Solutions to Problems in Chapter 10

Problems with Solutions

Problem 10.1

Consider the causal LTI state-space system:

$$\begin{aligned}\dot{x} &= Ax + Bu \\ y &= Cx\end{aligned}$$

where $A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$.

(a) Is the system stable? Justify your answer.

Answer:

We compute the eigenvalues of the A matrix:

$$\begin{aligned}&\det\begin{bmatrix} \lambda+1 & -1 \\ 1 & \lambda+1 \end{bmatrix} = (\lambda+1)(\lambda+1)+1 = \lambda^2 + 2\lambda + 2 \\ &\Rightarrow \lambda_1 = -1+j, \lambda_2 = -1-j\end{aligned}$$

The eigenvalues of the A matrix $\lambda_{1,2} = -1 \pm j$ have a negative real part, therefore the system is stable.

(b) Compute the transfer function $H(s)$ of the system. Specify its ROC.

$$
\begin{aligned}
H(s) &= C(sI_2 - A)^{-1}B \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 1 & s+1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{s^2+2s+2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 1 \\ -1 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{s^2+2s+2} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 \\ -1 \end{bmatrix} = \frac{s}{s^2+2s+2}, \quad \mathrm{Re}\{s\} > -1
\end{aligned}
$$

The poles of the transfer function are $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\frac{1}{\sqrt{2}}\sqrt{2} \pm j\sqrt{2}\sqrt{\frac{1}{2}} = -1 \pm j1$, equal to the eigenvalues as expected.

(c) Compute the impulse response $h(t)$ of the system using the matrix exponential.

Answer:

The eigenvectors of the A matrix are computed as follows:

$$
\begin{aligned}
&(\lambda_1 I - A)v_1 = \begin{bmatrix} -1+j+1 & -1 \\ 1 & -1+j+1 \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} j & -1 \\ 1 & j \end{bmatrix} \begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\
&\Rightarrow v_{11} = 1, v_{12} = j \\
&v_2 = v_1^* = \begin{bmatrix} 1 \\ -j \end{bmatrix}
\end{aligned}
$$

Thus $T = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$, $T^{-1} = \frac{1}{-2j}\begin{bmatrix} -j & -1 \\ -j & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -j\frac{1}{2} \\ \frac{1}{2} & j\frac{1}{2} \end{bmatrix}$, and

$$
\begin{aligned}
h(t) &= CT\mathrm{diag}\{e^{(-1+j)t}, e^{(-1-j)t}\}T^{-1}q(t) \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}\begin{bmatrix} e^{(-1+j)t} & 0 \\ 0 & e^{(-1-j)t} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & -j\frac{1}{2} \\ \frac{1}{2} & j\frac{1}{2} \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}q(t) \\
&= \begin{bmatrix} 1+j & 1-j \end{bmatrix}\begin{bmatrix} e^{(-1+j)t} & 0 \\ 0 & e^{(-1-j)t} \end{bmatrix}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}q(t) \\
&= \left[\left(\frac{1}{2}+j\frac{1}{2}\right)e^{(-1+j)t} + \left(\frac{1}{2}-j\frac{1}{2}\right)e^{(-1-j)t}\right]q(t) \\
&= e^{-t}\left[\left(\frac{1}{2}+j\frac{1}{2}\right)e^{jt} + \left(\frac{1}{2}-j\frac{1}{2}\right)e^{-jt}\right]q(t) \\
&= 2e^{-t}\,\mathrm{Re}\left[\left(\frac{1}{2}+j\frac{1}{2}\right)e^{jt}\right]q(t) = e^{-t}\left(\cos t - \sin t\right)q(t)
\end{aligned}
$$

Thus, $h(t) = e^{-t}\left(\cos t - \sin t\right)q(t)$.

