# Solutions to Problems in Chapter 17

## Problems with Solutions

### Problem 17.1

We want to implement a causal continuous-time LTI Butterworth filter of the second order as a discrete-time system. The transfer function of the Butterworth filter is given by: $H(s)=\dfrac{1}{\dfrac{1}{\omega_n^2}s^2+\dfrac{2\zeta}{\omega_n}s+1}$ where $\omega_n=2000\pi$ and $\zeta=\dfrac{1}{\sqrt{2}}$.

(a) Find a state-space realization of the Butterworth filter and discretize it with a sampling frequency ten times higher than the cutoff frequency of the filter. Use the bilinear transformation.

*Answer:*

$$H(s)=\frac{1}{\frac{1}{\omega_n^2}s^2+\frac{2\zeta}{\omega_n}s+1}=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$$
$$=\frac{4\pi^2\times10^6}{s^2+2000\pi\sqrt{2}s+4\pi^2\times10^6}$$

The controllable canonical state-space realization is given by:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\underbrace{\begin{bmatrix}0 & 1\\ -4\pi^2\times10^6 & -2000\pi\sqrt{2}\end{bmatrix}}_{A}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{B}u,$$

$$y=\underbrace{\begin{bmatrix}4\pi^2\times10^6 & 0\end{bmatrix}}_{C}\begin{bmatrix}x_1\\ x_2\end{bmatrix}.$$

and, $D=0$ .

The cutoff frequency of the filter is $\omega_n = 2000\pi$, thus we set the sampling frequency at $\omega_s = 20000\pi$, so that $T_s = 10^{-4}\,s$.

$$\begin{aligned}
A_{bilin} &:= \left( I_2 - \frac{T_s}{2} A \right)^{-1} \left( I_2 + \frac{T_s}{2} A \right) \\
&= \left( I_2 - 5\times10^{-5} \begin{bmatrix} 0 & 1 \\ -4\pi^2\times10^6 & -2000\pi\sqrt{2} \end{bmatrix} ] \right)^{-1} \left( I_2 + 5\times10^{-5} \begin{bmatrix} 0 & 1 \\ -4\pi^2\times10^6 & -2000\pi\sqrt{2} \end{bmatrix} \right) \\
&= \begin{bmatrix} 0.8721 & 0.00006481 \\ -2559 & 0.2962 \end{bmatrix}
\end{aligned}$$

$$B_{bilin} := T_s \left( I_n - \frac{T_s}{2} A \right)^{-2} B = \begin{bmatrix} 0.00000000513 \\ 0.0000379 \end{bmatrix}$$

$$C_{bilin} := C = \begin{bmatrix} 3.948\times10^7 & 0 \end{bmatrix}$$

$$D_{bilin} := D + \frac{T_s}{2} C \left( I_n - \frac{T_s}{2} A \right)^{-1} B = 0.06396$$

(b) Plot the frequency responses of both the continuous-time and the discrete-time filter on the same graph up to the Nyquist frequency (half of the sampling frequency) in radians/s. Discuss the results and how you would implement this filter.

*Answer:*

The transfer function is computed as:

$$\begin{aligned}
G_{bilin}(z) &= C_{bilin}(zI_n - A_{bilin})^{-1} B_{bilin} + D_{bilin} \\
&= \frac{0.06396 + 0.1279z^{-1} + 0.06396z^{-2}}{1 - 1.1683z^{-1} + 0.4241z^{-2}}, \quad |z| > 0.65
\end{aligned}$$

