Solutions to Problems in Chapter 3

Problems with Solutions

Problem 3.1

Consider the following first-order, causal LTI differential system S_1 initially at rest:

$$S_1: \qquad \frac{dy(t)}{dt} + ay(t) = \frac{dx(t)}{dt} - 2x(t), \quad a > 0 \text{ is real.}$$

(a) Calculate the impulse response $h_1(t)$ of the system S_1. Sketch it for $a = 2$.

Answer:

Step 1: Set up the problem to calculate the impulse response of the left-hand side of the equation:

$$\frac{dh_a(t)}{dt} + ah_a(t) = \delta(t).$$

Step 2: Find the initial condition of the corresponding homogeneous equation at $t = 0^+$ by integrating the above differential equation from $t = 0^-$ to $t = 0^+$. Note that the impulse will be in the term $\frac{dh_a(t)}{dt}$, so $h_a(t)$ will have a finite jump at most. Thus we have $\int_{0^-}^{0^+} \frac{dh_a(\tau)}{d\tau} d\tau = h_1(0^+) = 1$, and hence $h_1(0^+) = 1$ is our initial condition for the homogeneous equation for $t > 0$:

$$\frac{dh_a(t)}{dt} + ah_a(t) = 0.$$

Step 3: The characteristic polynomial is $p(s) = s + a$ and it has one zero at $s = -a$, which means that the homogeneous response has the form $h_a(t) = Ae^{-at}$ for $t > 0$. The initial condition allows us to determine the constant A: $h_a(0^+) = A = 1$, so that

$$h_a(t) = e^{at}u(t).$$

Step 4: LTI systems are commutative, so we can apply the right-hand side of the differential equation to $h_a(t)$ in order to obtain $h_1(t)$:

$$\begin{aligned} h_1(t) &= \frac{dh_a(t)}{dt} - 2h_a(t) \\ &= \frac{d}{dt}\left(e^{-at}u(t)\right) - 2e^{-at}u(t) \\ &= -(2+a)e^{-at}u(t) + \delta(t) \end{aligned}$$

This impulse response is plotted in Figure 3.1 for $a = 2$:

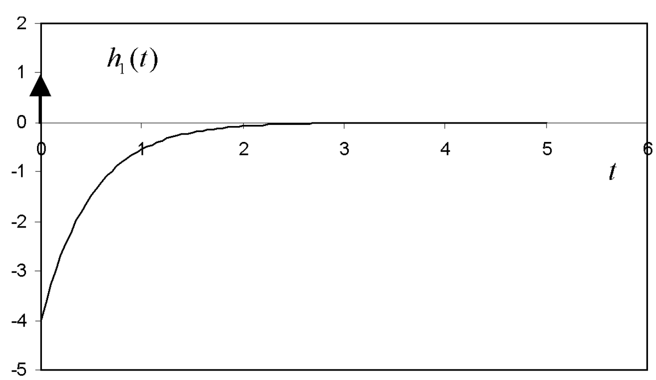

Figure 3.1: Impulse response of the first-order differential system.

(b) Is the system S_1 BIBO stable? Justify your answer.

Answer: Yes, it is stable. The single real zero of its characteristic polynomial is negative: $s = -a < 0$.

Problem 3.2

Consider the following second-order, causal LTI differential system S_2 initially at rest:

$$S_2: \qquad \frac{d^2y(t)}{dt^2} + 5\frac{dy(t)}{dt} + 6y(t) = x(t)$$

Calculate the impulse response $h_2(t)$ of the system S_2.

Answer:

Solution 1:

Step 1: Set up the problem to calculate the impulse response of the left-hand side of the equation

$$\frac{d^2h_2(t)}{dt^2} + 5\frac{dh_2(t)}{dt} + 6h_2(t) = \delta(t)$$

Step 2: Find the initial conditions of the corresponding homogeneous equation at $t = 0^+$ by integrating the above differential equation from $t = 0^-$ to $t = 0^+$. Note that the impulse will be in the term $\frac{d^2h_2(t)}{dt^2}$, so $\frac{dh_2(t)}{dt}$ will have a finite jump at most. Thus we have

$$\int_{0^-}^{0^+} \frac{d^2h_2(\tau)}{d\tau^2} d\tau = \frac{dh_2(0^+)}{dt} = 1,$$

hence $\frac{dh_2(0^+)}{dt} = 1$ is one of our two initial conditions for the homogeneous equation for $t > 0$:

$$\frac{d^2h_2(t)}{dt^2} + 5\frac{dh_2(t)}{dt} + 6h_2(t) = 0.$$

Since $\frac{dh_2(t)}{dt}$ has a finite jump from $t = 0^-$ to $t = 0^+$, the other initial condition is $h_2(0^+) = 0$.

