Solutions to Problems in Chapter 12

Problems with Solutions

Problem 12.1

Compute the Fourier series coefficients $\{a_k\}$ of the signal $x[n]$ shown in Figure 12.1. Sketch the magnitude and phase of the coefficients. Write $x[n]$ as a Fourier series.

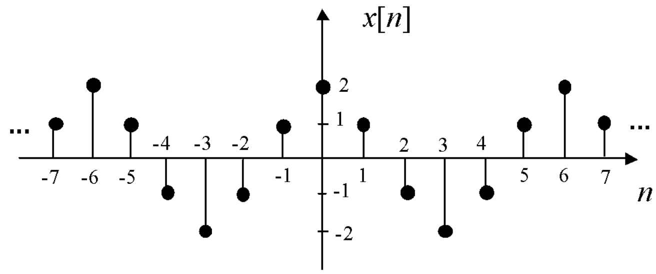


Figure 12.1: Periodic signal of Problem 12.1.

Answer:

The fundamental period of this signal is $N = 6$ and its fundamental frequency is $\omega_0 = \frac{2\pi}{6}$. The DC component is $a_0 = 0$. The other coefficients are obtained using the analysis equation of the DTFS:

$$
\begin{aligned}
a_k &= \frac{1}{6}\sum_{n=0}^{5} x[n]e^{-jk\frac{2\pi}{6}n} \\
&= \frac{1}{6}\left(2 + e^{-jk\frac{2\pi}{6}} - e^{-jk\frac{4\pi}{6}} - 2e^{-jk\frac{6\pi}{6}} - e^{-jk\frac{8\pi}{6}} + e^{-jk\frac{10\pi}{6}}\right) \\
&= \frac{1}{6}\left(2(1-(-1)^k) + e^{-jk\frac{\pi}{3}} - e^{-jk\frac{2\pi}{3}} - e^{-jk\frac{4\pi}{3}} + e^{-jk\frac{5\pi}{3}}\right) \\
&= \frac{1}{6}\left(2(1-(-1)^k) + (1-(-1)^k)e^{-jk\frac{\pi}{3}} - (1-(-1)^k)e^{-jk\frac{2\pi}{3}}\right) \\
&= \frac{(1-(-1)^k)}{6}\left(2 + e^{-jk\frac{\pi}{3}} - e^{-jk\frac{2\pi}{3}}\right)
\end{aligned}
$$

Numerically, $a_0 = 0,\ a_1 = 1,\ a_2 = 0,\ a_3 = 0,$
$a_4 = 0,\ a_5 = 1$

All coefficients are real as the signal is real and even. The magnitude and phase of the spectrum are shown in Figure 12.2.

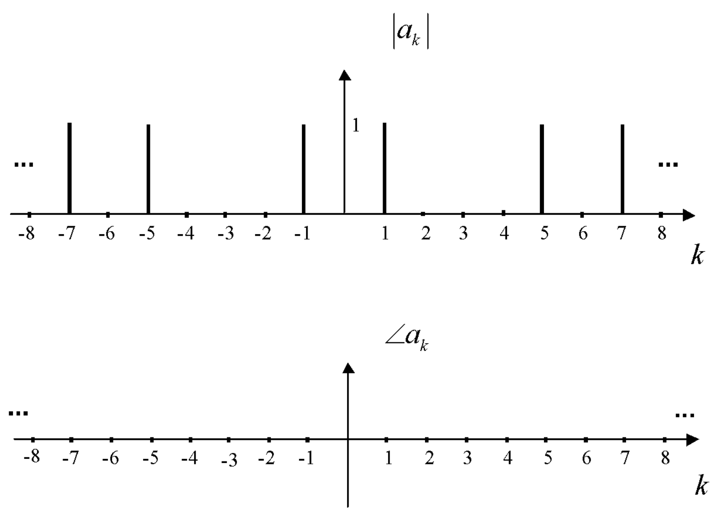


Figure 12.2: Magnitude and phase of DTFS of Problem 12.1.

We can write the Fourier series of $x[n]$ as:

$$x[n]=\sum_{k=0}^{5} a_k e^{jk\omega_0 n} = e^{j\frac{2\pi}{6}n} + e^{j\frac{2\pi}{6}5n}.$$

Problem 12.2

(a) Compute the Fourier transform $X(e^{j\omega})$ of the signal $x[n]$ shown below and plot its magnitude and phase over the interval $\omega \in [-\pi,\pi]$.