The Bode plots of $H(j\omega)$ and $H_{bilin}(e^{j0.0001\omega})$ up to the Nyquist frequency $10000\pi$ radians/s are computed using the following MATLAB script:

```
%% Problem 17.1 discretization of Butterworth filter
% transfer function and CT state-space model
num=[1];
den=[1/(2000*pi)^2 sqrt(2)/(2000*pi) 1];
 [A,B,C,D]=tf2ss(num,den);
T=[0 1; 1 0]; % permutation matrix to get same form as in Chapter 10
A=inv(T)*A*T;
B=inv(T)*B;
C=C*T;
H=ss(A,B,C,D);

% bilinear transf
Ts=0.0001
Ab=inv(eye(2)-0.5*Ts*A)*(eye(2)+0.5*Ts*A);
Bb=Ts*inv(eye(2)-0.5*Ts*A)*inv(eye(2)-0.5*Ts*A)*B;
Cb=C;
Db=D+0.5*Ts*C*inv(eye(2)-0.5*Ts*A)*B;
Hb=ss(Ab,Bb,Cb,Db,Ts);

% Frequency response of CT Butterworth and its discretized version
w=logspace(1,log10(10000*pi),200);
w=w(1,1:199);
[MAG,PHASE] = bode(H,w);
[MAGbilin,PHASEbilin] = bode(Hb,w);
figure(1)
semilogx(w,20*log10(MAGbilin(:,:)),w,20*log10(MAG(:,:)))
figure(2)
semilogx(w,PHASEbilin(:,:),w,PHASE(:,:))
```

The resulting Bode plots are shown in Figure 17.1.

