Solutions to Problems in Chapter 2

Problems with Solutions

Problem 2.1

Compute the convolutions $y[n] = x[n] * h[n]$:

(a) $x[n] = \alpha^n u[n],\ h[n] = \beta^n u[n],\quad \alpha \neq \beta$. Sketch the output signal $y[n]$ for the case $\alpha = 0.8$, $\beta = 0.9$.

Answer:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \\
&= \sum_{k=-\infty}^{+\infty} \alpha^k u[k] \beta^{n-k} u[n-k] \\
&= \beta^n \sum_{k=0}^{n} \left(\frac{\alpha}{\beta}\right)^k, \quad n \geq 0 \\
&= \beta^n \left(\frac{1-\left(\frac{\alpha}{\beta}\right)^{n+1}}{1-\left(\frac{\alpha}{\beta}\right)} \right) u[n] \\
&= \left(\frac{\beta^{n+1} - \alpha^{n+1}}{\beta - \alpha} \right) u[n], \quad \alpha \neq \beta
\end{aligned}
$$

For $\alpha = 0.8,\ \beta = 0.9$, we obtain:

$$
y[n] = \left(\frac{(0.9)^{n+1} - (0.8)^{n+1}}{0.1} \right) u[n] = \left[9(0.9)^n - 8(0.8)^n \right] u[n],
$$

which is plotted in Figure 2.1.

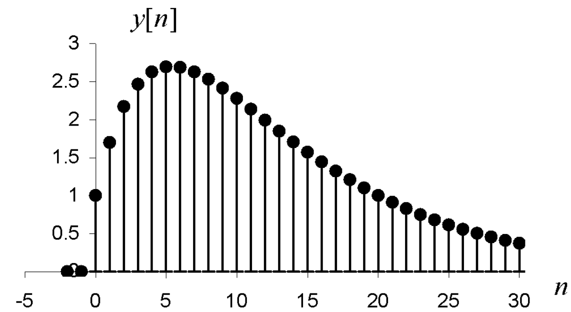


Figure 2.1: Output of discrete-time LTI system obtained by convolution in Problem 2.1(a).

(b) $x[n] = \delta[n] - \delta[n-2], \ \ h[n] = u[n]$

Answer:

$$
\begin{aligned}
y[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = \sum_{k=-\infty}^{+\infty} \left(\delta[k] - \delta[k-2]\right)u[n-k] \\
&= \sum_{k=-\infty}^{+\infty} \delta[k]u[n-k] - \sum_{k=-\infty}^{+\infty} \delta[k-2]u[n-k] \qquad \text{(by the sampling property)} \\
&= u[n]\sum_{k=-\infty}^{+\infty} \delta[k] - u[n-2]\sum_{k=-\infty}^{+\infty} \delta[k-2] \\
&= u[n] - u[n-2] = \delta[n] + \delta[n-1]
\end{aligned}
$$

(c) The input signal and impulse response depicted in Figure 2.2. Sketch the output signal $y[n]$.

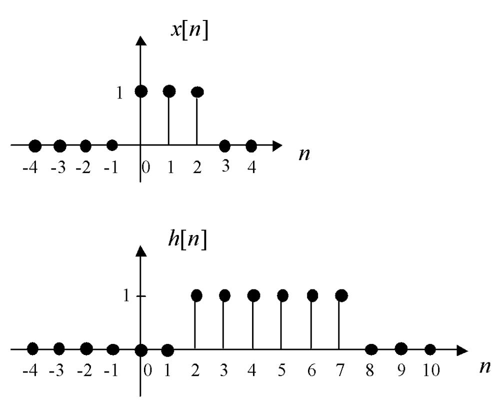


Figure 2.2: Input signal and impulse response in Problem 2.1(c).

Answer:

Let us compute this one by time-reversing and shifting $x[k]$ (note that time-reversing and shifting $h[k]$ would lead to the same answer) as shown in Figure 2.3.