Problem 10.2

Find controllable and observable canonical state-space realizations for the following LTI system:

$$H(s) = \frac{s^3 + s + 2}{s^3 + 3s}, \quad \mathrm{Re}\{s\} > 0.$$

Answer:

The system's controllable canonical state-space realization is shown in Figure 10.1

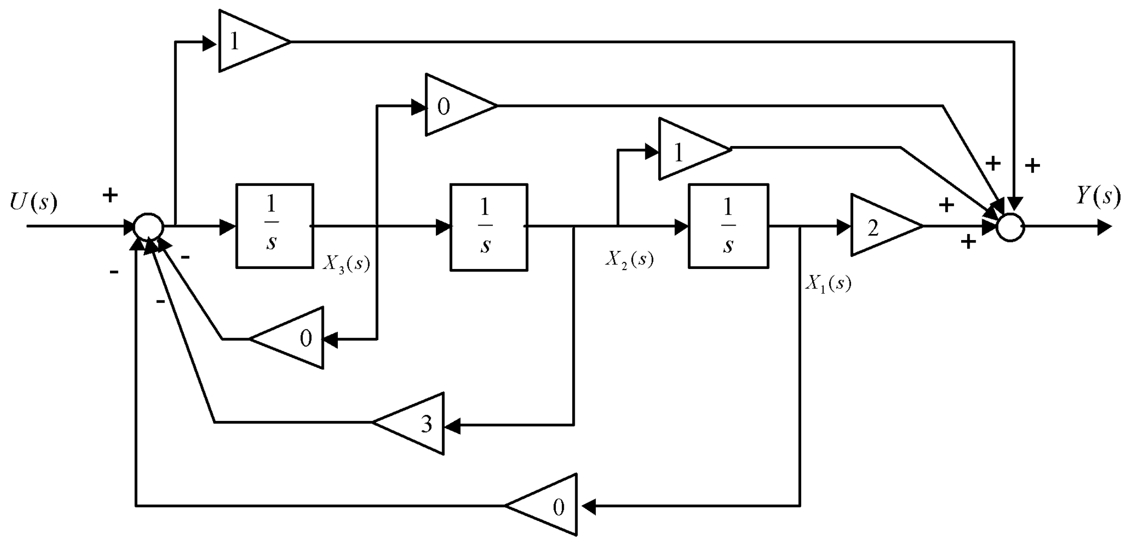


Figure 10.1: Controllable canonical realization of system of Problem 10.2.

Referring to Figure 10.1, we can write down the state-space equations of the controllable canonical realization of the system:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -3 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 2 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + u(t)\,.$$

The observable canonical realization is the block diagram of Figure 10.2

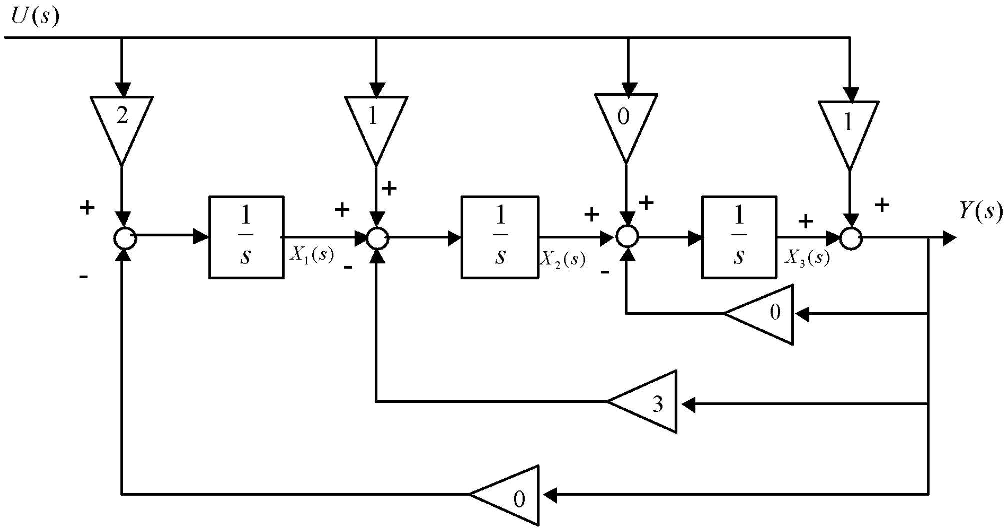


Figure 10.2: Observable canonical realization of system of Problem 10.2.