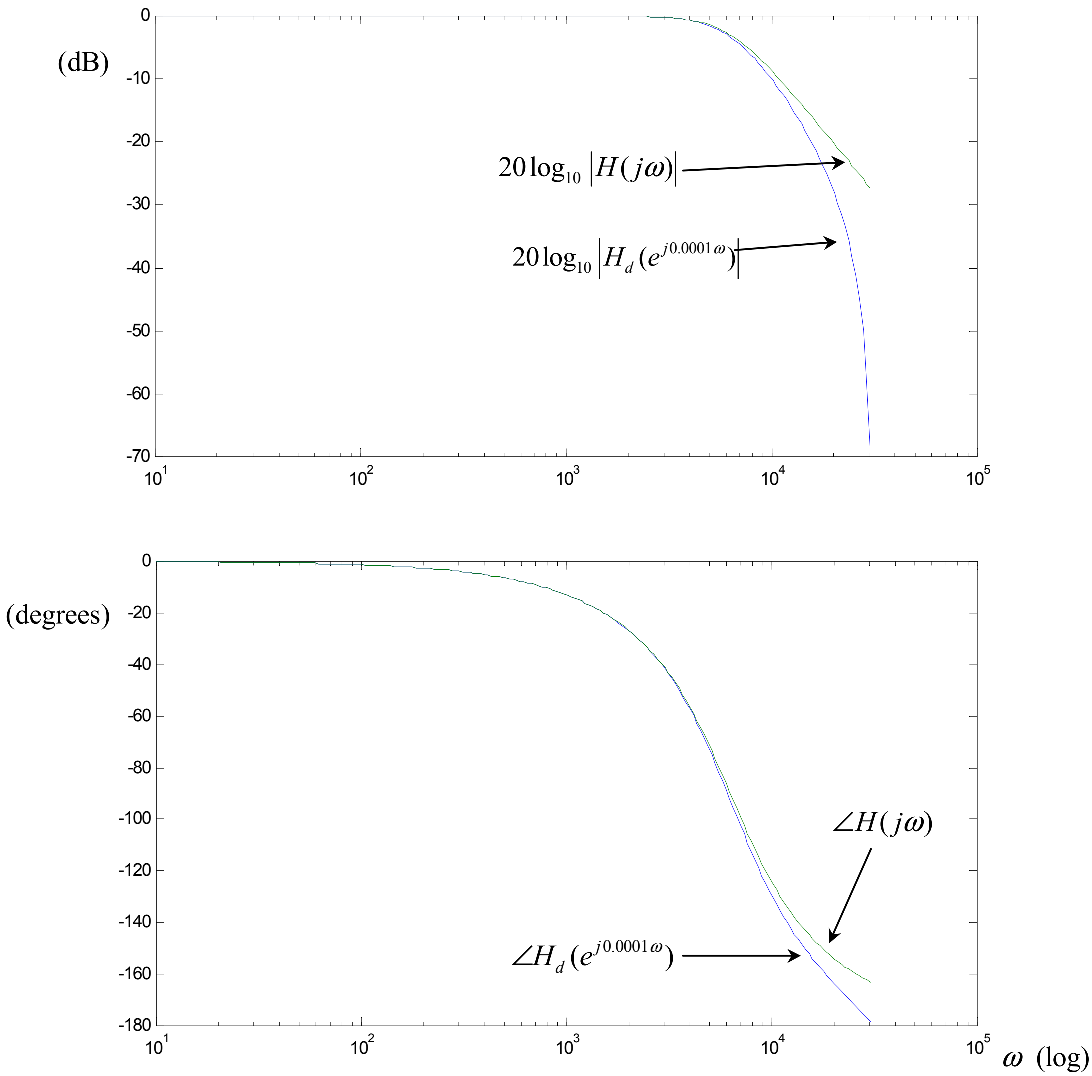


Figure 17.1: Bode plots of second-order Butterworth filter and its bilinear discretization up to the Nyquist frequency.

The filter can be implemented as a second-order recursive difference equation:

$$y[n] = 1.1683y[n-1] - 0.4241y[n-2] + 0.06396x[n] + 0.1279x[n-1] + 0.06396x[n-2]$$

(c) Compute and plot on the same graph the first 30 points of the step response of the Butterworth and its discretized version.

*Answer:*

The following MATLAB script (run after the script given in (b)) computes the continous-time step response using the *lsim* command (which internally uses a c2d discretization) with a sampling period ten times shorter than $T_s$.

```
% Step responses
figure(3)
t=[0:0.00001:.00299];  % time vector to plot step resp of CT system
 [y,ts,x]=lsim(H,ones(1,300),t);
plot(ts,y)
hold on
 [yb,tsb,xb]=lsim(Hb,ones(1,30));
plot(tsb,yb,'o')
hold off
```

The resulting plot is shown in Figure 17.2.

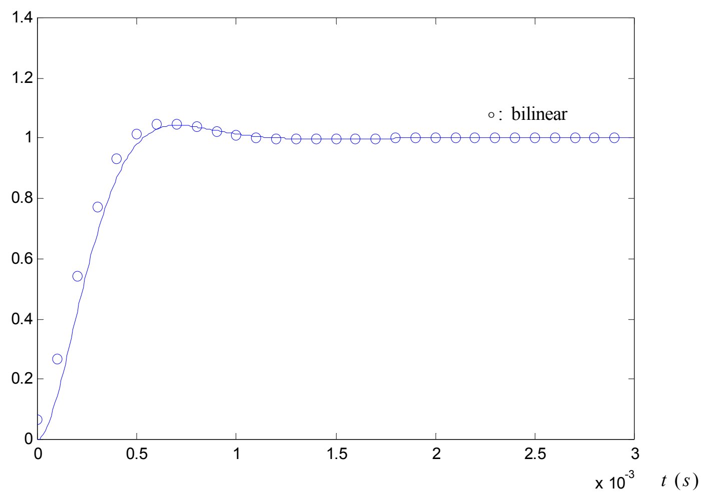


Figure 17.2: Step responses of second-order Butterworth filter and its bilinear discretization.

**Problem 17.2**

Consider the causal DLTI system given by its transfer function

$$H(z) = \frac{z^2 - z}{z^2 + 0.1z - 0.72}, \quad |z| > 0.9.$$

(a) Find the controllable canonical state-space realization (A,B,C,D) of the system (draw the block diagram and give the realization.) Assess its stability based on the eigenvalues of A.

*Answer:*

The direct form realization is shown in Figure 17.3.