Step 3: The characteristic polynomial is $p(s) = s^2 + 5s + 6$ and it has zeros at $s_1 = -2, s_2 = -3$, which means that the homogeneous response has the form $h_2(t) = Ae^{-2t} + Be^{-3t}$ for $t > 0$. The initial condition allows us to determine the constant A:

$$h_2(0^+) = 0 = A + B,$$

$$\frac{dh_2(0^+)}{dt} = 1 = -2A - 3B$$

so that $A = 1, B = -1$ and finally:

$$h_2(t) = \left(e^{-2t} - e^{-3t}\right)u(t).$$

Solution 2: (step response approach)

Step 1: Set up the problem to calculate the step response of the left-hand side of the equation

$$\frac{d^2 h_2(t)}{dt^2} + 5\frac{dh_2(t)}{dt} + 6h_2(t) = u(t)$$

Step 2: Compute the step response as the sum of a forced response and a homogeneous response.

The characteristic polynomial of this system is

$$p(s) = s^2 + 5s + 6 = (s+2)(s+3),$$

Hence, the homogeneous solution has the form

$$y_h(t) = Ae^{-2t} + Be^{-3t}.$$

We look for a particular solution of the form $y_p(t) = K$ for $t > 0$ when $x(t) = 1$. We find

$$y_p(t) = \frac{1}{6}.$$

Adding the homogeneous and particular solutions, we obtain the overall step response for $t > 0$:

$$s_a(t) = Ae^{-2t} + Be^{-3t} + \frac{1}{6}.$$

The initial conditions at $t = 0^-$ are $s_a(0^-) = 0, \frac{ds_a}{dt}(0^-) = 0$. Thus,

$$s_a(0^-) = 0 = A + B + \frac{1}{6}$$
$$\frac{ds_a}{dt}(0^-) = 0 = -2A - 3B.$$
$$\Rightarrow \quad A = -\frac{1}{2}, B = \frac{1}{3}$$

Which means that the step response of the system is:

$$s_a(t) = \left(-\frac{1}{2}e^{-2t} + \frac{1}{3}e^{-3t} + \frac{1}{6}\right)u(t).$$

Step 3: Differentiating, we obtain the impulse response:

$$h(t) = \frac{d}{dt}s(t) = \left(e^{-2t} - e^{-3t}\right)u(t)$$

Problem 3.3

Consider the following second-order, causal LTI differential system S initially at rest:

$$\frac{d^2y(t)}{dt^2}+\frac{dy(t)}{dt}+y(t)=x(t)$$

(a) Compute the response $y(t)$ of the system to the input $x(t)=2u(t)$ using the basic approach of the sum of a forced response and a natural response.

Answer:

First we seek to find a forced response of the same form as the input: $y_p(t)=A$. This yields $A=2$. Then, the natural response of the homogeneous equation

$$\frac{d^2y_h(t)}{dt^2}+\frac{dy_h(t)}{dt}+y_h(t)=0$$

will be a linear combination of terms of the form e^{st}. Substituting, we get the characteristic equation with complex roots:

$$s^2+s+1=(s+\frac{1}{2}-j\frac{\sqrt{3}}{2})(s+\frac{1}{2}+j\frac{\sqrt{3}}{2})=0.$$

Thus, the natural response is given by:

$$y_h(t)=A_1e^{(-\frac{1}{2}-j\frac{\sqrt{3}}{2})t}+A_2e^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t}.$$

The response of the system is the sum of the forced response and the natural response:

$$y(t) = y_h(t) + y_p(t) = A_1 e^{(-\frac{1}{2} - j\frac{\sqrt{3}}{2})t} + A_2 e^{(-\frac{1}{2} + j\frac{\sqrt{3}}{2})t} + 2 .$$

The initial conditions (zero) allow us to compute the remaining two unknown coefficients:

$$\begin{aligned} &y(0^+) = A_1 + A_2 + 2 = 0 \\ &\frac{dy}{dt}(0^+) = (-\frac{1}{2} - j\frac{\sqrt{3}}{2})A_1 + (-\frac{1}{2} + j\frac{\sqrt{3}}{2})A_2 = 0 \end{aligned}$$

