u(t) + \left[-\frac{1}{6}e^{-3t} - \frac{1}{2}e^{-t} \right] u(t)$$

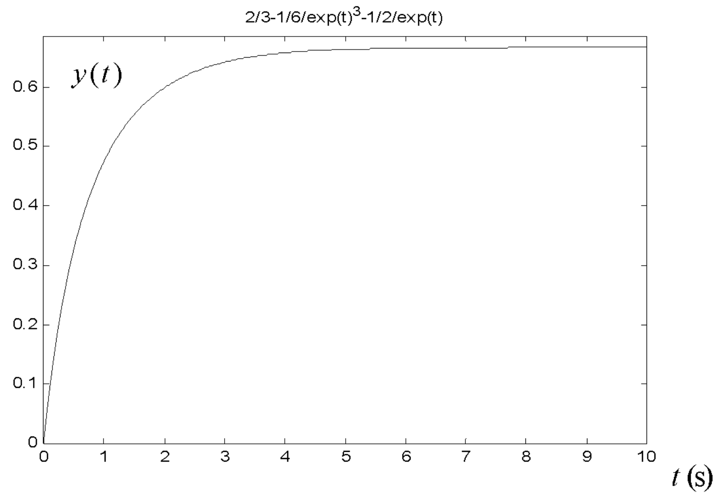

Figure 5.6: Step response of feedback system in Problem 5.4(b).

Exercises

Problem 5.5

Compute the energy-density spectrum of the signal $x(t) = e^{-5(t-2)}u(t-2)$. Now, suppose that this signal is filtered by a unit-gain ideal bandpass filter with cutoff frequencies $\omega_{c1} = 2\frac{rd}{s}$, $\omega_{c2} = 4\frac{rd}{s}$. Compute the total energy contained in the output signal of the filter.

Answer:

The Fourier transform of the signal is computed first:

$$X(j\omega) = \underbrace{e^{-j\omega}}_{\substack{\text{time-delay} \\ \text{property}}} \int_{-\infty}^{+\infty} e^{-5t}u(t)e^{-j\omega t}\,dt = e^{-j\omega}\int_{0}^{+\infty} e^{-(5+j\omega)t}\,dt = \frac{e^{-j\omega}}{j\omega+5}.$$

The energy-density spectrum is the squared magnitude of the FT:

$$|X(j\omega)|^2 = \left|\frac{e^{-j\omega}}{j\omega+5}\right|^2 = \left|\frac{1}{j\omega+5}\right|^2 = \frac{1}{\omega^2+25}.$$

The total energy at the output of the bandpass filter is computed using the Parseval Equality:

$$E_{\infty y} = \frac{1}{2\pi}\int_{-\infty}^{\infty}|Y(j\omega)|^2\,d\omega = \frac{2}{2\pi}\int_{2}^{4}\frac{1}{\omega^2+25}\,d\omega = \frac{1}{5\pi}\left[\arctan\frac{\omega}{5}\right]_2^4 = \frac{1}{5\pi}\left[\arctan\frac{4}{5} - \arctan\frac{2}{5}\right] = 0.0187$$

Problem 5.6

Compute the Fourier transform of the signal $x(t)$ shown in Figure 5.7.

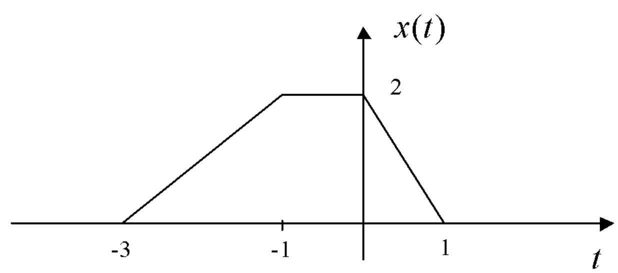

Figure 5.7: Signal in Problem 5.6.

Problem 5.7

Find the time-domain signal corresponding to the Fourier transform: $X(j\omega) = \dfrac{j\omega+1}{j4\omega+4-\omega^2}$.