From Figure 10.2, the state-space equations of the observable canonical realization of the system can be written as:

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -3 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + u(t)$$

Exercises

Problem 10.3

Compute $e^A + I$ for $A = \begin{bmatrix} -2 & 0 \\ 3 & 1 \end{bmatrix}$.

Answer:

The eigenvalues of the A matrix are computed:

$$\det(\lambda I - A) = 0$$

$$\det\begin{bmatrix} \lambda+2 & 0 \\ -3 & \lambda-1 \end{bmatrix} = (\lambda+2)(\lambda-1) = 0$$
$$\Rightarrow \lambda_1 = -2, \lambda_2 = 1$$

Next, we compute the eigenvectors of A:

Eigenvector v_1 corresponding to $\lambda_1 = -2$:

$$\begin{bmatrix} (-2)+2 & 0 \\ -3 & (-2)-1 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -3 & -3 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow v_{11} = 1, v_{12} = -1$$

Eigenvector v_2 corresponding to $\lambda_1 = -3$:

$$\begin{bmatrix} (1)+2 & 0 \\ -3 & (1)-1 \end{bmatrix}\begin{bmatrix} v_{11} \\ v_{12} \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} v_{21} \\ v_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\Rightarrow v_{21} = 0, v_{22} = 1$$

Diagonalizing matrix $T = [v_1 \quad v_2] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$

$$
\begin{aligned}
e^{A} &= T\text{diag}\{e^{-2}, e^{1}\}T^{-1} \\
&= \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2} & 0 \\ 0 & e^{1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} e^{-2} & 0 \\ e^{1} & e^{1} \end{bmatrix} \\
&= \begin{bmatrix} e^{-2} & 0 \\ e^{1} - e^{-2} & e^{1} \end{bmatrix}
\end{aligned}
$$

Finally, $$e^{A} + I = \begin{bmatrix} 1 + e^{-2} & 0 \\ e^{1} - e^{-2} & 1 + e^{1} \end{bmatrix}.$$

Problem 10.4

Consider the causal LTI state-space system

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx
\end{aligned}
\qquad \text{where } A = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.
$$

(a) Is the system stable? Justify.

(b) Compute the transfer function $\mathcal{H}(s)$ of the system. Specify its ROC.

(c) Compute the impulse response $h(t)$ of the system using the matrix exponential.

Problem 10.5

Repeat Problem 2 with $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 1 \end{bmatrix}$. What type of system is this?

(a) Is the system stable? Justify.

Answer:

The eigenvalues of the A matrix $\lambda_{1,2} = 0$ sit on the imaginary axis, therefore the system is unstable.

(b) Compute the transfer function $\mathcal{H}(s)$ of the system. Specify its ROC.

Answer:

$$\begin{aligned}
\mathcal{H}(s) &= C(sI_2 - A)^{-1}B \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ -1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{s^2} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} s & 0 \\ 1 & s \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\
&= \frac{1}{s^2}, \quad \mathrm{Re}\{s\} > 0
\end{aligned}$$

This is a double integrator. There is a double pole at 0, same as the eigenvalues so the system is minimal.

(c) Compute the impulse response $h(t)$ of the system using the matrix exponential.

Answer:

Use the power series for the matrix exponential:

$$\begin{aligned}
e^{At} &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \underbrace{\frac{1}{2} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}^2 + \cdots}_{0} \\
&= \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}
\end{aligned}$$

$$
\begin{aligned}
h(t) &= Ce^{At}Bq(t) \\
&= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} q(t) \\
&= tq(t)
\end{aligned}
$$

Problem 10.6

Find controllable and observable canonical state-space realizations for each of the following LTI systems.

(a) $h(t) = e^{-2t}q(t) + te^{2t}q(t)$

(b) $\dfrac{s^3 + s^2 + 2s + 1}{s^3 + 5s^2 + 2s}, \ \ \mathrm{Re}\{s\} > 0$

Problem 10.7

Find controllable and observable canonical state-space realizations for each of the following LTI systems.