We find $A_1 = -1 - j\frac{1}{\sqrt{3}}, A_2 = -1 + j\frac{1}{\sqrt{3}}$ which are complex conjugate of each other, as expected. Finally the response of the system is the signal:

$$\begin{aligned} y(t) = y_h(t) + y_p(t) &= \left((-1 - j\frac{1}{\sqrt{3}})e^{(-\frac{1}{2} - j\frac{\sqrt{3}}{2})t} + (-1 + j\frac{1}{\sqrt{3}})e^{(-\frac{1}{2} + j\frac{\sqrt{3}}{2})t} + 2 \right) u(t) \\ &= \left(2\,\mathrm{Re}\left\{ (-1 + j\frac{1}{\sqrt{3}})e^{j\frac{\sqrt{3}}{2}t} \right\} e^{-\frac{1}{2}t} + 2 \right) u(t) \\ &= 2e^{-\frac{1}{2}t} \left(-\cos\frac{\sqrt{3}}{2}t - \frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}t \right) u(t) + 2u(t) \end{aligned}$$

(b) Calculate the impulse response $h(t)$ of the system.

Answer:

Step 1: Set up the problem to calculate the impulse response of the left-hand side of the equation. Note that this will directly give us the impulse response of the system.

$$\frac{d^2h(t)}{dt^2} + \frac{dh(t)}{dt} + h(t) = \delta(t)$$

Step 2: Find the initial conditions of the corresponding homogeneous equation at $t = 0^+$ by integrating the above differential equation from $t = 0^-$ to $t = 0^+$. Note that the impulse will be in the term $\frac{d^2 h(t)}{dt^2}$, so $\frac{dh(t)}{dt}$ will have a finite jump at most. Thus we have

$$\int_{0^-}^{0^+} \frac{d^2 h(\tau)}{d\tau^2} d\tau = \frac{dh(0^+)}{dt} = 1,$$

hence $\frac{dh(0^+)}{dt} = 1$ is one of our two initial conditions for the homogeneous equation for $t > 0$:

$$\frac{d^2 h(t)}{dt^2} + \frac{dh(t)}{dt} + h(t) = 0.$$

Since $\frac{dh(t)}{dt}$ has a finite jump from $t = 0^-$ to $t = 0^+$, the other initial condition is $h(0^+) = 0$.

Step 3: From (a) the natural response is given by:

$$h(t) = A_1 e^{(-\frac{1}{2} - j\frac{\sqrt{3}}{2})t} + A_2 e^{(-\frac{1}{2} + j\frac{\sqrt{3}}{2})t}, \; t > 0.$$

The initial conditions allow us to determine the constants:

$$h(0^+) = 0 = A_1 + A_2,$$

$$\frac{dh(0^+)}{dt} = 1 = A_1(-\frac{1}{2} - j\frac{\sqrt{3}}{2}) + A_2(-\frac{1}{2} + j\frac{\sqrt{3}}{2})$$

so that $A_1 = \frac{j}{\sqrt{3}}$, $A_2 = -\frac{j}{\sqrt{3}}$, and finally,

$$\begin{aligned} h(t) &= \left(\frac{j}{\sqrt{3}} e^{(-\frac{1}{2}-j\frac{\sqrt{3}}{2})t} - \frac{j}{\sqrt{3}} e^{(-\frac{1}{2}+j\frac{\sqrt{3}}{2})t} \right) u(t) \\ &= 2\,\mathrm{Re}\left(\frac{j}{\sqrt{3}} e^{-j\frac{\sqrt{3}}{2}t} \right) e^{-\frac{1}{2}t} u(t) \\ &= \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t \right) e^{-\frac{1}{2}t} u(t) \end{aligned}.$$

Problem 3.4

Consider the following second-order, causal difference LTI system S initially at rest:

$$S: \qquad y[n] - 0.64 y[n-2] = x[n] + x[n-1]$$

Compute the response of the system to the input $x[n] = (0.2)^n u[n]$.

Answer:

The characteristic polynomial is $z^2 - 0.64 = (z-0.8)(z+0.8)$ with zeros $z_1 = -0.8, z_2 = 0.8$.