Answer:

$$X(j\omega)=\frac{j\omega+1}{j4\omega+4-\omega^2}=\frac{j\omega+1}{(j\omega)^2+j4\omega+4}=\frac{j\omega+1}{(j\omega+2)^2}$$
$$=\frac{A}{(j\omega+2)}+\frac{B}{(j\omega+2)^2}$$

We find the coefficients:

$$B=\left.\frac{(s+2)^2(s+1)}{(s+2)^2}\right|_{s=-2}-(s+2)A\Big|_{s=-2}=-1$$
$$A=\left.\frac{(s+2)(s+1)}{(s+2)^2}\right|_{s=+\infty}+\left.\frac{(s+2)}{(s+2)^2}\right|_{s=+\infty}=1$$

and finally, from Table D.1 of FT pairs:

$$X(j\omega)=\frac{1}{(j\omega+2)}-\frac{1}{(j\omega+2)^2}$$
$$\overset{\mathcal{FT}}{\leftrightarrow}$$
$$x(t)=e^{-2t}u(t)-te^{-2t}u(t)$$

Problem 5.8

Sketch the signal $x(t)=e^{(t-1)}\left[u(t)-u(t-1)\right]+e^{-(t-1)}\left[u(t-1)-u(t-3)\right]$ and compute its Fourier transform.

Problem 5.9

Find the time-domain signals corresponding to the following Fourier transforms. You can use the Table D.1 of Fourier transform pairs.

(a) $X(j\omega)=\frac{j\omega+2}{j\omega(j\omega+5)}+\frac{2\pi}{5}\delta(\omega)$

Answer:

$$X(j\omega) = \frac{j\omega+2}{j\omega(j\omega+5)} + \frac{2\pi}{5}\delta(\omega)$$
$$= \frac{3}{5}\frac{1}{j\omega+5} + \frac{2}{5j\omega} + \frac{2\pi}{5}\delta(\omega)$$

From Table D.1 of Fourier transform pairs,

$$X(j\omega) \overset{\mathcal{FT}}{\leftrightarrow}$$
$$x(t) = \frac{3}{5}e^{-5t}u(t) + \frac{2}{5}u(t)$$

(b) $X(j\omega) = \dfrac{j\omega+1}{-\omega^2 + j2\sqrt{2}\omega + 4}$

Answer:

Let $s = j\omega$. Partial fraction expansion:

$$X(s) = \frac{s+1}{s^2 + 2\sqrt{2}s + 4}$$
$$= \frac{s+1}{(s+\sqrt{2} - j\sqrt{2})(s+\sqrt{2} + j\sqrt{2})}$$
$$= \frac{(1-\sqrt{2} + j\sqrt{2})/(j2\sqrt{2})}{s+\sqrt{2} - j\sqrt{2}} + \frac{(1-\sqrt{2} - j\sqrt{2})/(-j2\sqrt{2})}{s+\sqrt{2} + j\sqrt{2}}$$
$$= \frac{\frac{1}{2} + j(\frac{1}{2} - \frac{1}{2\sqrt{2}})}{s+\sqrt{2} - j\sqrt{2}} + \frac{\frac{1}{2} - j(\frac{1}{2} - \frac{1}{2\sqrt{2}})}{s+\sqrt{2} + j\sqrt{2}}$$
$$X(j\omega) = \frac{\frac{1}{2} + j(\frac{1}{2} - \frac{1}{2\sqrt{2}})}{j\omega+\sqrt{2} - j\sqrt{2}} + \frac{\frac{1}{2} - j(\frac{1}{2} - \frac{1}{2\sqrt{2}})}{j\omega+\sqrt{2} + j\sqrt{2}}$$

Thus,

$$X(j\omega) \overset{\mathcal{FT}}{\leftrightarrow} x(t) = \left\{ \left(\frac{1}{2} + j(\frac{1}{2} - \frac{1}{2\sqrt{2}}) \right) e^{(-\sqrt{2}+j\sqrt{2})t} + \left(\frac{1}{2} - j(\frac{1}{2} - \frac{1}{2\sqrt{2}}) \right) e^{(-\sqrt{2}-j\sqrt{2})t} \right\} u(t)$$

$$= 2\,\mathrm{Re}\left\{ \left(\frac{1}{2} + j(\frac{1}{2} - \frac{1}{2\sqrt{2}}) \right) e^{(-\sqrt{2}+j\sqrt{2})t} \right\} u(t)$$

$$= 2\,\mathrm{Re}\left\{ \left(\frac{1}{2} + j(\frac{1}{2} - \frac{1}{2\sqrt{2}}) \right) e^{j\sqrt{2}t} \right\} e^{-\sqrt{2}t} u(t)$$

$$= e^{-\sqrt{2}t} \left[\cos(\sqrt{2}t) - (1 - \frac{1}{\sqrt{2}}) \sin(\sqrt{2}t) \right] u(t)$$

(c) $X(j\omega) = \dfrac{\sin(\omega)}{\omega}$

Answer:

This is a sinc function which we recognize as the FT of a rectangular pulse. From the table:

$$w(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T \end{cases} \xleftrightarrow{FT} \frac{2 \sin \omega T_1}{\omega},$$

hence,

$$x(t) = \begin{cases} 1/2, & |t| < 1 \\ 0, & |t| > 1 \end{cases}.$$

(d) $H(j\omega) = \begin{cases} 1, & \omega_{c1} < |\omega| < \omega_{c2} \\ 0, & \text{otherwise} \end{cases}$, where $\omega_{c1} < \omega_{c2}$.