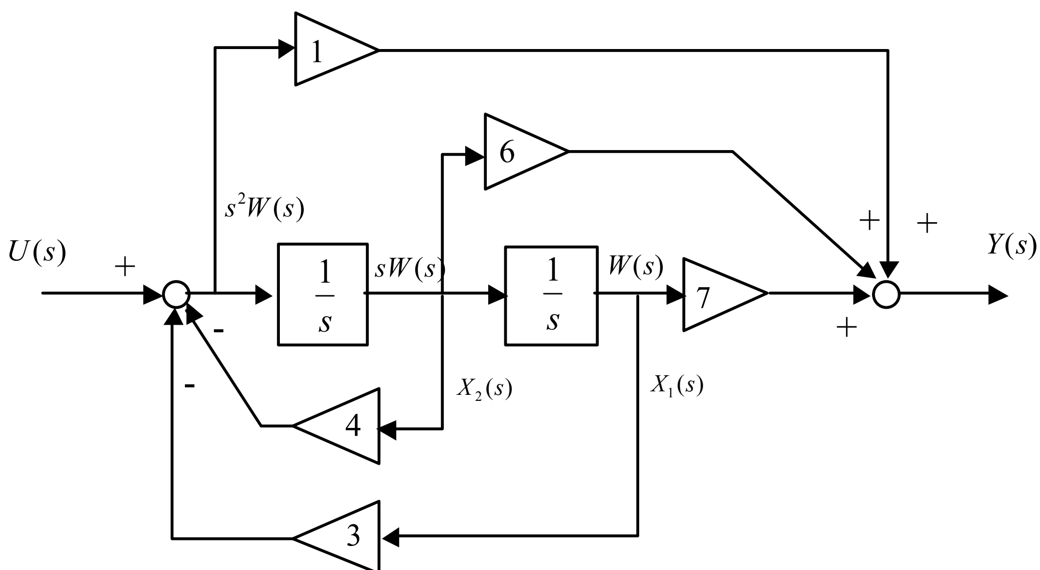
(a) $h(t) = e^{-3t}q(t) + e^{-t}q(t) + \delta(t)$

Answer:

This system is causal. Its transfer function is given by:

$$
\begin{aligned}
H(s) &= \mathcal{L}\left\{e^{-3t}q(t) + e^{-t}q(t) + \delta(t)\right\} \\
&= \underbrace{\frac{1}{s+3}}_{\mathrm{Re}\{s\}>-3} + \underbrace{\frac{1}{s+1}}_{\mathrm{Re}\{s\}>-1} + 1 \\
&= \frac{2s+4}{(s+3)(s+1)} + 1,\ \mathrm{Re}\{s\} > -1 \\
&= \frac{2s+4}{s^2+4s+3} + 1,\ \mathrm{Re}\{s\} > -1 \\
&= \frac{s^2+6s+7}{s^2+4s+3},\ \mathrm{Re}\{s\} > -1
\end{aligned}
$$