The homogeneous response is given by:

$$h_a[n] = A(-0.8)^n + B(0.8)^n, \ n \geq 0.$$

The forced response for $n \geq 1$ has the form $y_p[n] = C(0.2)^n$:

$$\begin{aligned} C(0.2)^n - 0.64C(0.2)^{n-2} &= (0.2)^n + (0.2)^{n-1} \\ C(1 - 0.64(0.2)^{-2}) &= 1 + (0.2)^{-1} \\ C &= \frac{6}{-15} = -0.4 \end{aligned}$$

Notice here that we assume that $n \geq 1$ so that all the terms in the right-hand side of the differential equation are present in the computation of the coefficient C. The assumption of initial rest implies $y[-2] = y[-1] = 0$, but we need to use initial conditions at times when the forced response exists (for $n \geq 1$), i.e., $y[1]$, $y[2]$ which can be obtained by a simple recursion:

$$
\begin{aligned}
&y[n] = 0.64y[n-2] + (0.2)^n u[n] + (0.2)^{n-1} u[n-1] \\
&n = 0: \quad y[0] = 0.64(0) + (0.2)^0 + 0 = 1 \\
&n = 1: \quad y[1] = 0.64(0) + (0.2)^1 + (0.2)^0 = 1.2 \\
&n = 2: \quad y[2] = 0.64(1) + (0.2)^2 + (0.2)^1 = 0.88
\end{aligned}
$$

Now, we can compute the coefficients A and B:

$$
\begin{aligned}
&y[1] = A(-0.8)^1 + B(0.8)^1 - 0.4(0.2)^1 = 1.2 \\
&\Rightarrow -0.8A + 0.8B = 1.28 \\
&y[2] = A(-0.8)^2 + B(0.8)^2 - 0.4(0.2)^2 = 0.88 \\
&\Rightarrow 0.64A + 0.64B = 0.896
\end{aligned}
$$

This yields: $A = -0.1, \quad B = 1.5$. Finally, the overall response is:

$$y[n] = \left[-0.1(-0.8)^n + 1.5(0.8)^n - 0.4(0.2)^n \right] u[n].$$

Problem 3.5

Consider the causal LTI system initially at rest and described by the difference equation

$$y[n] + 0.4y[n-1] = x[n] + x[n-1].$$

Find the response of this system to the input depicted in Figure 3.2 by convolution.

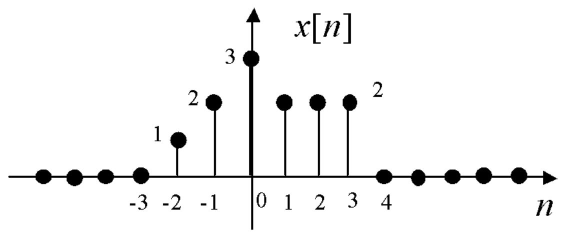

Figure 3.2: Input signal of the difference system

Answer:

First we need to find the impulse response of the difference system. The characteristic polynomial is given by:

$$p(z) = z + 0.4$$

and its zero is $z_1 = -0.4$. The homogeneous response is given by:

$$h_a[n] = A(-0.4)^n, \ \ n > 0 .$$

The initial condition for the homogeneous equation for $n > 0$ is $h_a[0] = \delta[0] = 1$. Now, we can compute the coefficient A:

$$h_a[0] = A = 1 .$$

Hence, $h_a[n] = (-0.4)^n u[n]$ and the impulse response is obtained as follows:

$$h[n] = h_a[n] + h_a[n-1] = (-0.4)^n u[n] + (-0.4)^{n-1} u[n-1] .$$

Secondly, we compute the convolution $y[n] = h[n] * x[n]$. Perhaps the easiest way to compute it is to write:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[-2]h[n+2] + x[-1]h[n+1] + x[0]h[n] + x[1]h[n-1] + x[2]h[n-2] + x[3]h[n-3] \\
&= h[n+2] + 2h[n+1] + 3h[n] + 2h[n-1] + 2h[n-2] + 2h[n-3] \\
&= (-0.4)^{n+2}u[n+2] + (-0.4)^{n+1}u[n+1] + 2(-0.4)^{n+1}u[n+1] + 2(-0.4)^{n}u[n] + 3(-0.4)^{n}u[n] \\
&\quad +3(-0.4)^{n-1}u[n-1] + 2(-0.4)^{n-1}u[n-1] + 2(-0.4)^{n-2}u[n-2] + 2(-0.4)^{n-2}u[n-2] \\
&\quad +2(-0.4)^{n-3}u[n-3] + 2(-0.4)^{n-3}u[n-3] + 2(-0.4)^{n-4}u[n-4] \\
&= (-0.4)^{n+2}u[n+2] + 3(-0.4)^{n+1}u[n+1] + 5(-0.4)^{n}u[n] + 5(-0.4)^{n-1}u[n-1] \\
&\quad +4(-0.4)^{n-2}u[n-2] + 4(-0.4)^{n-3}u[n-3] + 2(-0.4)^{n-4}u[n-4]
\end{aligned}
$$