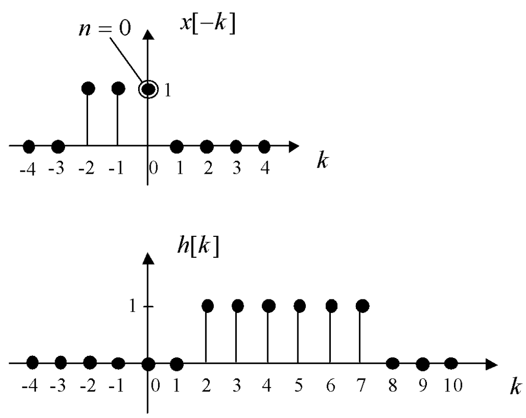


Figure 2.3: Time-reversing and shifting the input signal to compute the convolution in Problem 2.1(c).

Intervals:

$n < 2$ $h[k]x[k-n] = 0, \forall k$ $y[n] = 0$

$2 \le n \le 4:$ $h[k]x[k-n] = 1,\ 2 \le k \le n$ $y[n] = \sum_{k=2}^{n} 1 = n-(2)+1 = n-1$

$5 \le n \le 7:$ $h[k]x[k-n] = 1,\ n-2 \le k \le n$ $y[n] = \sum_{k=n-2}^{n} 1 = 3$

$8 \le n \le 9:$ $h[k]x[k-n] = 1,\ n-2 \le k \le 7$ $y[n] = \sum_{k=n-2}^{7} 1 = 7-(n-2)+1 = 10-n$

$n \ge 10$ $h[k]x[k-n] = 0, \forall k$ $y[n] = 0$

Figure 2.4 shows a plot of the output signal:

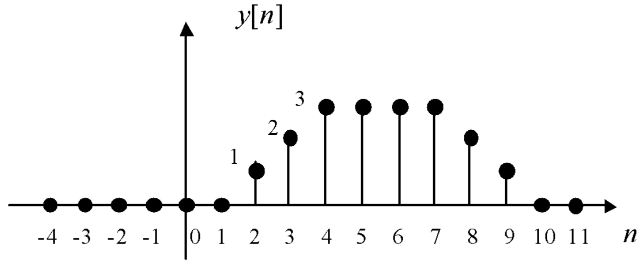


Figure 2.4: Output of discrete-time LTI system obtained by convolution in Problem 2.1(c).

(d) $x[n]=u[n],\ \ h[n]=u[n]$

Answer:

$$\begin{aligned}
y[n] &= \sum_{k=-\infty}^{+\infty} x[k]h[n-k] \\
&= \sum_{k=-\infty}^{+\infty} u[k]u[n-k] \\
&= \sum_{k=-\infty}^{+\infty} u[k]u[-(k-n)] \\
&= \begin{cases} \sum_{k=0}^{n} 1 = n+1, & n \geq 0 \\ 0, & n<0 \end{cases} \\
&= (n+1)u[n]
\end{aligned}$$

Problem 2.2

Compute and sketch the output $y(t)$ of the continuous-time LTI system with impulse response $h(t)$ for an input signal $x(t)$ as depicted in Figure 2.5.

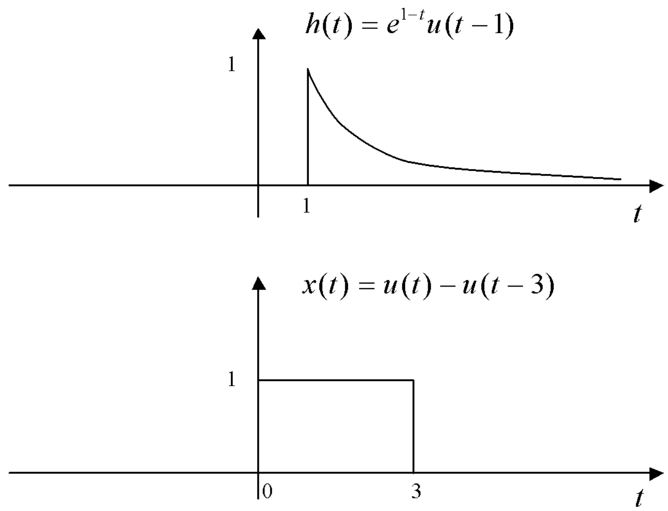


Figure 2.5: Input signal and impulse response in Problem 2.2.

Answer:

Let us time-reverse and shift the impulse response. The intervals of interest are:

$t < 1$: no overlap as seen in Figure 2.6, so $y(t) = 0$.