$$H(z) = \frac{1 - z^{-1}}{1 + 0.1z^{-1} - 0.72z^{-2}}, \quad |z| > 0.9$$

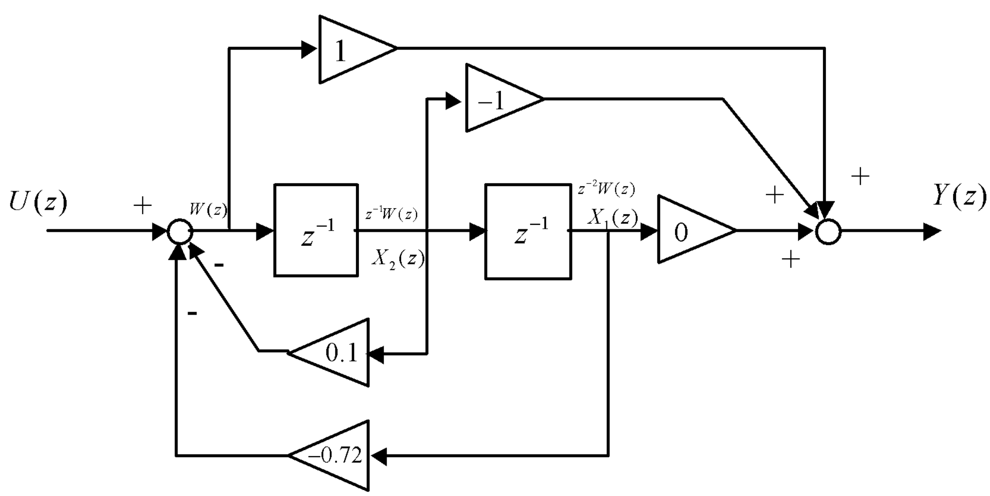


Figure 17.3: Direct form realization in Problem 17.2.

Controllable canonical state-space realization:

$$x[n+1]=\underbrace{\begin{bmatrix}0 & 1\\ 0.72 & -0.1\end{bmatrix}}_{A}x[n]+\underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{B}u[n]$$

$$y[n]=\underbrace{\begin{bmatrix}0.72 & -1.1\end{bmatrix}}_{C}x[n]+\underbrace{1}_{D}u[n]$$

Eigenvalues of $A$: $0.8, -0.9$ are inside the unit circle, and therefore the system is stable.

(b) Compute the impulse response of the system $h[n]$ by diagonalizing the $A$ matrix.

*Answer:*

The impulse response of the system $h[n]$ is given by:

$$\begin{aligned}
h[n] &= CA^{n-1}Bq[n-1]+D\delta[n]\\
&= \begin{bmatrix}0.72 & -1.1\end{bmatrix}\begin{bmatrix}0 & 1\\ 0.72 & -0.1\end{bmatrix}^{n-1}\begin{bmatrix}0\\ 1\end{bmatrix}q[n-1]+\delta[n]\\
&= \begin{bmatrix}0.72 & -1.1\end{bmatrix}\left(\begin{bmatrix}0.7809 & -0.7433\\ 0.6247 & 0.6690\end{bmatrix}\begin{bmatrix}0.8 & 0\\ 0 & -0.9\end{bmatrix}\begin{bmatrix}0.7809 & -0.7433\\ 0.6247 & 0.6690\end{bmatrix}^{-1}\right)^{n-1}\begin{bmatrix}0\\ 1\end{bmatrix}q[n-1]+\delta[n]\\
&= \begin{bmatrix}0.72 & -1.1\end{bmatrix}\left(\begin{bmatrix}0.7809 & -0.7433\\ 0.6247 & 0.6690\end{bmatrix}\begin{bmatrix}0.8^{n-1} & 0\\ 0 & (-0.9)^{n-1}\end{bmatrix}\begin{bmatrix}0.7809 & -0.7433\\ 0.6247 & 0.6690\end{bmatrix}^{-1}\right)\begin{bmatrix}0\\ 1\end{bmatrix}q[n-1]+\delta[n]\\
&= [-0.0941(0.8)^{n-1}-1.0059(-0.9)^{n-1}]q[n-1]+\delta[n]
\end{aligned}$$

## Exercises

### Problem 17.3

The causal continuous-time LTI system with transfer function $G(s)=\dfrac{1}{0.1s+1}$ is discretized with a sampling period of $T_s=0.01$ s for simulation purposes. Use the c2d transformation first to get $G_{c2d}(z)$, then the bilinear transformation to get $G_{bilin}(z)$.