Exercises

Problem 3.6

Determine whether the following causal LTI second-order differential system is stable.

$$2\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 24y(t) = \frac{d}{dt}x(t) - 4x(t).$$

Problem 3.7

Consider the following first-order, causal LTI difference system:

$$2y[n] + 1.2y[n-1] = x[n-1].$$

Compute the impulse response $h[n]$ of the system by using recursion.

Answer:

$$
\begin{aligned}
n = 0: \quad & y[0] = -0.6y[-1] + 0.5\delta[-1] \\
& = 0 \\
n = 1: \quad & y[1] = -0.6y[0] + 0.5\delta[0] \\
& = -0.6(0) + 0.5(1) \\
& = 0.5 \\
n = 2: \quad & y[2] = -0.6y[1] + 0.5\delta[1] \\
& = -0.6(0.5) + 0.5(0) \\
& = -0.3 \\
n = 3: \quad & y[3] = -0.6y[2] + 0.5\delta[2] \\
& = -0.6(-0.6(0.5)) + 0.5(0) \\
& = (-0.6)^2(0.5) = 0.18 \\
\vdots \quad & \quad \vdots \\
n = k: \quad & y[k] = -0.6y[k-1] + 0.5\delta[k-1] \\
& = 0.5(-0.6)^{k-1} \\
\vdots \quad & \quad \vdots
\end{aligned}
$$

Hence the impulse response is $h[n] = \frac{1}{2}(-0.6)^{n-1}u[n-1]$.

Problem 3.8

Suppose that a \$1000 deposit is made at the beginning of each year in a bank account carrying an annual interest rate of $r = 6\%$. The interest is vested in the account at the end of each year. Write the difference equation describing the evolution of the account and find the amount accrued at the end of the 50^{th} year.

Problem 3.9

Find the impulse response $h(t)$ of the following second-order, causal LTI differential system:

$$\frac{d^2y(t)}{dt^2} + \sqrt{2}\frac{dy(t)}{dt} + y(t) = \frac{dx(t)}{dt} + x(t).$$

Answer:

Step 1: Set up the problem to calculate the impulse response of the left-hand side of the equation.

$$\frac{d^2 h_a(t)}{dt^2} + \sqrt{2}\frac{dh_a(t)}{dt} + h_a(t) = \delta(t)$$

Step 2: Find the initial conditions of the corresponding homogeneous equation at $t = 0^+$ by integrating the above differential equation from $t = 0^-$ to $t = 0^+$. Note that the impulse will be in the term $\frac{d^2 h_a(t)}{dt^2}$, so $\frac{dh_a(t)}{dt}$ will have a finite jump at most. Thus we have

$$\int_{0^-}^{0^+} \frac{d^2 h_a(\tau)}{d\tau^2} d\tau = \frac{dh_a(0^+)}{dt} = 1,$$

hence $\frac{dh_a(0^+)}{dt} = 1$ is one of our two initial conditions for the homogeneous equation for $t > 0$

$$\frac{d^2 h_a(t)}{dt^2} + \sqrt{2}\frac{dh_a(t)}{dt} + h_a(t) = 0.$$

Since $\frac{dh_a(t)}{dt}$ has a finite jump from $t = 0^-$ to $t = 0^+$, the other initial condition is $h_a(0^+) = 0$.