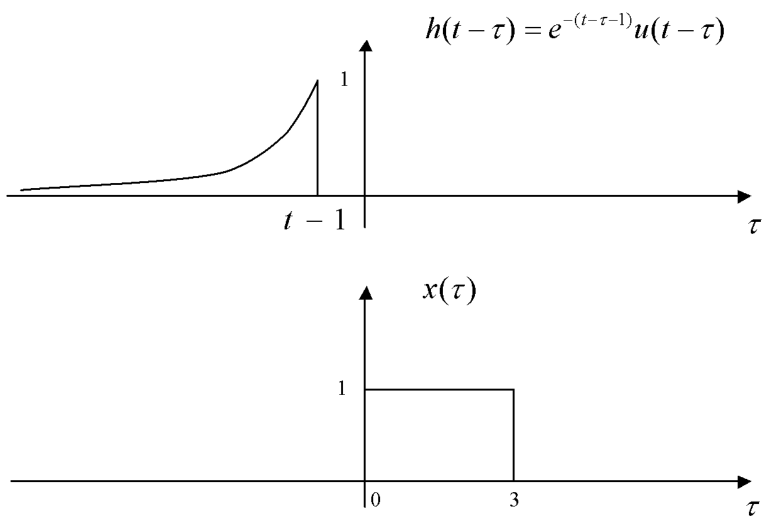


Figure 2.6: Time-reversed and shifted impulse response and input signal do not overlap for $t < 1$, Problem 2.2.

$1 \le t < 4$: overlap for $0 < \tau < t - 1$ as shown in Figure 2.7. Then,

$$y(t) = \int_{0}^{t-1} h(t-\tau)x(\tau)d\tau = \int_{0}^{t-1} e^{-(t-\tau-1)}d\tau = e^{-(t-\tau-1)}\Big|_{0}^{t-1} = (1-e^{-(t-1)}), \quad 1 \le t < 4.$$

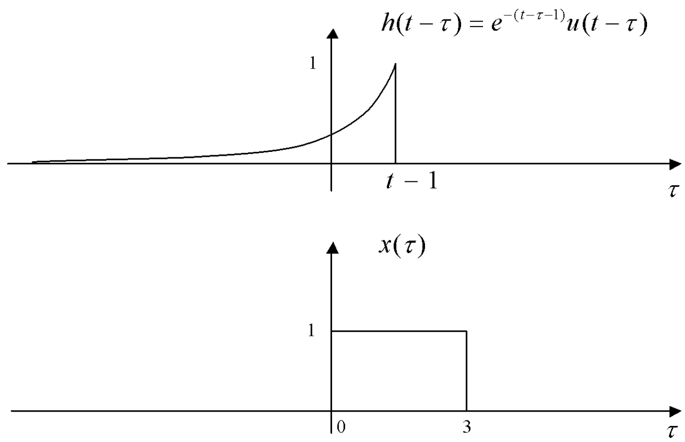


Figure 2.7: Overlapping betwee the impulse response and the input for $1 \le t < 4$ in Problem 2.2.

$t \ge 4$: overlap for $0 < \tau < 3$ as shown in Figure 2.8. Then,

$$y(t) = \int_{0}^{3} h(t-\tau)x(\tau)d\tau = \int_{0}^{3} e^{-(t-\tau-1)}d\tau = e^{-(t-\tau-1)}\Big|_{0}^{3} = (e^{4-t} - e^{1-t}) = (e^{4} - e^{1})e^{-t}, \quad t > 4.$$

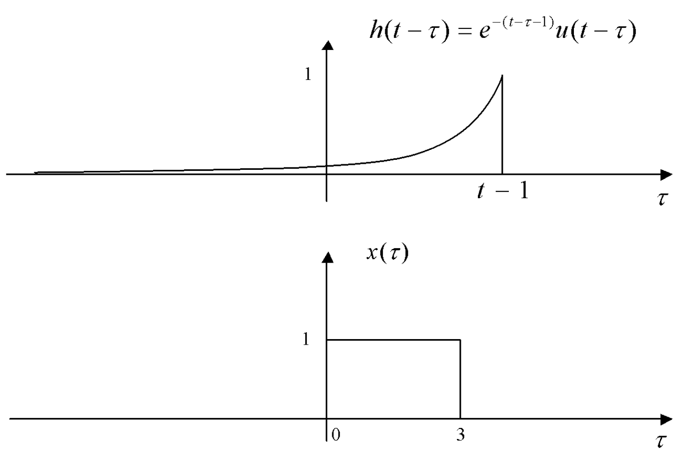


Figure 2.8: Overlap between the impulse response and the input for $t > 4$ in Problem 2.2.