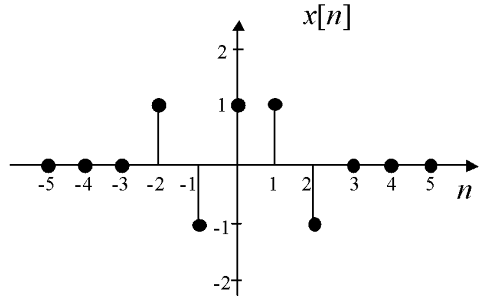


Figure 12.3: Signal of Problem 12.2.

Answer:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= e^{j\omega 2} - e^{j\omega} + 1 + e^{-j\omega} - e^{-j\omega 2} \\ &= 1 - 2j\sin\omega + 2j\sin 2\omega = 1 - 2j(\sin\omega - \sin 2\omega) \end{aligned}$$

The magnitude given below is shown in Figure 12.4 for $\omega \in [-\pi,\pi]$.

$$\begin{aligned} \left|X(e^{j\omega})\right| &= \sqrt{1+4\left[\sin\omega - \sin 2\omega\right]^2} \\ &= \sqrt{1+4\sin^2\omega - 8\sin(2\omega)\sin\omega + 4\sin^2(2\omega)} \end{aligned}$$

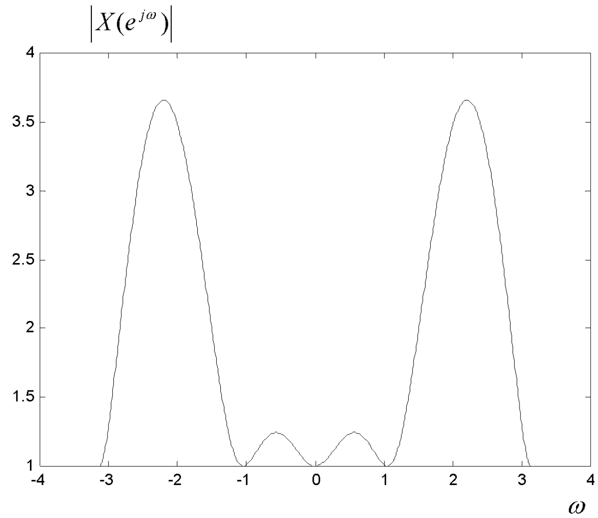


Figure 12.4: Magnitude of DTFT of Problem 12.2.

The phase is given by:

$$\angle X(e^{j\omega}) = \arctan \frac{2\sin(2\omega) - 2\sin\omega}{1},$$

and is plotted in Figure 12.5 for $\omega \in [-\pi, \pi]$.

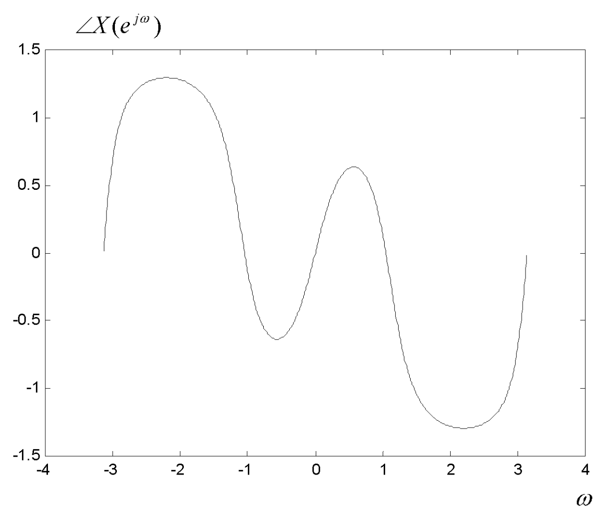


Figure 12.5: Phase of DTFT of Problem 12.2.

Problem 12.3

Compute the Fourier transforms $X(e^{j\omega})$ of the following discrete-time signals.

(a) $x[n] = [\alpha^n \sin(\omega_0 n) + \beta^n]u[n], \quad |\alpha| < 1, |\beta| < 1.$

Answer:

$$\begin{aligned} x[n] &= \left[\frac{1}{2j}\alpha^n \left(e^{j\omega_0 n} - e^{-j\omega_0 n} \right) + \beta^n \right] u[n] \\ &= \frac{1}{2j}(\alpha e^{j\omega_0})^n u[n] - \frac{1}{2j}(\alpha e^{-j\omega_0})^n u[n] + (\beta)^n u[n] \end{aligned}$$

Using the table, we obtain the DTFT:

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{2j}\left[\frac{1}{1-\alpha e^{-j(\omega-\omega_0)}} - \frac{1}{1-\alpha e^{-j(\omega+\omega_0)}} \right] + \frac{1}{1-\beta e^{-j\omega}} \\ &= \frac{(-0.5j)(1-\alpha e^{-j(\omega+\omega_0)}) + (0.5j)(1-\alpha e^{-j(\omega-\omega_0)})}{1-2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega}} + \frac{1}{1-\beta e^{-j\omega}} \\ &= \frac{-(0.5j)\alpha e^{-j(\omega+\omega_0)} + (0.5j)\alpha e^{-j(\omega-\omega_0)})}{1-2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega}} + \frac{1}{1-\beta e^{-j\omega}} \\ &= \frac{-\alpha\sin\omega_0 e^{-j\omega}}{1-2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega}} + \frac{1}{1-\beta e^{-j\omega}} \\ &= \frac{-\alpha\sin\omega_0 e^{-j\omega}(1-\beta e^{-j\omega}) + 1 - 2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega}}{(1-2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega})(1-\beta e^{-j\omega})} \\ &= \frac{1-\alpha(\sin\omega_0 + 2\cos\omega_0)e^{-j\omega} + \alpha(\alpha+\beta\sin\omega_0)e^{-j2\omega}}{(1-2\alpha\cos\omega_0 e^{-j\omega} + \alpha^2 e^{-j2\omega})(1-\beta e^{-j\omega})} \end{aligned}$$

(b) $x[n] = (u[n+2] - u[n-3]) * (u[n+2] - u[n-3])$ where $*$ is the convolution operator.

Answer:

First consider the DTFT of the pulse signal $y[n] := u[n+2] - u[n-3]$ which is:

$$Y(e^{j\omega}) = \frac{\sin(5/2\,\omega)}{\sin(\omega/2)}.$$

Now , $x[n] = y[n] * y[n]$ and hence,

$$\begin{aligned} X(e^{j\omega}) &= Y(e^{j\omega})Y(e^{j\omega}) \\ &= \left[\frac{\sin(5/2\,\omega)}{\sin(\omega/2)}\right]^2 \end{aligned}$$

Exercises

Problem 12.4

Given an LTI system with $h[n] = u[n]$, and an input $x[n] = (0.8)^n u[n]$, compute the DTFT of the output $Y(e^{j\omega})$ and its inverse DTFT $y[n]$.

Problem 12.5

Consider the signal $x[n] = \cos(\frac{\pi}{5}n + 2)(j)^n$.

(a) Is this signal periodic? If the signal is periodic, what is its fundamental period?

Answer:

$$
\begin{aligned}
x[n] &= \frac{e^{j(\frac{\pi}{5}n+2)} + e^{-j(\frac{\pi}{5}n+2)}}{2} e^{j\frac{\pi}{2}n} \\
&= \frac{1}{2}(e^{j((\frac{\pi}{5}+\frac{\pi}{2})n+2)} + e^{-j((\frac{\pi}{5}-\frac{\pi}{2})n+2)}) \\
&= \frac{1}{2}(e^{j(\frac{7\pi}{10}n+2)} + e^{-j(-\frac{3\pi}{10}n+2)}) = \frac{1}{2}(e^{j(\frac{7(2\pi)}{20}n+2)} + e^{-j(-\frac{3(2\pi)}{20}n+2)})
\end{aligned}
$$

Each one of these exponentials is periodic of fundamental period 20, thus the signal is periodic and its fundamental period is $N = 20$.

(b) Compute the discrete-time Fourier series coefficients of $x[n]$

Answer:

The expression:

$$
\begin{aligned}
x[n] &= \frac{1}{2}(e^{j(\frac{7(2\pi)}{20}n+2)} + e^{-j(-\frac{3(2\pi)}{20}n+2)}) \\
&= \frac{1}{2}e^{j2}e^{j\frac{7(2\pi)}{20}n} + \frac{1}{2}e^{-j2}e^{j\frac{3(2\pi)}{20}n}
\end{aligned}
$$

is already a Fourier series representation.Hence, the 20 Fourier series coefficients are:

$$
a_3 = \frac{1}{2}e^{-j2},\ a_7 = \frac{1}{2}e^{j2};\ a_k = 0,\quad k = 0,1,2,4,5,6,8,9,10,11,12,13,14,15,16,17,18,19
$$

The signal $x[n]$ is complex-valued, so the spectral coefficients have neither an even magnitude, nor an odd phase.

Problem 12.6

Compute the Fourier transforms $X(e^{j\omega})$ of the following signals

(a) $x[n] = \alpha^n [\sin(\omega_0 n) + 2\cos(\omega_0 n)]u[n], \quad |\alpha| < 1.$

(b) $x[n] = (u[n+2] - u[n-3]) * \sum_{k=-\infty}^{+\infty} e^{j\pi k}\delta[n-15k],$ where $*$ is the convolution operator.