Controllable canonical state-space realization



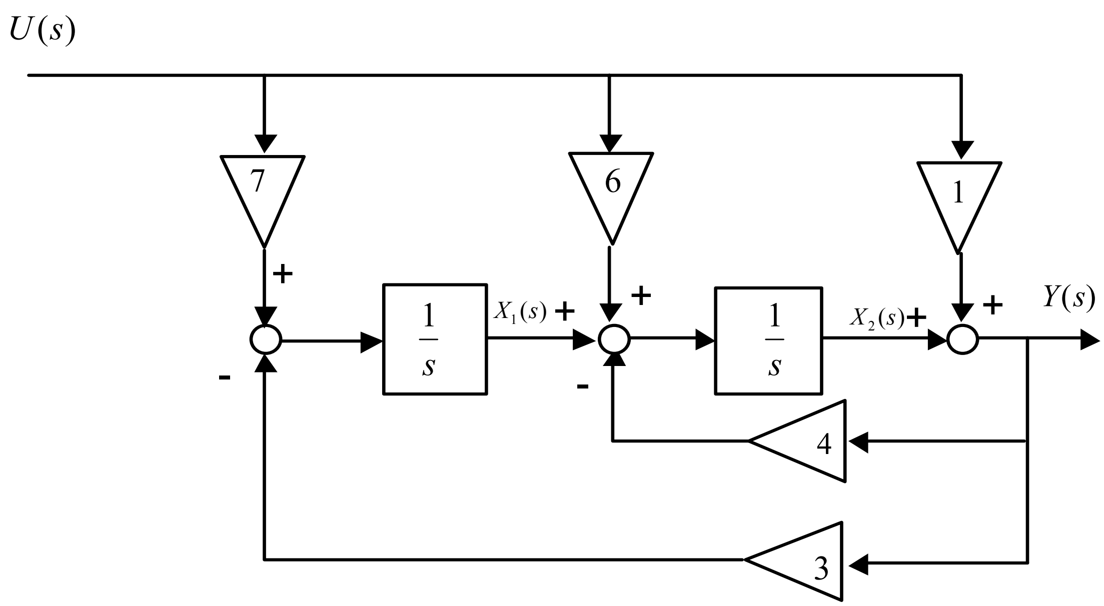
Thus,

$$
\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),
$$

$$
y(t) = \begin{bmatrix} 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1u(t)
$$

Observable canonical state-space realization:

$$H(s) = \frac{2s+4}{s^2+4s+3} + 1 = \frac{2s^{-1}+4s^{-2}}{1+4s^{-1}+3s^{-2}} + 1$$
$$= \frac{1+6s^{-1}+7s^{-2}}{1+4s^{-1}+3s^{-2}}, \ \mathrm{Re}\{s\} > -1$$



Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 4 \\ 2 \end{bmatrix} u(t),$$

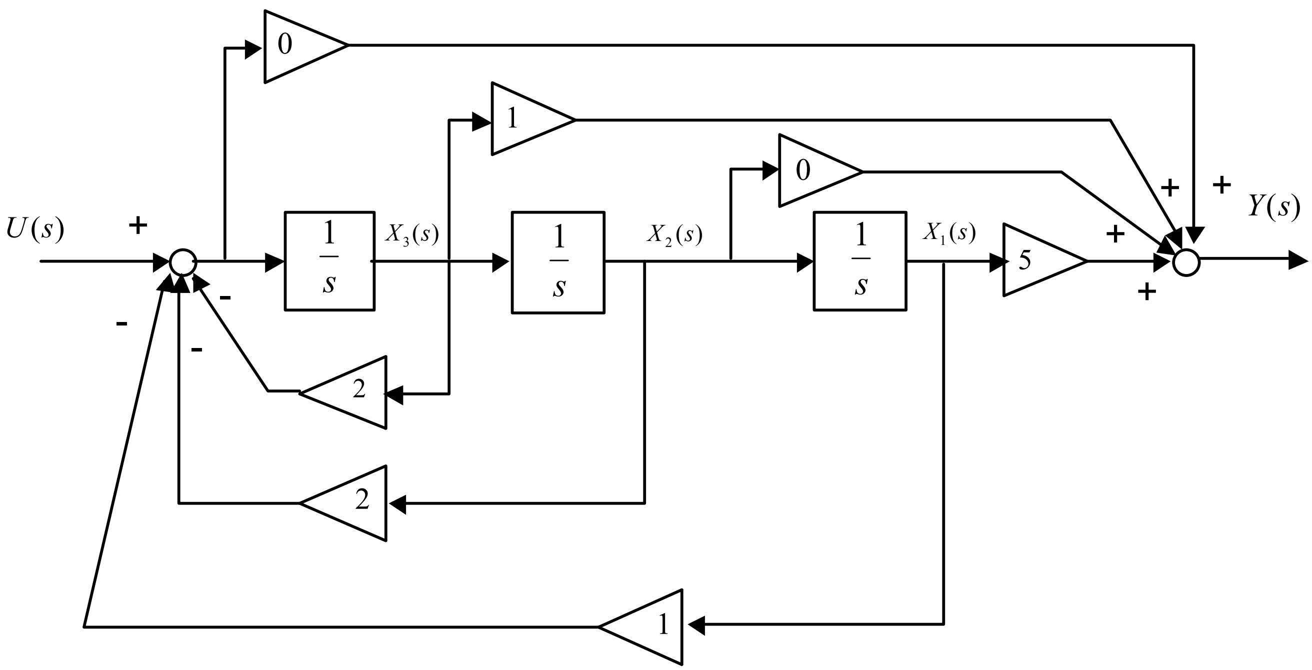
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 1u(t)$$