Finally, the output signal shown in Figure 2.9 can be written as follows:

$$y(t)=\begin{cases} 0 & t<1 \\ 1-e^{-(t-1)} & 1\le t<4 \\ (e^{4}-e^{1})e^{-t}, & t\ge 4 \end{cases}$$

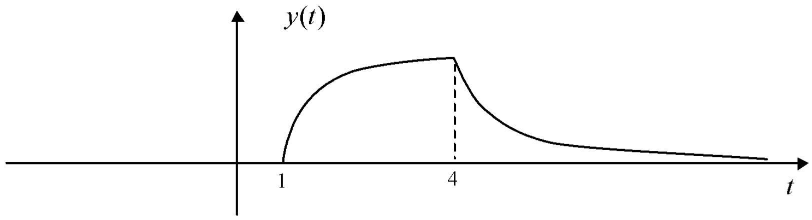


Figure 2.9: Output signal in Problem 2.2.

Exercises

Problem 2.3

Compute the output $y(t)$ of the continuous-time LTI system with impulse response $h(t)=u(t+1)-u(t-1)$ subjected to the input signal $x(t)=u(t+1)-u(t-1)$.

Answer:

$$\begin{aligned}
y(t) &= \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{+\infty}\left[u(\tau+1)-u(\tau-1)\right]\left[u(t-\tau+1)-u(t-\tau-1)\right]d\tau \\
&= \int_{-\infty}^{+\infty} u(\tau+1)u(t-\tau+1)d\tau - \int_{-\infty}^{+\infty} u(\tau+1)u(t-\tau-1)d\tau - \int_{-\infty}^{+\infty} u(\tau-1)u(t-\tau+1)d\tau + \int_{-\infty}^{+\infty} u(\tau-1)u(t-\tau-1)d\tau \\
&= \int_{-\infty}^{+\infty} u(\tau+1)u(-(\tau-(t+1)))d\tau - \int_{-\infty}^{+\infty} u(\tau+1)u(-(\tau-(t-1)))d\tau \\
&\quad - \int_{-\infty}^{+\infty} u(\tau-1)u(-(\tau-(t+1)))d\tau + \int_{-\infty}^{+\infty} u(\tau-1)u(-(\tau-(t-1)))d\tau \\
&= u(t+2)\int_{-1}^{t+1} d\tau - u(t)\int_{-1}^{t-1} d\tau - u(t)\int_{1}^{t+1} d\tau + u(t-2)\int_{1}^{t-1} d\tau \\
&= \left[\tau\right]_{-1}^{t+1} u(t+2) - \left[\tau\right]_{-1}^{t-1} u(t) - \left[\tau\right]_{1}^{t+1} u(t) + \left[\tau\right]_{1}^{t-1} u(t-2) \\
&= (t+2)u(t+2) - tu(t) - tu(t) + (t-2)u(t-2) \\
&= (t+2)u(t+2) - 2tu(t) + (t-2)u(t-2)
\end{aligned}$$

Problem 2.4

Determine whether the discrete-time LTI system with impulse response $h[n]=(-0.9)^n u[n-4]$ is BIBO stable. Is it causal?

Problem 2.5

Compute the step response of the LTI system with impulse response $h(t)=e^{-t}\cos(2t)u(t)$.

Answer:

$$\int_{-\infty}^{t} h(\tau)d\tau = \int_{-\infty}^{t} e^{-\tau}\cos(2\tau)u(\tau)d\tau = \frac{1}{2}u(t)\int_{0}^{t} e^{-\tau}(e^{j2\tau}+e^{-j2\tau})d\tau = \frac{1}{2}u(t)\int_{0}^{t} e^{(-1+j2)\tau}d\tau + \frac{1}{2}u(t)\int_{0}^{t} e^{(-1-j2)\tau}d\tau$$

$$= \frac{1}{2(-1+j2)}\left[e^{(-1+j2)\tau}\right]_0^t u(t) + \frac{1}{2(-1-j2)}\left[e^{(-1-j2)\tau}\right]_0^t u(t)$$