*Answer:*

For the c2d discretization, a state-space model of $G(s)=\dfrac{1}{0.1s+1}=\dfrac{10}{s+10}$ is readily obtained:

$A=-10,\ B=1,\ C=10,\ D=0$, which yields $A_d=e^{-0.1},\ B_d=0.1(e^{-0.1}-1),\ C_d=10,\ D_d=0$ whose

transfer function is: $G_{c2d}(z)=\dfrac{10(-0.1)(e^{-0.1}-1)}{z-e^{-0.1}}=\dfrac{-(e^{-0.1}-1)z^{-1}}{1-e^{-0.1}z^{-1}}=\dfrac{0.0952z^{-1}}{1-0.9048z^{-1}}$.

Bilinear:

$$G_{bilin}(z)=\left.\frac{1}{0.1s+1}\right|_{s=200\frac{1-z^{-1}}{1+z^{-1}}}=\frac{1}{20\dfrac{1-z^{-1}}{1+z^{-1}}+1}=\frac{1+z^{-1}}{20-20z^{-1}+1+z^{-1}}$$

$$=\frac{1+z^{-1}}{21-19z^{-1}}=\frac{0.0476(1+z^{-1})}{(1-\frac{19}{21}z^{-1})}=\frac{0.0476(1+z^{-1})}{(1-0.9048z^{-1})}$$

**Problem 17.4**

The causal continuous-time LTI system with transfer function $G(s)=\dfrac{s}{(s+1)(s+2)}$ is discretized with a sampling period of $T_s=0.1\text{ s}$ for simulation purposes. The c2d transformation will be used first, and then it will be compared to the bilinear transformation.

(a) Find a continuous-time state-space realization of the system.

(b) Compute the discrete-time state-space system $(A_{c2d},B_{c2d},C_{c2d},D_{c2d})$ for $G(s)$ and its associated transfer function $G_{c2d}(z)$, specifying its ROC.

(c) Compute the discrete-time state-space system representing the bilinear transformation of $G(s)$ $(A_{bilin},B_{bilin},C_{bilin},D_{bilin})$ and its associated transfer function $G_{bilin}(z)$, specifying its ROC.

(d) Use MATLAB to plot the frequency responses $G(j\omega), G_{bilin}(e^{j\omega T_s}), G_{c2d}(e^{j\omega T_s})$ up to frequency $\frac{\omega_s}{2}$ on the same graph, where $\omega$ is the *continuous-time* frequency. Use a dB scale for the magnitude, and a log frequency scale for both magnitude and phase plots. Discuss any difference that you might observe.

**Problem 17.5**

Consider the causal DLTI system specified by its transfer function:

$$H(z) = \frac{z^2 - z}{z^2 - z + 0.5}, \quad |z| > \frac{1}{\sqrt{2}}.$$

(a) Find the controllable canonical state-space realization $(A, B, C, D)$ of the system. Assess its stability based on the eigenvalues of $A$.

*Answer:*

$$x[n+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ -0.5 & 1 \end{bmatrix}}_{A} x[n] + \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{B} u[n]$$

$$y[n] = \underbrace{\begin{bmatrix} -0.5 & 0 \end{bmatrix}}_{C} x[n] + \underbrace{1}_{D} u[n]$$

Eigenvalues of $A$: $\frac{1}{2} \pm j\frac{1}{2}$ are inside the unit circle, and therefore the system is stable.

(b) Compute the zero-input response of the system $y_{zi}[n]$ for the initial state $x[0]=\begin{bmatrix}1\\0\end{bmatrix}$ by diagonalizing the $A$ matrix.

