

Solutions to Problems in Chapter 14

Problems with Solutions

Problem 14.1

(a) Design a discrete-time second-order IIR bandpass filter approximating the ideal bandpass filter with frequencies $\omega_{c1} = \frac{\pi}{4}$, $\omega_{c2} = \frac{3\pi}{4}$ and normalize it to have a passband-frequency gain of

1.

Answer:

We compute $\omega_p = \frac{\omega_{c1} + \omega_{c2}}{2} = \frac{\pi}{2}$. The pole angle is independent of r as:

$$\theta = \arccos\left\{\frac{2r}{1+r^2} \cos \omega_p\right\} = \arccos\{0\} = \frac{\pi}{2}.$$

The transfer function is:

$$\begin{aligned} H(z) &= \frac{\left[1+r^2-2r\cos(\omega_p+\theta)\right]^{\frac{1}{2}} \left[1+r^2-2r\cos(\omega_p-\theta)\right]^{\frac{1}{2}}}{1-2r\cos\theta z^{-1}+r^2 z^{-2}}, \quad |z|>r \\ &= \frac{\left[1+r^2-2r\cos(\pi)\right]^{\frac{1}{2}} \left[1+r^2-2r\cos(0)\right]^{\frac{1}{2}}}{1-2r\cos\frac{\pi}{2} z^{-1}+r^2 z^{-2}}, \quad |z|>r \\ &= \frac{\left[1+r^2+2r\right]^{\frac{1}{2}} \left[1+r^2-2r\right]^{\frac{1}{2}}}{1+r^2 z^{-2}} = \frac{\left[(1+r^2)^2-4r^2\right]}{1+r^2 z^{-2}} = \frac{(1-r^2)^2}{1+r^2 z^{-2}}, \quad |z|>r \end{aligned}$$

The magnitude plot in Figure 14.1 shows that a "reasonable" design is obtained with $r = 0.52$.

(b) Plot the magnitude of its frequency response and the magnitude of the ideal bandpass filter on the same graph.

Answer:

The frequency response magnitudes of the bandpass filters are plotted in Figure 14.1.

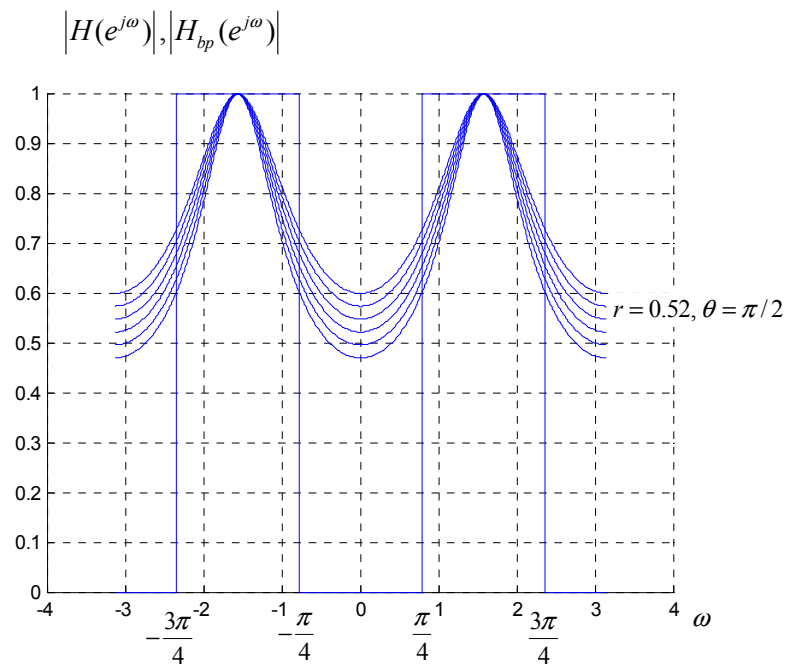


Figure 14.1: Magnitudes of second-order IIR and ideal bandpass filters, Problem 14.1.

Problem 14.2

(a) First, design a lowpass FIR filter of length $M + 1 = 17$ with cutoff frequency $\omega_c = \frac{\pi}{4}$ using a Hamming window. Plot its magnitude over the magnitude of the ideal filter.