$$= \frac{1}{2(-1+j2)}\left[e^{(-1+j2)t}-1\right]u(t) + \frac{1}{2(-1-j2)}\left[e^{(-1-j2)t}-1\right]u(t)$$

$$= \left[\frac{-1}{2(-1+j2)} - \frac{1}{2(-1-j2)} + \frac{1}{2(-1+j2)}e^{(-1+j2)t} + \frac{1}{2(-1-j2)}e^{(-1-j2)t}\right]u(t)$$

$$= \left[2\operatorname{Re}\left\{\frac{-1}{2(-1+j2)}\right\} + 2\operatorname{Re}\left\{\frac{1}{2(-1+j2)}e^{(-1+j2)t}\right\}\right]u(t)$$

$$= \left[\operatorname{Re}\left\{\frac{1+j2}{5}\right\} + \operatorname{Re}\left\{\frac{1-j2}{5}e^{(-1+j2)t}\right\}\right]u(t) = \left[\frac{1}{5} + e^{-t}\left\{\frac{1}{5}\cos(2t)+\frac{2}{5}\sin(2t)\right\}\right]u(t)$$

Problem 2.6

Compute the output $y(t)$ of the continuous-time LTI system with impulse response $h(t)$ for an input signal $x(t)$ as depicted in Figure 2.10.

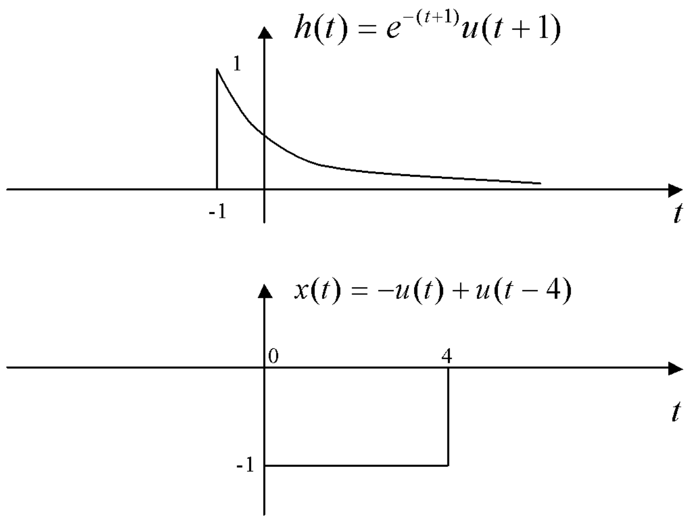


Figure 2.10: Impulse response and input signal in Problem 2.3.

Problem 2.7

Compute the convolutions $y[n] = x[n] * h[n]$.

(a) The input signal $x[n]$ and impulse response $h[n]$ are depicted in Figure 2.11. Sketch the output signal $y[n]$.