Step 3: From (a) the natural response is given by

$$h_a(t) = A_1 e^{(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})t} + A_2 e^{(-\frac{1}{2} + j\frac{1}{\sqrt{2}})t}, \ t > 0.$$

The initial conditions allow us to determine the constants:

$$h_a(0^+) = 0 = A_1 + A_2,$$

$$\frac{dh_a(0^+)}{dt} = 1 = A_1(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}) + A_2(-\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})$$

so that $A_1 = \frac{j}{\sqrt{2}}$, $A_2 = -\frac{j}{\sqrt{2}}$, thus

$$\begin{aligned} h_a(t) &= \left(\frac{j}{\sqrt{2}} e^{(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}})t} - \frac{j}{\sqrt{2}} e^{(-\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}})t} \right) u(t) \\ &= 2\,\mathrm{Re}\left(\frac{j}{\sqrt{2}} e^{-j\frac{1}{\sqrt{2}}t} \right) e^{-\frac{1}{\sqrt{2}}t} u(t) \\ &= \sqrt{2} \sin\left(\frac{1}{\sqrt{2}} t \right) e^{-\frac{1}{\sqrt{2}}t} u(t) \end{aligned}.$$

And finally, we apply the right hand side of the differential equation:

$$\begin{aligned} h(t) &= \frac{dh_a(t)}{dt} + h_a(t) \\ &= \frac{d}{dt}\left[\sqrt{2} \sin\left(\frac{1}{\sqrt{2}} t \right) e^{-\frac{1}{\sqrt{2}}t} u(t) \right] + \sqrt{2} \sin\left(\frac{1}{\sqrt{2}} t \right) e^{-\frac{1}{\sqrt{2}}t} u(t) \\ &= \left[\cos\left(\frac{1}{\sqrt{2}} t \right) e^{-\frac{1}{\sqrt{2}}t} u(t) - \sin\left(\frac{1}{\sqrt{2}} t \right) e^{-\frac{1}{\sqrt{2}}t} u(t) + 0\delta(t) \right] + \sqrt{2} \sin\left(\frac{1}{\sqrt{2}} t \right) e^{-\frac{1}{\sqrt{2}}t} u(t) \\ &= \left[\cos\left(\frac{1}{\sqrt{2}} t \right) + (\sqrt{2} - 1) \sin\left(\frac{1}{\sqrt{2}} t \right) \right] e^{-\frac{1}{\sqrt{2}}t} u(t) \end{aligned}$$

Problem 3.10

Compute and sketch the impulse response $h(t)$ of the following causal LTI first-order differential system initially at rest:

$$2\frac{dy(t)}{dt} + 4y(t) = 3\frac{d}{dt}x(t) + 2x(t).$$

Problem 3.11

Compute the impulse response $h[n]$ of the following causal LTI second-order difference system initially at rest:

$$y[n]+\sqrt{3}y[n-1]+y[n-2]=x[n-1]+x[n-2].$$

Simplify your expression of $h[n]$ to obtain a real function of time.

Answer:

Write:

$$y[n]+\sqrt{3}y[n-1]+y[n-2]=\delta[n].$$

Initial conditions for the homogeneous equation for $n>0$ are $y[0]=1$, $y[-1]=0$.

characteristic polynomial and zeros:

$$p(z)=z^2+\sqrt{3}z+1=(z-e^{j\frac{5\pi}{6}})(z-e^{-j\frac{5\pi}{6}}).$$

The zeros are $z_1=e^{j\frac{5\pi}{6}}$, $z_2=e^{-j\frac{5\pi}{6}}$.

The homogeneous response for $n>0$ is given by:

$$h_a[n]=A(e^{j\frac{5\pi}{6}})^n+B(e^{-j\frac{5\pi}{6}})^n.$$

Use initial conditions to compute the coefficients A and B:

$$h_a[-1] = 0 = Ae^{-j\frac{5\pi}{6}} + Be^{j\frac{5\pi}{6}}$$

$$h_a[0] = 1 = A + B$$

We get $A = e^{j\frac{\pi}{3}}$, $B = e^{-j\frac{\pi}{3}}$, and the intermediate homogeneous response is

$$\begin{aligned} h_a[n] &= \left[e^{j\frac{\pi}{3}} (e^{j\frac{5\pi}{6}})^n + e^{-j\frac{\pi}{3}} (e^{-j\frac{5\pi}{6}})^n \right] u[n] \\ &= 2\,\mathrm{Re}\left\{ e^{j(\frac{5\pi}{6}n + \frac{\pi}{3})} \right\} u[n] = 2\cos(\frac{5\pi}{6}n + \frac{\pi}{3}) u[n] \end{aligned}.$$