Answer:

Windowed ideal lowpass filter:

$$h_0[n] = w[n]h_p[n] = w[n] \frac{\sin \omega_c n}{\pi n}.$$

Where the Hamming window is:

$$w[n] = \left[0.54 + 0.46 \cos\left(\frac{2\pi n}{16}\right) \right] (u[n+8] - u[n-9]).$$

After the shift:

$$h[n] = w[n-8] \frac{\sin \omega_c (n-8)}{\pi (n-8)} = w[n-8] \frac{\omega_c}{\pi} \operatorname{sinc} \frac{\omega_c}{\pi} (n-8).$$

Frequency response:

$$\begin{aligned} H(e^{j\omega}) &= h[0] + h[1]e^{-j\omega} + \dots + h[16]e^{-j\omega 16} \\ &= e^{-j\omega 8} \left(h[8] + 2 \sum_{k=0}^7 h[k] \cos \omega (8-k) \right) \\ &= e^{-j\omega 8} \left(\frac{1}{4} + 2 \sum_{k=0}^7 \frac{\sin\left(\frac{\pi}{4}(k-8)\right)}{\pi(k-8)} \left[0.54 + 0.46 \cos\left(\frac{2\pi(k-8)}{16}\right) \right] \cos \omega (8-k) \right) \end{aligned}$$

The magnitude of this frequency response, which is just the absolute value of the big bracket in the last line of the above equation, is plotted in Figure 14.2.

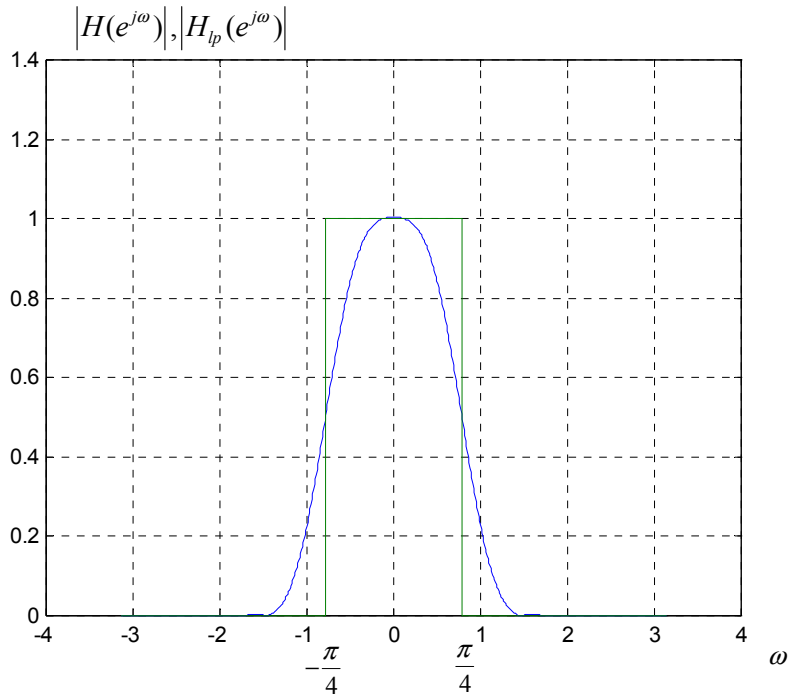


Figure 14.2: Magnitudes of sixteenth-order Hamming-windowed FIR and ideal lowpass filters, Problem 14.2.

Matlab script used to design the FIR filter:

```

%% Freq resp of FIR filter
% number of points in FIR filter is M+1
M=16;
% cutoff frequency of lowpass
wc=pi/4;
% freq vector
w=[-pi:0.01:pi]';
% FR computation using cosine formula
Hjw=(wc/pi)*ones(length(w),1);
for k=0:M/2-1
    Hjw=Hjw+2*sin((k-M/2)*wc)/((k-M/2)*pi)*(0.54+0.46*cos(2*pi*(k-
    M/2)/M))*cos(w*(M/2-k));
end

```

```

% Ideal lowpass
Hlp=zeros(length(w),1);
for k=1:length(w)
    if (abs(w(k)) < wc)
        Hlp(k)=1;
    end
end
%plot frequency response
plot(w,abs(Hjw),w,Hlp);

```

(b) Use the filter in (a) to form a real bandpass FIR filter with cutoff frequencies $\frac{\pi}{4}, \frac{3\pi}{4}$. Give its impulse response and plot its magnitude over the magnitude of the ideal bandpass filter, and that of the IIR filter in Problem 14.1.