Problem 12.7

Compute the Fourier transform $H(e^{j\omega})$ of the impulse response $h[n]$ shown in Figure 12.6 and find its magnitude over the interval $\omega \in [-\pi, \pi]$.

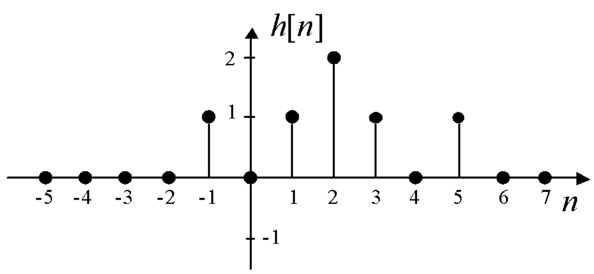


Figure 12.6: Signal in Problem 12.7.

Answer:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = e^{j\omega} + e^{-j\omega} + 2e^{-j2\omega} + e^{-j\omega 3} + e^{-j\omega 5}$$
$$= e^{-j2\omega}\left(e^{j3\omega} + e^{j\omega} + 2 + e^{-j\omega} + e^{-j3\omega}\right)$$
$$= 2e^{-j2\omega}\left(1 + \cos\omega + \cos 3\omega\right)$$

Magnitude is $|H(e^{j\omega})| = 2\left|1 + \cos\omega + \cos 3\omega\right|$.

Problem 12.8

Compute the Fourier transform $X(e^{j\omega})$ of the signal $x[n]$ shown in Figure 12.7 and sketch its magnitude and phase over the interval $\omega \in [-\pi, \pi]$.

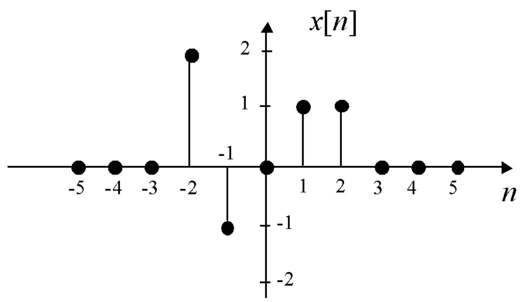


Figure 12.7: Signal in Problem 12.7.

Problem 12.9

Compute the Fourier series coefficients $\{a_k\}$ of the signal $x[n]$ shown in Figure 12.8. Sketch the magnitude and phase of the coefficients. Write $x[n]$ as a Fourier series.

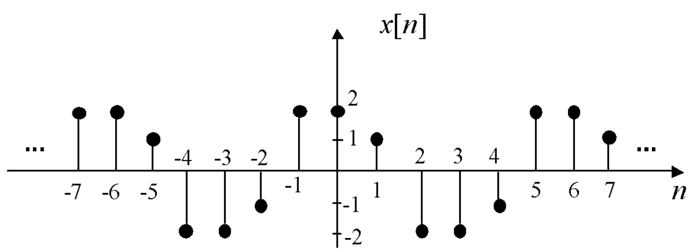


Figure 12.8: Periodic signal in Problem 12.9.

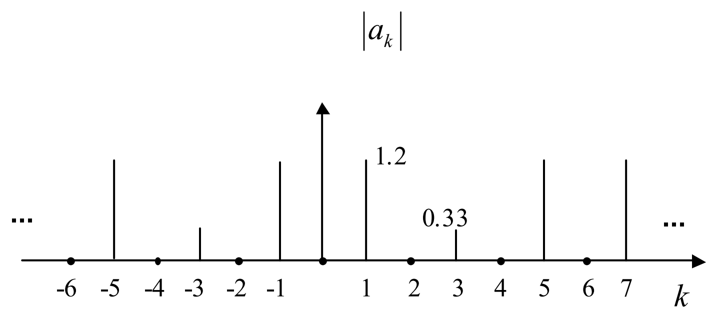
Answer:

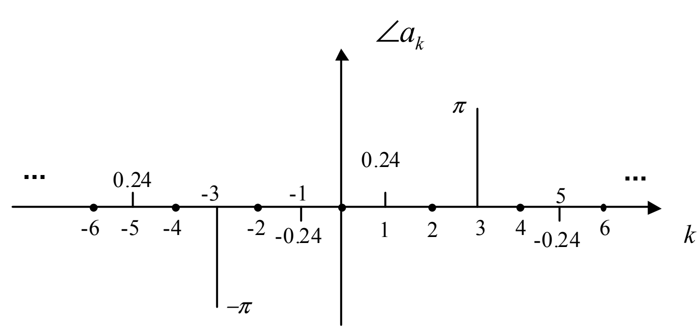
The fundamental period of this signal is *N*=6 and the fundamental frequency is $\omega_0 = \frac{2\pi}{6}$. The DC component is $a_0 = 0$. The other coefficients are obtained using the analysis equation of the DTFS:

$$
\begin{aligned}
a_k &= \frac{1}{6}\sum_{n=0}^{5} x[n] e^{-jk\frac{2\pi}{6}n} \\
&= \frac{1}{6}\left(2 + e^{-jk\frac{2\pi}{6}} - 2e^{-jk\frac{4\pi}{6}} - 2e^{-jk\frac{6\pi}{6}} - e^{-jk\frac{8\pi}{6}} + 2e^{-jk\frac{10\pi}{6}}\right) \\
&= \frac{1}{6}\left(2(1 + e^{-jk\frac{10\pi}{6}}) + (e^{-jk\frac{2\pi}{6}} - e^{-jk\frac{8\pi}{6}}) - 2(e^{-jk\frac{4\pi}{6}} + e^{-jk\frac{6\pi}{6}})\right) \\
&= \frac{1}{6}\left(2(1 + e^{-jk\frac{5\pi}{3}}) + (e^{-jk\frac{\pi}{3}} - e^{-jk\frac{4\pi}{3}}) - 2(e^{-jk\frac{2\pi}{3}} + (-1)^k)\right) \\
&= \frac{1}{6}\left(2(1 - (-1)^k) + (1 - (-1)^k) e^{-jk\frac{\pi}{3}} - 2(1 - (-1)^k) e^{-jk\frac{2\pi}{3}}\right) \\
&= \frac{(1 - (-1)^k)}{6}\left(2 + e^{-jk\frac{\pi}{3}} - 2e^{-jk\frac{2\pi}{3}}\right)
\end{aligned}
$$

Numerically,

$$
a_0 = 0,\ a_1 = \frac{7}{6} + j\frac{\sqrt{3}}{6} = 1.202 e^{j0.243},\ a_2 = 0,\ a_3 = -\frac{1}{3} = \frac{1}{3} e^{j\pi},
$$
$$
a_4 = 0,\ a_5 = \frac{7}{6} - j\frac{\sqrt{3}}{6} = 1.202 e^{-j0.243}
$$





We can write the Fourier series of $x[n]$ as:

$$x[n] = \sum_{k=\langle 6 \rangle} a_k e^{jk\omega_0 n}$$
$$= \left(\frac{7}{6} + j\frac{\sqrt{3}}{6}\right) e^{j\frac{2\pi}{6}n} - \frac{1}{3} e^{j\frac{2\pi}{6}3n} + \left(\frac{7}{6} - j\frac{\sqrt{3}}{6}\right) e^{j\frac{2\pi}{6}5n}$$

Problem 12.10

Consider a DLTI system with impulse response $h[n] = (-0.4)^n u[n] - (0.5)^{n-2} u[n-2]$. Compute the output signal $y[n]$ for the input $x[n] = (0.2)^n u[n]$. Use the DTFT.

Problem 12.11

Compute the Fourier transform $X(e^{j\omega})$ of the signal $x[n] = ne^{j\frac{\pi}{8}n} \alpha^{n-3} u[n-3], \quad |\alpha| < 1.$

Answer:

$$X(e^{j\omega}) = j\frac{d}{d\omega}\left[\frac{e^{-j3(\omega-\frac{\pi}{8})}}{1-\alpha e^{-j(\omega-\frac{\pi}{8})}}\right] = j\frac{-j3e^{-j3(\omega-\frac{\pi}{8})}\left(1-\alpha e^{-j(\omega-\frac{\pi}{8})}\right) + j\alpha e^{-j(\omega-\frac{\pi}{8})} e^{-j3(\omega-\frac{\pi}{8})}}{\left(1-\alpha e^{-j(\omega-\frac{\pi}{8})}\right)^2}$$

$$= j\frac{-j3e^{-j3(\omega-\frac{\pi}{8})} + j3\alpha e^{-j(4\omega-\frac{\pi}{2})} + j\alpha e^{-j(4\omega-\frac{\pi}{2})}}{\left(1-\alpha e^{-j(\omega-\frac{\pi}{8})}\right)^2}$$

$$= \frac{3e^{-j3(\omega-\frac{\pi}{8})} - 4\alpha e^{-j(4\omega-\frac{\pi}{2})}}{\left(1-\alpha e^{-j(\omega-\frac{\pi}{8})}\right)^2}$$