*Answer:*

The zero-input response of the system $y_{zi}[n]$ is given by:

$$
\begin{aligned}
y_{zi}[n] &= CA^n x[0]q[n] \\
&= \begin{bmatrix}-0.5 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\ -0.5 & 1\end{bmatrix}^n \begin{bmatrix}1\\0\end{bmatrix} q[n] \\
&= \begin{bmatrix}-0.5 & 0\end{bmatrix}\left(\begin{bmatrix}1+j & 1-j\\ j & -j\end{bmatrix}\begin{bmatrix}0.5+j0.5 & 0\\ 0 & 0.5-j0.5\end{bmatrix}\begin{bmatrix}1+j & 1-j\\ j & -j\end{bmatrix}^{-1}\right)^n \begin{bmatrix}1\\0\end{bmatrix} q[n] \\
&= \begin{bmatrix}-0.5 & 0\end{bmatrix}\left(\begin{bmatrix}1+j & 1-j\\ j & -j\end{bmatrix}\begin{bmatrix}0.5+j0.5 & 0\\ 0 & 0.5-j0.5\end{bmatrix}\begin{bmatrix}0.5 & -0.5-j0.5\\ 0.5 & -0.5+j0.5\end{bmatrix}\right)^n \begin{bmatrix}1\\0\end{bmatrix} q[n \\
&= \begin{bmatrix}-0.5 & 0\end{bmatrix}\begin{bmatrix}1+j & 1-j\\ j & -j\end{bmatrix}\begin{bmatrix}\left(\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}\right)^n & 0\\ 0 & \left(\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\right)^n\end{bmatrix}\begin{bmatrix}0.5 & -0.5-j0.5\\ 0.5 & -0.5+j0.5\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix} q[n] \\
&= \begin{bmatrix}-\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}} & -\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\end{bmatrix}\begin{bmatrix}\left(\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}\right)^n & 0\\ 0 & \left(\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\right)^n\end{bmatrix}\begin{bmatrix}0.5\\0.5\end{bmatrix} q[n] \\
&= -0.5\left[\left(\frac{1}{\sqrt{2}}e^{j\frac{\pi}{4}}\right)^{n+1} + \left(\frac{1}{\sqrt{2}}e^{-j\frac{\pi}{4}}\right)^{n+1}\right] q[n] \\
&= -\left(\frac{1}{\sqrt{2}}\right)^{n+1}\cos\left(\frac{\pi}{4}(n+1)\right) q[n]
\end{aligned}
$$

**Problem 17.6**

Consider the causal DLTI system with transfer function:

$$H(z) = \frac{z+0.5}{z^2-0.64}, \ |z| > 0.8 .$$

(a) Find the observable canonical state-space realization $(A,B,C,D)$ of the system. Assess its stability based on the eigenvalues of $A$.

(b) Compute the zero-input response of the system $y_{zi}[n]$ for the initial state $x[0] = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

**Problem 17.7**

Find the controllable canonical state-space realization of the causal LTI system defined by the difference equation $y[n] - 0.4y[n-1] = 2x[n] - 0.4x[n-1]$, and compute its transfer function from the state-space model, specifying its ROC.

*Answer:*

The state-space "matrices" (here they are scalars) are:

$$A = 0.4, \ B = 1, \ C = -0.4 + 2(0.4) = 0.4, \ D = 2 .$$

The transfer function is computed as follows:

$$\begin{aligned} H(z) &= C(z-A)^{-1}B + D = 0.4(z-0.4)^{-1}1 + 2 \\ &= \frac{0.4}{z-0.4} + 2 = \frac{2(z-0.2)}{z-0.4}, \ |z| > 0.4 \end{aligned}$$

**Problem 17.8**

The causal continuous-time LTI system with transfer function $G(s)=\dfrac{s+2}{s+1}$ is discretized with sampling period $T_s = 0.1$ s for simulation purposes. The c2d transformation is used first, and then it is compared to the bilinear transformation.

(a) Find a state-space realization of $G(s)=\dfrac{s+2}{s+1}$, $\mathrm{Re}\{s\}>-1$.

(b) Compute the discrete-time state-space system $(A_d, B_d, C_d, D_d)$ for $G(s)$ and its associated transfer function $G_{c2d}(z)$, specifying its ROC.

(c) Find the bilinear transformation $G_{bilin}(z)$ of $G(s)$, specifying its ROC. Compute the difference between the two frequency responses obtained with c2d and the bilinear transformation, i.e., compute $E(e^{j\omega}) := G_{bilin}(e^{j\omega}) - G_{c2d}(e^{j\omega})$. Evaluate this difference at DC and at the highest discrete-time frequency.