Answer:

Using frequency shifting:

$$\begin{aligned}
 H_1(e^{j\omega}) &= H(e^{j(\omega+\frac{\pi}{2})}) + H(e^{j(\omega-\frac{\pi}{2})}) \\
 &= \left\{ h[0] + h[1]e^{-j(\omega+\frac{\pi}{2})} + \dots + h[16]e^{-j(\omega+\frac{\pi}{2})16} \right\} + \left\{ h[0] + h[1]e^{-j(\omega-\frac{\pi}{2})} + \dots + h[16]e^{-j(\omega-\frac{\pi}{2})16} \right\} \\
 &= 2h[0] + \sum_{k=1}^{16} (e^{-j\frac{\pi}{2}k} + e^{j\frac{\pi}{2}k})h[k]e^{-jk\omega} \\
 &= 2h[0] + \sum_{k=1}^{16} 2\cos\left(\frac{\pi}{2}k\right)h[k]e^{-jk\omega}
 \end{aligned}$$

The impulse response is given by: $h_1[k] = 2\cos\left(\frac{\pi}{2}k\right)h[k]$, thus:

$$\begin{aligned}
 H_1(e^{j\omega}) &= e^{-j\omega 8} \left(h_1[8] + 2 \sum_{k=0}^7 h_1[k] \cos \omega(8-k) \right) \\
 &= e^{-j\omega 8} \left(\frac{1}{2} + 2 \sum_{k=0}^7 2 \cos\left(\frac{\pi}{2}k\right) \frac{\sin\left(\frac{\pi}{4}(k-8)\right)}{\pi(k-8)} \left[0.54 + 0.46 \cos\left(\frac{2\pi(k-8)}{16}\right) \right] \cos \omega(8-k) \right)
 \end{aligned}$$

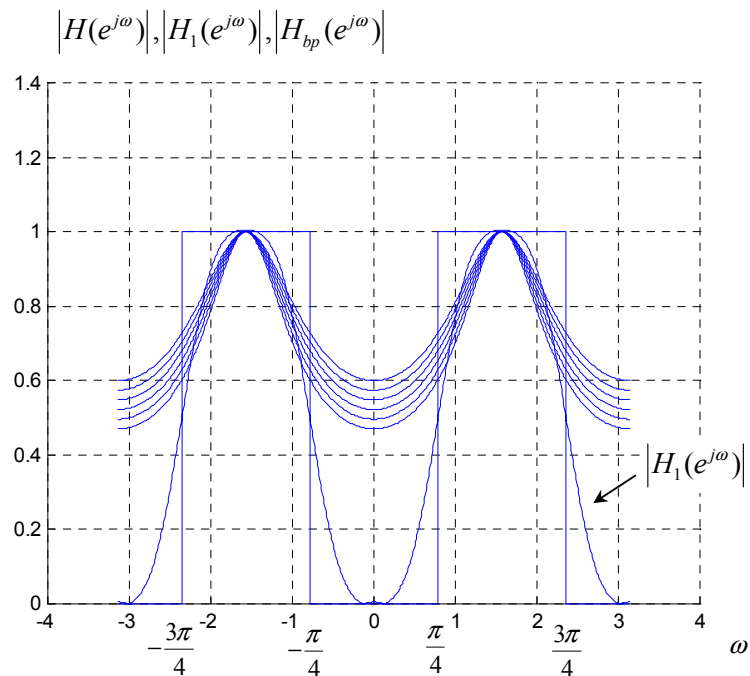


Figure 14.3: Magnitudes of bandpass filters, Problem 14.2.

(c) Give an expression for the phase in the passband of the FIR bandpass filter.

Answer:

Phase in passband is -8ω .

Exercises

Problem 14.3

Consider the following moving-average filter S initially at rest:

$$S: \quad y[n] = \frac{1}{4}x[n+1] + \frac{1}{4}x[n] + \frac{1}{4}x[n-1] + \frac{1}{4}x[n-2].$$

(a) Find the impulse response $h[n]$ of the filter. Is it causal?

Answer:

Let $x[n] = \delta[n]$. Then the impulse response is simply:

$$h[n] = \frac{1}{4}\delta[n+1] + \frac{1}{4}\delta[n] + \frac{1}{4}\delta[n-1] + \frac{1}{4}\delta[n-2].$$

It is nonzero at $n = -1$, therefore the system is noncausal.

(b) Find the transfer function of the filter $H(z)$ and give its region of convergence. Sketch the pole-zero plot.