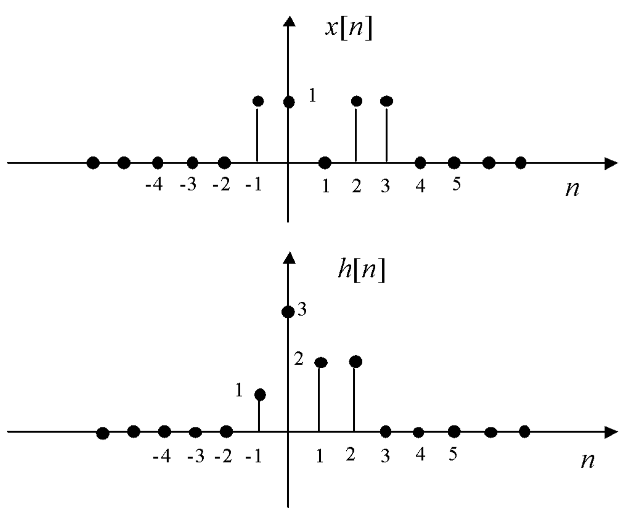
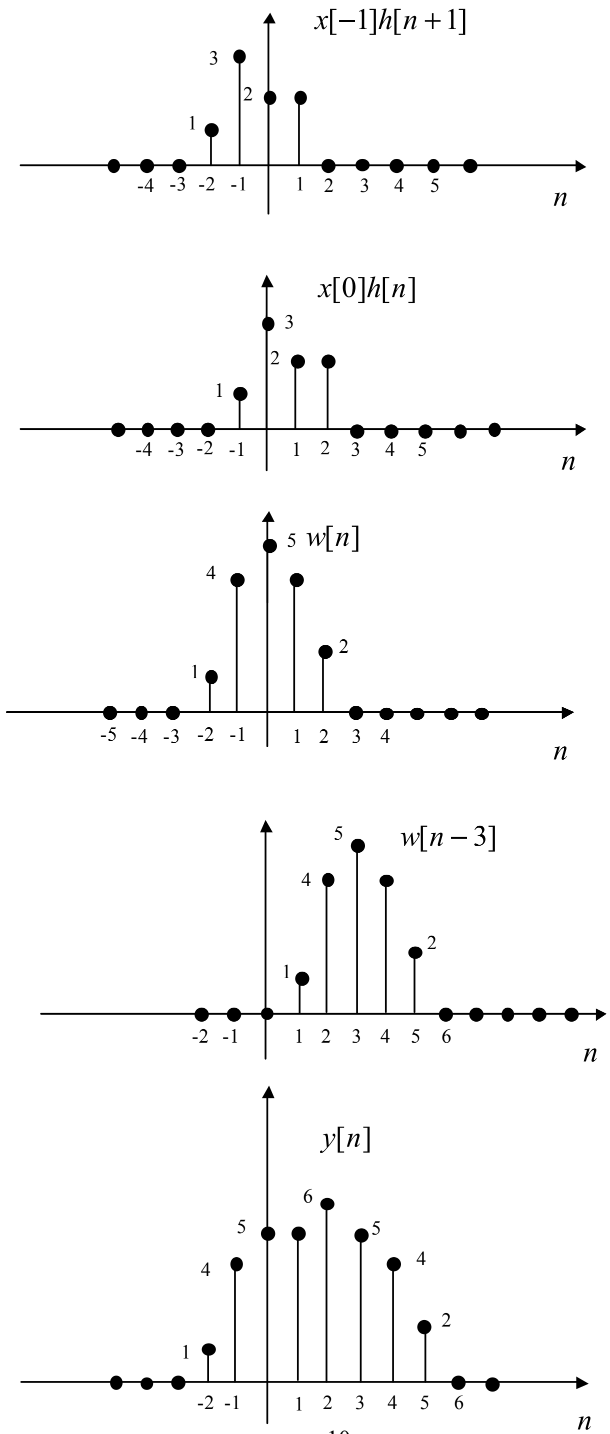


Figure 2.11: Input signal and impulse response in Problem 2.4.

Answer: The simplest technique here is probably to plot the two signals:

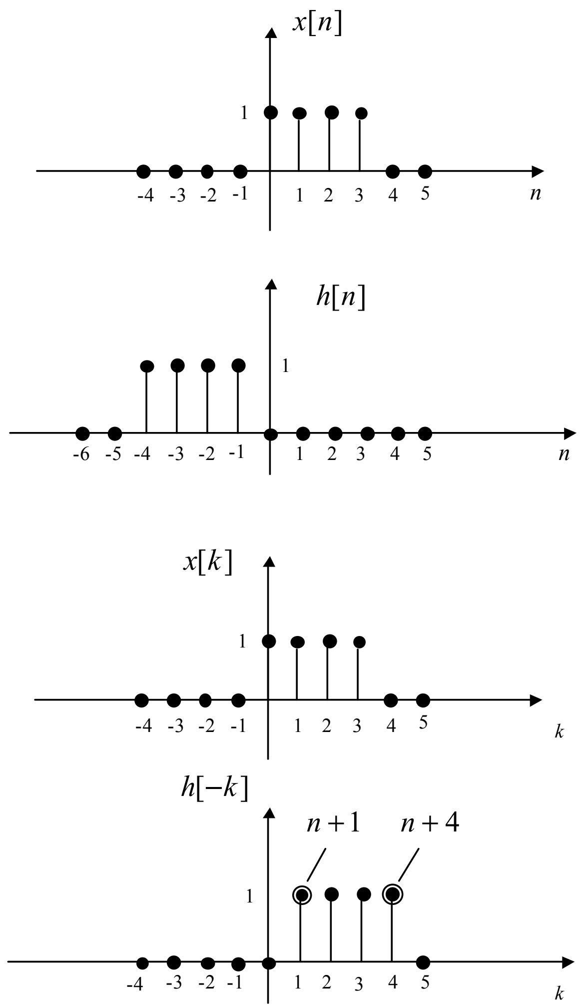
$$x[-1]h[n+1] = h[n+1], \;\; x[0]h[n] = h[n],$$

to add them up graphically, and then to add to the resulting signal $w[n] = h[n+1] + h[n]$ a version of this same signal time-delayed by three time steps. This gives $y[n] = w[n] + w[n-3]$:



(b) $x[n]=u[n]-u[n-4]$, $h[n]=u[n+4]-u[n]$. Sketch the input signal $x[n]$, the impulse response $h[n]$, and the output signal $y[n]$.

Answer:



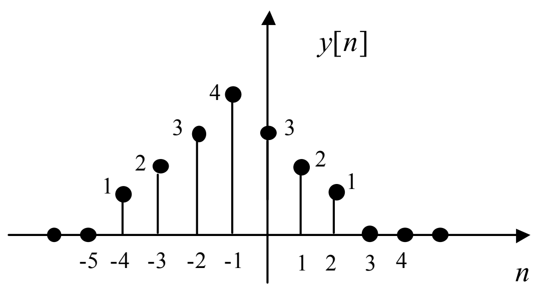
Intervals of interest:

$$n < -4: \qquad y[n] = 0$$

$$-4 \le n \le -1: \qquad y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k] = \sum_{k=0}^{n+4} 1 = n+5$$

$$0 \le n \le 2: \qquad y[n] = \sum_{k=n+1}^{3} 1 = 3-(n+1)+1 = 3-n$$

$$n \ge 3: \qquad y[n] = 0$$



Problem 2.8

Compute and sketch the output $y(t)$ of the continuous-time LTI system with impulse response $h(t)$ for an input signal $x(t)$ as depicted in Figure 2.12.

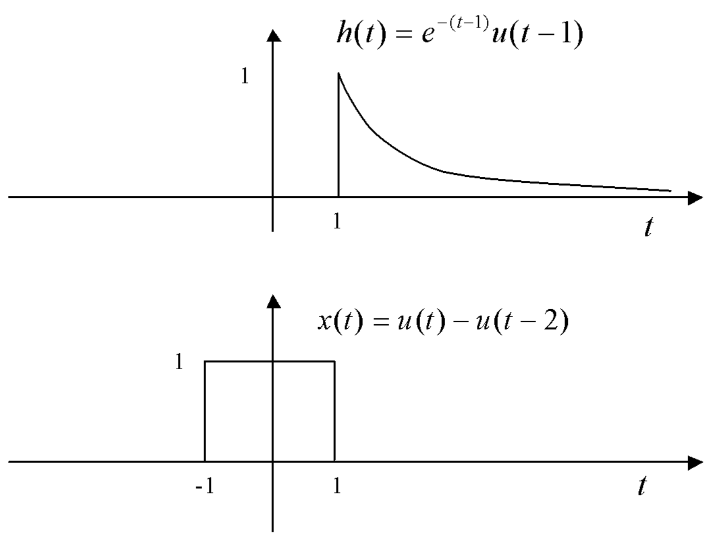


Figure 2.12: Impulse response and input signal in Problem 2.5.

Problem 2.9

Compute the response of an LTI system described by its impulse response

$h[n]=\begin{cases}(0.8)^n, & 0\le n\le 5\\ 0, & \text{otherwise}\end{cases}$ to the input signal shown in Figure 2.13.

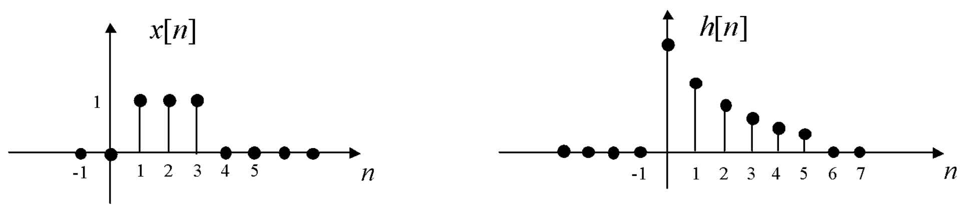
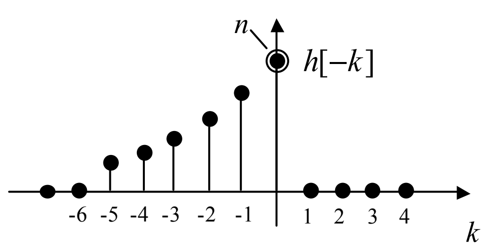


Figure 2.13: Input signal and impulse response in Problem 2.6.