Finally the impulse response is found as:

$$\begin{aligned} h[n] &= h_a[n-1] + h_a[n-2] \\ &= 2\cos(\frac{5\pi}{6}(n-1) + \frac{\pi}{3}) u[n-1] + 2\cos(\frac{5\pi}{6}(n-2) + \frac{\pi}{3}) u[n-2] \end{aligned}$$

Problem 3.12

Calculate the impulse response $h(t)$ of the following second-order, causal LTI differential system initially at rest:

$$\frac{d^2 y(t)}{dt^2} + 2\frac{dy(t)}{dt} + 2y(t) = -3\frac{dx(t)}{dt} + x(t) .$$

Problem 3.13

Consider the following second-order, causal difference LTI system S initially at rest:

$$S: \qquad y[n] - 0.9y[n-1] + 0.2y[n-2] = x[n-1]$$

(a) What is the characteristic polynomial of S? What are its zeros? Is the system stable?

Answer:

$$p(z) = z^2 - 0.9z + 0.2 = (z - 0.4)(z - 0.5)$$

The zeros of the characteristic polynomial are $z_1 = 0.4,\ z_2 = 0.5$. Both have a magnitude less than one, hence the system is stable.

(b) Compute the impulse response of S for all n.

Answer:

We first consider the difference equation $y[n] - 0.9y[n-1] + 0.2y[n-2] = \delta[n]$. Its homogeneous response is given by:

$$y[n] = A(0.4)^n + B(0.5)^n,\ \ n > 0.$$

The initial conditions for the homogeneous equation for $n > 0$ are $y[-1] = 0$ and $y[0] = \delta[0] = 1$. Now, we can compute coefficients A and B:

$$y[-1] = A(0.4)^{-1} + B(0.5)^{-1} = 2.5A + 2B = 0$$

$$y[0] = A + B = 1$$

From which we find: $A = -4, \quad B = 5$, and the intermediate impulse response is

$$h_a[n] = \left[-4(0.4)^n + 5(0.5)^n\right]u[n].$$

Applying the right-hand side of the difference equation, we get the impulse response of *S*:

$$
\begin{aligned}
h[n] = h_a[n-1] &= \left[-4(0.4)^{n-1} + 5(0.5)^{n-1}\right]u[n-1] \\
&= \left[-10(0.4)^n + 10(0.5)^n\right]u[n-1]
\end{aligned}
$$

(c) Compute the response of S for all n for the input signal $x[n] = 2u[n]$ using the conventional solution (sum of particular solution and homogeneous solution.)

Answer:

First, let's treat the unit delay on the right-hand side as a separate subsystem that we will apply at the end, so we consider the subsystem S_1 : $y[n] - 0.9y[n-1] + 0.2y[n-2] = x[n]$.

The particular solution of S_1 has the form $y_p[n] = Y, \; n \geq 0$:

$$
\begin{aligned}
&y_p[n] - 0.9y_p[n-1] + 0.2y_p[n-2] = 2 \\
&Y(1 - 0.9 + 0.2) = 2 \\
&\Rightarrow Y = 6.67
\end{aligned}
$$

The homogeneous equation from (b):

$$
y_h[n] = A(0.4)^n + B(0.5)^n, \; n > 0
$$

So the overall response of S_1 is given by:

$$
y[n] = A(0.4)^n + B(0.5)^n + 6.67, \; n > 0
$$

The two initial conditions at $n = 0$ and $n = 1$ are obtained by recursion:

$$
\begin{aligned}
&y[0] = 0.9y[-1] - 0.2y[-2] + 2 = 2 \\
&y[1] = 0.9y[0] - 0.2y[-1] + 2 = 1.8 + 2 = 3.8
\end{aligned}
$$

We compute the coefficients A and B :

$$\begin{aligned}
&2 = y[0] = A + B + 6.67 \\
&\Rightarrow A + B = -4.67 \\
&3.8 = y[1] = A(0.4) + B(0.5) + 6.67 \\
&\Rightarrow 0.4A + 0.5B = -2.87 \\
&A = 5.35, \quad B = -10.02
\end{aligned}$$

So the overall response of S_1 is given by:

$$y[n] = \left[5.35(0.4)^n - 10.02(0.5)^n + 6.67\right]u[n],$$

and finally the overall response of S (after applying the delay) is given by:

$$y[n] = \left[5.35(0.4)^{n-1} - 10.02(0.5)^{n-1} + 6.67\right]u[n-1].$$