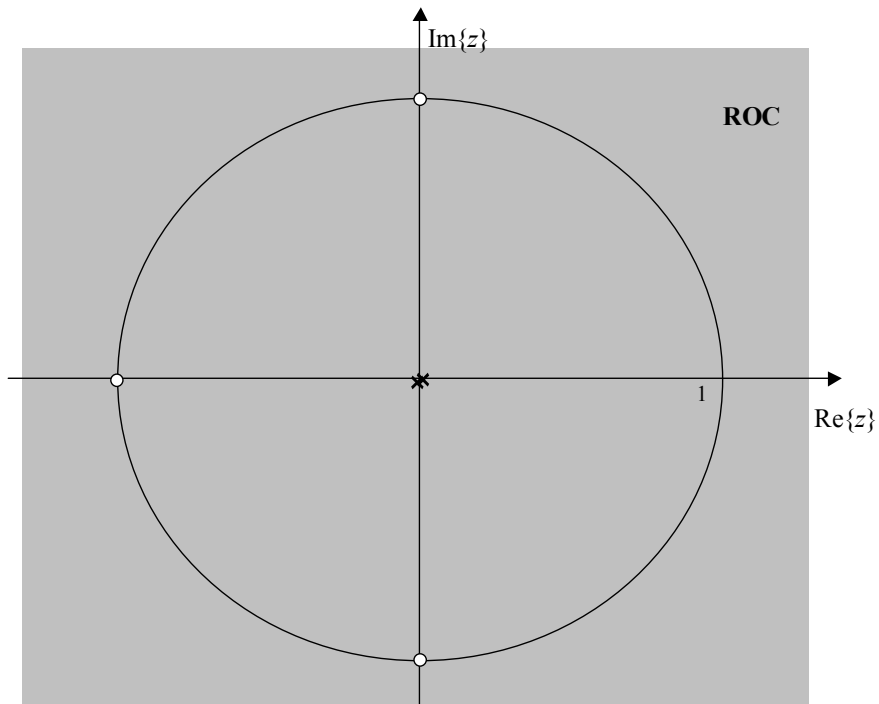
Answer:

The z -transform of this filter is given by:

$$\begin{aligned} H(z) &= \frac{1}{4}z + \frac{1}{4} + \frac{1}{4}z^{-1} + \frac{1}{4}z^{-2} \\ &= \frac{1}{4} \frac{z^3 + z^2 + z + 1}{z^2} \end{aligned}$$

with ROC $\{z \in \mathbb{C} : z \neq 0, z \neq \infty\}$.

The three zeros of the filter are on the unit circle, equally spaced as though there were four, then the one at 1 is removed. This means that the zeros are at $j, -j, -1$.



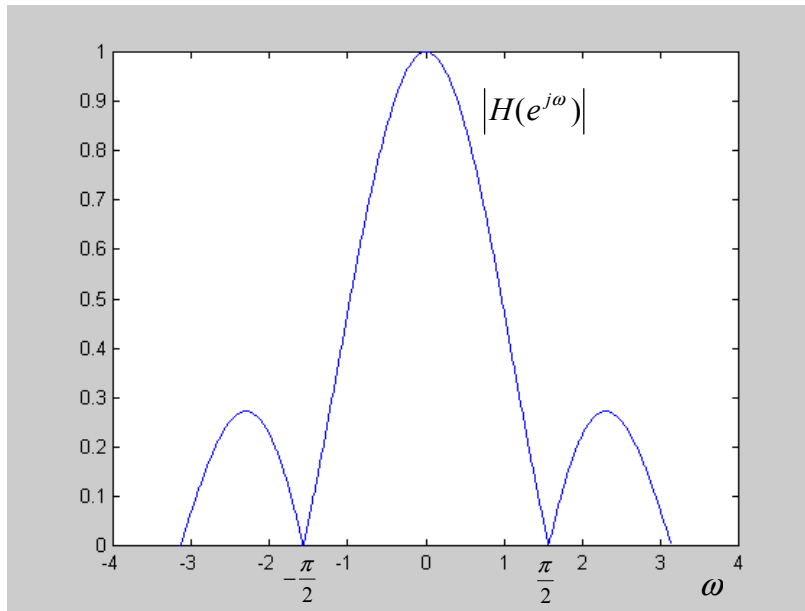
(c) Find the frequency response of the filter. Sketch its magnitude.

Answer:

Frequency response is:

$$\begin{aligned}
 H(e^{j\omega}) &= \frac{1}{4}e^{j\omega} + \frac{1}{4} + \frac{1}{4}e^{-j\omega} + \frac{1}{4}e^{-j2\omega} \\
 &= \frac{1}{4}e^{-j\frac{1}{2}\omega} \left(e^{j\frac{3}{2}\omega} + e^{j\frac{1}{2}\omega} + e^{-j\frac{1}{2}\omega} + e^{-j\frac{3}{2}\omega} \right) \\
 &= \frac{1}{2}e^{-j\frac{1}{2}\omega} \left(\cos \frac{1}{2}\omega + \cos \frac{3}{2}\omega \right)
 \end{aligned}$$

Magnitude: $\left| H(e^{j\omega}) \right| = \frac{1}{2} \left| \cos \frac{1}{2}\omega + \cos \frac{3}{2}\omega \right|$



(d) Compute the -3dB cutoff frequency ω_c .