Answer:

After time-reversing $h[k]$, we see that we can break down the problem into five intervals for n.



For $n<1$: $x[k]h[n-k]$ is zero for all k, hence $y[n]=0$.

For $1\le n\le 3$: Then $g[k]=x[k]h[n-k]\ne 0$ for $k=1,\ldots,n$. We get

$$y[n]=\sum_{k=1}^{n}g[k]=\sum_{k=1}^{n}(0.8)^{n-k}=(0.8)^n\sum_{m=0}^{n-1}(0.8)^{-(m+1)}=(0.8)^{n-1}\sum_{m=0}^{n-1}(0.8)^{-m}$$
$$=(0.8)^{n-1}\left(\frac{1-(0.8^{-1})^n}{1-(0.8)^{-1}}\right)=\frac{(0.8)^n-1}{-0.2}=5-5(0.8)^n$$

For $4 \le n \le 6$: Then $g[k] = x[k]h[n-k] \ne 0$ for $k = 1,\ldots,3$. We get

$$y[n] = \sum_{k=1}^{3} g[k] = \sum_{k=1}^{3} (0.8)^{n-k} = (0.8)^n \sum_{m=0}^{2} (0.8)^{-(m+1)} = (0.8)^{n-1} \sum_{m=0}^{2} (0.8)^{-m}$$
$$= (0.8)^{n-1}\left(\frac{1-(0.8^{-1})^3}{1-(0.8)^{-1}}\right) = \frac{(0.8)^n - (0.8)^{n-3}}{-0.2} = 5(0.8)^{n-3} - 5(0.8)^n = 4.7656(0.8)^n$$

For $7 \le n \le 8$: Then $g[k] = x[k]h[n-k] \ne 0$ for $n-5 \le k \le 3$. We have

$$y[n] = \sum_{k=n-5}^{3} g[k] = \sum_{k=n-5}^{3} (0.8)^{n-k} = \sum_{m=0}^{8-n} (0.8)^{n-(m+n-5)} = (0.8)^5 \sum_{m=0}^{8-n} (0.8)^{-m}$$
$$= (0.8)^5\left(\frac{1-(0.8^{-1})^{9-n}}{1-(0.8)^{-1}}\right) = \frac{(0.8)^6 - (0.8)^{n-3}}{-0.2} = 5(0.8)^{n-3} - 5(0.8)^6$$

Finally for $n \ge 9$ the two signals $x[k], h[n-k]$ don't overlap, so $y[n] = 0$.

In summary, the output signal of the LTI system is given by:

$$y[n] = \begin{cases} 0, & n \le 0 \\ 5 - 5(0.8)^n, & 1 \le n \le 3 \\ \left[5(0.8)^{-3} - 5\right](0.8)^n, & 4 \le n \le 6 \\ 5(0.8)^{n-3} - 5(0.8)^6, & 7 \le n \le 8 \\ 0, & n \ge 9 \end{cases}$$

Problem 2.10

The input signal of the LTI system shown in Figure 2.14 is the following:

$$x(t) = u(t) - u(t-2) + \delta(t+1).$$

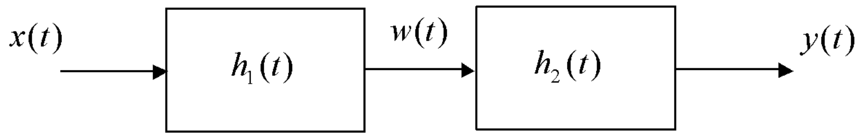


Figure 2.14: Cascaded LTI systems in Problem 2.7.

The impulse responses of the subsystems are: $h_1(t) = e^{-t}u(t), \quad h_2(t) = e^{-2t}u(t)$.

(a) Compute the impulse response $h(t)$ of the overall system.

(b) Find an equivalent system (same impulse response) configured as a parallel interconnection of two LTI subsystems.

(c) Sketch the input signal $x(t)$. Compute the output signal $y(t)$.