Answer:

This is a bandpass filter. We can use the impulse response of the ideal lowpass and the frequency shifting property to obtain $h_{bp}(t)$. We know that:

$$h_{lp}(t) = \frac{\omega_c}{\pi}\operatorname{sinc}\left(\frac{\omega_c}{\pi}t\right) \xleftrightarrow{FT} H_{lp}(j\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$

Let $\omega_1 := \frac{\omega_{c1} + \omega_{c2}}{2}$, and $\omega_c := \omega_{c2} - \omega_1$. Then, we can write the frequency response of the ideal bandpass filter as follows: $H(j\omega) = H_{lp}(j(\omega + \omega_1)) + H_{lp}(j(\omega - \omega_1))$,

which yields:

$$\begin{aligned} h(t) &= e^{j\omega_1 t}\frac{\omega_c}{\pi}\operatorname{sinc}\left(\frac{\omega_c}{\pi}t\right) + e^{-j\omega_1 t}\frac{\omega_c}{\pi}\operatorname{sinc}\left(\frac{\omega_c}{\pi}t\right) \\ &= \frac{2\omega_c}{\pi}\cos(\omega_1 t)\operatorname{sinc}\left(\frac{\omega_c}{\pi}t\right) \end{aligned}$$

Problem 5.10

Compute the Fourier transform of the periodic signal $x(t)$ shown in Figure 5.8.

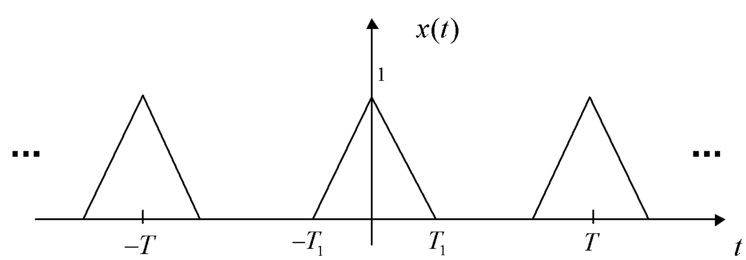

Figure 5.8: Periodic triangular waveform of Problem 5.10.

Problem 5.11

Find the inverse Fourier transform $x(t)$ of $X(j\omega)$ whose magnitude and phase are shown in Figure 5.9.

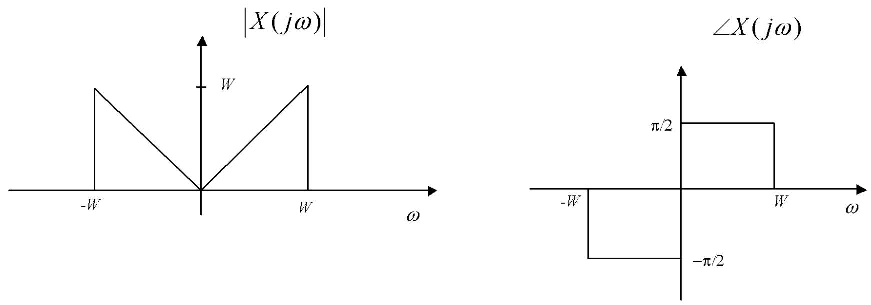

Figure 5.9: Magnitude and phase of Fourier transform in Problem 5.11.

Answer:

Let $Y(j\omega)$ be the Fourier transform of a rectangular window of unit magnitude and zero phase (i.e., it is real) from $-W$ to W. Then the Fourier transform of $x(t)$ is

$$X(j\omega) = j\omega Y(j\omega).$$

Using the differentiation property, the signal $x(t)$ is given by:

$$\begin{aligned} x(t) &= \frac{dy(t)}{dt} = \frac{d\frac{W}{\pi}\operatorname{sinc}(\frac{W}{\pi}t)}{dt} \\ &= \frac{W}{\pi}\frac{d}{dt}\frac{\sin(Wt)}{Wt} \\ &= \frac{W}{\pi}\left[\frac{W^2 t\cos(Wt) - W\sin(Wt)}{W^2 t^2}\right] \\ &= \frac{W\cos(Wt)}{\pi t} - \frac{\sin(Wt)}{\pi t^2} \end{aligned}$$