Answer: $|H(e^{j\omega_c})| = \frac{1}{2} \left| \cos \frac{1}{2} \omega_c + \cos \frac{3}{2} \omega_c \right| = \frac{1}{\sqrt{2}}$ is solved numerically to obtain $\omega_c = 0.715$.

Problem 14.4

Suppose we want to design a causal, stable, first-order highpass filter of the type

$$H(z) = \frac{B}{1 - az^{-1}}, |z| > |a|$$

with -3dB cutoff frequency $\omega_c = \frac{2\pi}{3}$ and a real.

(a) Express the real constant B in terms of the pole a to obtain unity gain at the highest frequency $\omega = \pi$.

(b) Design the filter, i.e., find the numerical values of the pole a and the constant B .

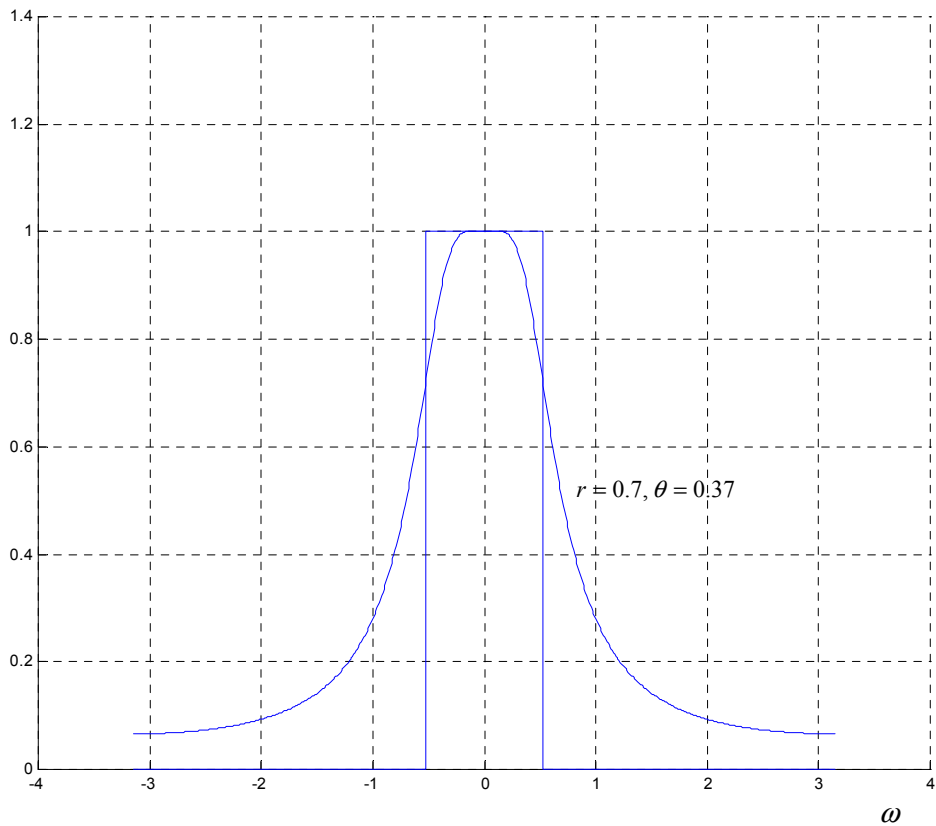
(c) Sketch the magnitude of the filter's frequency response.

Problem 14.5

(a) Design a discrete-time second-order IIR lowpass filter approximating the ideal lowpass filter with -3dB cutoff frequency $\omega_c = \frac{\pi}{6}$ and normalize it to have a DC gain of 1. You can proceed by trial and error. Plot the magnitude of its frequency response (use Matlab) and the magnitude of the ideal lowpass filter on the same figure.

Answer:

The transfer function is: $H(z) = \frac{1+r^2-2r\cos(\theta)}{1-2r\cos\theta z^{-1}+r^2 z^{-2}}$, $|z| > r$.

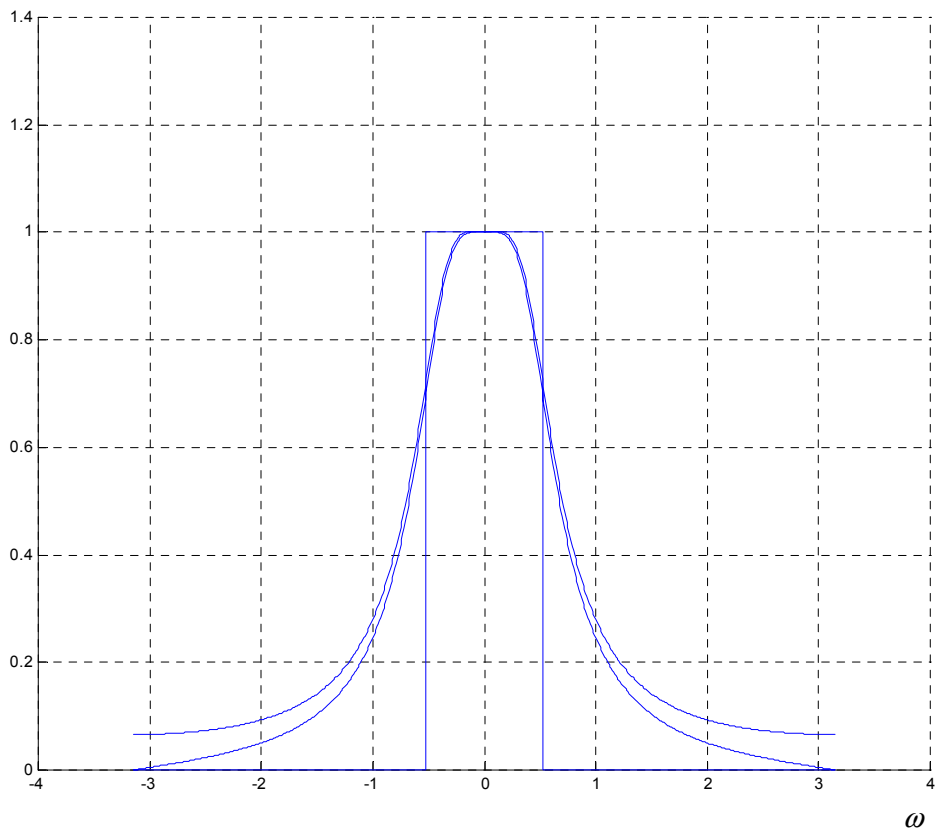


(b) Add a zero at $\omega = \pi$ to your design to improve the response at high frequency. Plot the resulting frequency response magnitude as well as the response in (a) and discuss the results.

Answer:

The transfer function is now:
$$H(z) = \frac{0.5[1 + r^2 - 2r \cos(\theta)](1 + z^{-1})}{1 - 2r \cos \theta z^{-1} + r^2 z^{-2}}, \quad |z| > r,$$

where the gain 0.5 is added to cancel off the DC gain of 2 of $1 + z^{-1}$. We can see that the magnitudes are almost the same except at high frequencies where the addition of a zero at $\omega = \pi$ in the second filter forces the magnitude to drop down to zero. Thus the second filter has better rejection in the stopband.



Problem 14.6

FIR lowpass filter design

- (a) First, design a lowpass FIR filter of order $M = 256$ with cutoff frequency $\omega_c = \frac{\pi}{6}$ radians using a Hamming window. Plot its magnitude over the magnitudes of the ideal filter and the filters in Problem 14.5.
- (b) Give an expression for the phase in the passband of the FIR bandpass filter.
- (c) Simulate the filter for the input $x[n] = \cos(\frac{\pi}{8}n)u[n] + \cos(\frac{\pi}{4}n)u[n]$ until the output reaches steady-state and plot the results. Also plot the input on a separate graph.

Problem 14.7

- (a) Design a first-order DT lowpass filter with a -3dB cutoff frequency $\omega_c = \frac{\pi}{12}$ and a DC gain of 1.

Answer:

The magnitude of the frequency response of the first-order lowpass filter has the form

$$|H(e^{j\omega})| = \left| (1-a) \frac{1}{1-ae^{-j\omega}} \right| = \frac{|1-a|}{(1+a^2-2a\cos\omega)^{1/2}}.$$

We can compute its pole by using the equation:

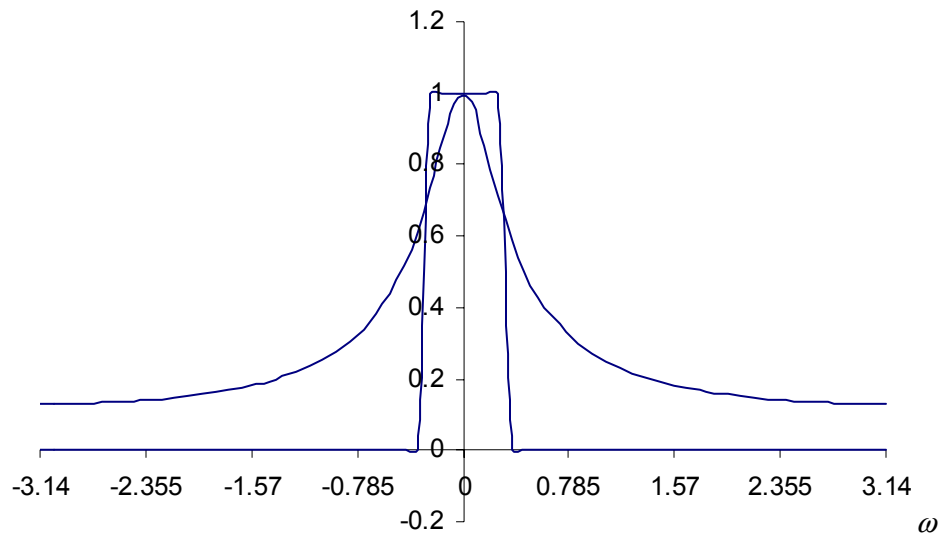
$$\begin{aligned}
 a &= \frac{(2 - 1.0024 \cos \omega_c)}{0.9976} - \frac{1}{0.9976} \sqrt{(2 - 1.0024 \cos \omega_c)^2 - 0.9952} \\
 &= \frac{(2 - 1.0024 \cos \frac{\pi}{12})}{0.9976} - \frac{1}{0.9976} \sqrt{(2 - 1.0024 \cos \frac{\pi}{12})^2 - 0.9952} \\
 &= 0.770317
 \end{aligned}$$

Therefore, the transfer function of the lowpass filter is

$$\begin{aligned}
 H(z) &= (1-a) \frac{1}{1-az^{-1}} \\
 &= \frac{0.229683}{1-0.770317z^{-1}}, \quad |z| > 0.770317
 \end{aligned}$$

(b) Plot the magnitude of its frequency response.

Answer:



(c) Write the filter as a difference equation, and plot the first 20 values of the filtered signal $y[n]$ for the input signal $x[n] = u[n] - u[n - 10]$ with the filter being initially at rest.

Answer:

$$y[n] = 0.770317y[n-1] + 0.229683x[n]$$

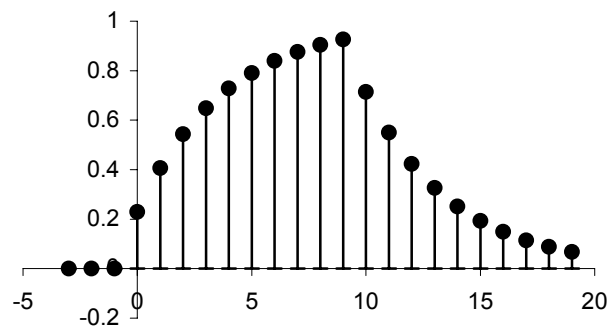
We can do the computation recursively:

$$y[0] = 0.770317y[-1] + x[0] = 0.229683$$

$$y[1] = 0.770317y[0] + 0.229683x[1] = 0.406612$$

⋮

Plot of first 20 values:



Problem 14.8

(a) Design a DT second-order IIR bandpass filter approximating the ideal bandpass filter with

frequencies $\omega_{c1} = \frac{3\pi}{8}$, $\omega_{c2} = \frac{\pi}{2}$ and normalize it to have a passband-frequency gain of 5.

(b) Plot the magnitude of its frequency response and the magnitude of the ideal bandpass filter on the same figure.

Problem 14.9

(a) Design a first-order DT lowpass filter with a -3dB cutoff frequency $\omega_c = \frac{\pi}{12}$ and a DC gain of 1.

Answer:

The magnitude of the frequency response of the first-order lowpass filter has the form

$$|H(e^{j\omega})| = \left| (1-a) \frac{1}{1-ae^{-j\omega}} \right| = \frac{|1-a|}{(1+a^2-2a\cos\omega)^{1/2}}.$$

We can compute its pole by using the equation:

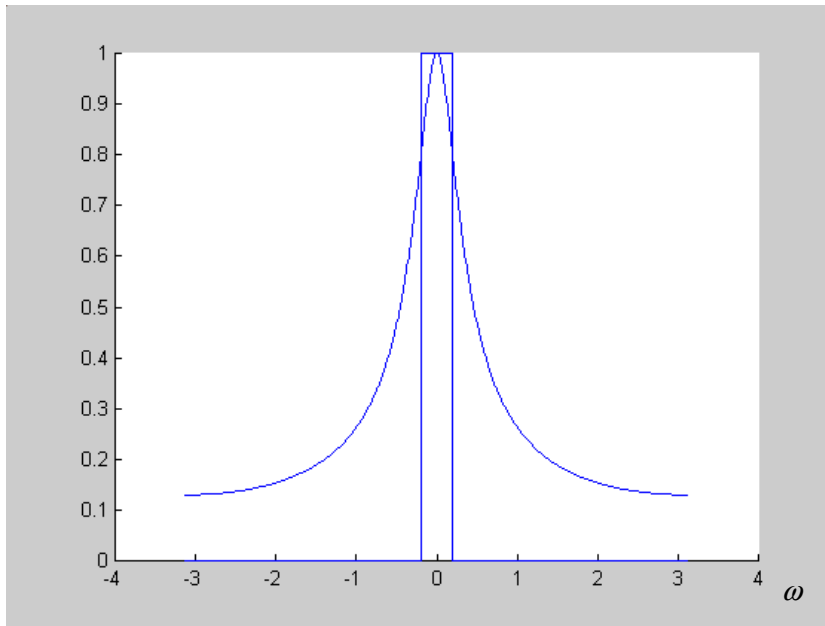
$$\begin{aligned} a &= \frac{(2 - 1.0024 \cos \omega_c)}{0.9976} - \frac{1}{0.9976} \sqrt{(2 - 1.0024 \cos \omega_c)^2 - 0.9952} \\ &= \frac{(2 - 1.0024 \cos \frac{\pi}{12})}{0.9976} - \frac{1}{0.9976} \sqrt{(2 - 1.0024 \cos \frac{\pi}{12})^2 - 0.9952} \\ &= 0.770317 \end{aligned}$$

Therefore, the transfer function of the lowpass filter is

$$\begin{aligned} H(z) &= (1-a) \frac{1}{1-az^{-1}} \\ &= \frac{0.229683}{1-0.770317z^{-1}}, \quad |z| > 0.770317 \end{aligned}$$

(b) Plot the magnitude of its frequency response.

Answer:



(c) Write the filter as a difference equation, and plot the first 15 values of the filtered signal $y[n]$ for the input signal $x[n] = u[n - 2]$.

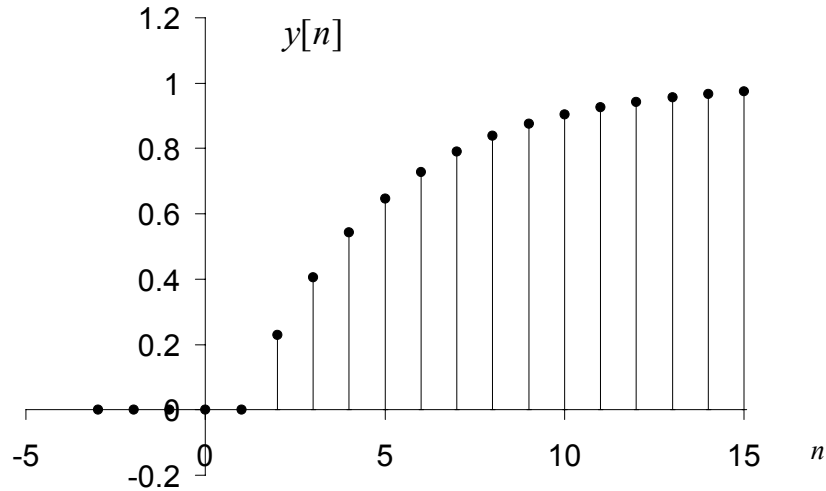
Answer:

$$y[n] = 0.770317y[n - 1] + 0.229683x[n]$$

We can do the computation recursively:

$$\begin{aligned}y[0] &= 0.770317y[-1] + x[0] = 0 \\y[1] &= 0.770317y[0] + 0.229683x[1] = 0 \\y[2] &= 0.770317y[1] + 0.229683x[2] = 0.229683 \\y[3] &= 0.770317y[2] + 0.229683x[3] = 0.406611 \\y[4] &= 0.770317y[3] + 0.229683x[4] = 0.542903 \\&\vdots\end{aligned}$$

Plot of first 15 values:



Problem 14.10

(a) Design a 21th-order (i.e., $M = 20$) DT causal FIR highpass filter approximating the ideal highpass filter with $\omega_c = \frac{2\pi}{3}$ and normalized to have a high-frequency gain of 1. Use the straight truncation (rectangular window) technique.

(b) Repeat (a) using the Hamming window.

(c) Plot the magnitudes of the frequency responses of the two FIR filters, the magnitude of the IIR filter obtained in Problem 14.4, and the magnitude of the ideal high-pass filter on the same figure.

(d) Plot the first 20 values of the two filtered signals $y[n]$, i.e., the output of each filter, for the step input signal $x[n] = u[n]$.