The state-space matrices are: $A = \begin{bmatrix} 0 & -3 \\ 1 & -4 \end{bmatrix}, \ B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \ C = \begin{bmatrix} 0 & 1 \end{bmatrix}, \ D = 1$

(b) $\dfrac{s^2+5}{s^3+2s^2+2s+1}$, $\quad \mathrm{Re}\{s\} > -0.5$

Answer:

Controllable canonical state-space realization

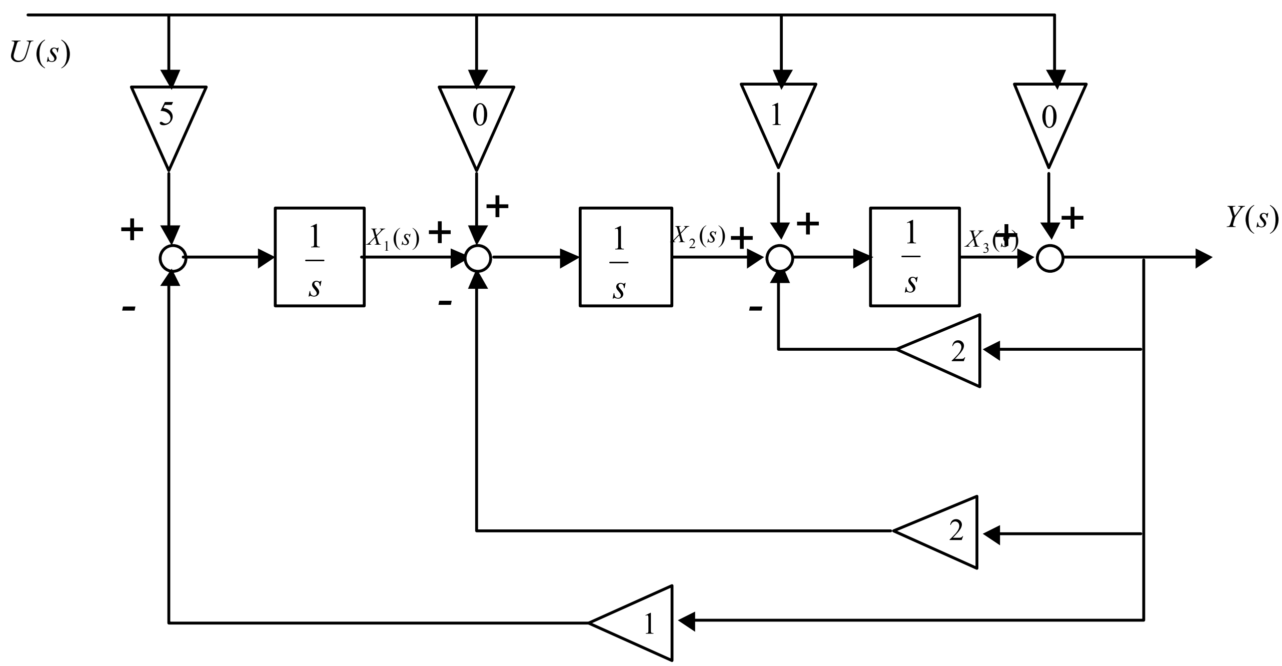


We have:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + u(t)$$

Observable canonical state-space realization



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

c) $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 1 \end{bmatrix} u(t), \;\; y(t) = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$, causal.

Answer:

Let's compute the transfer function first:

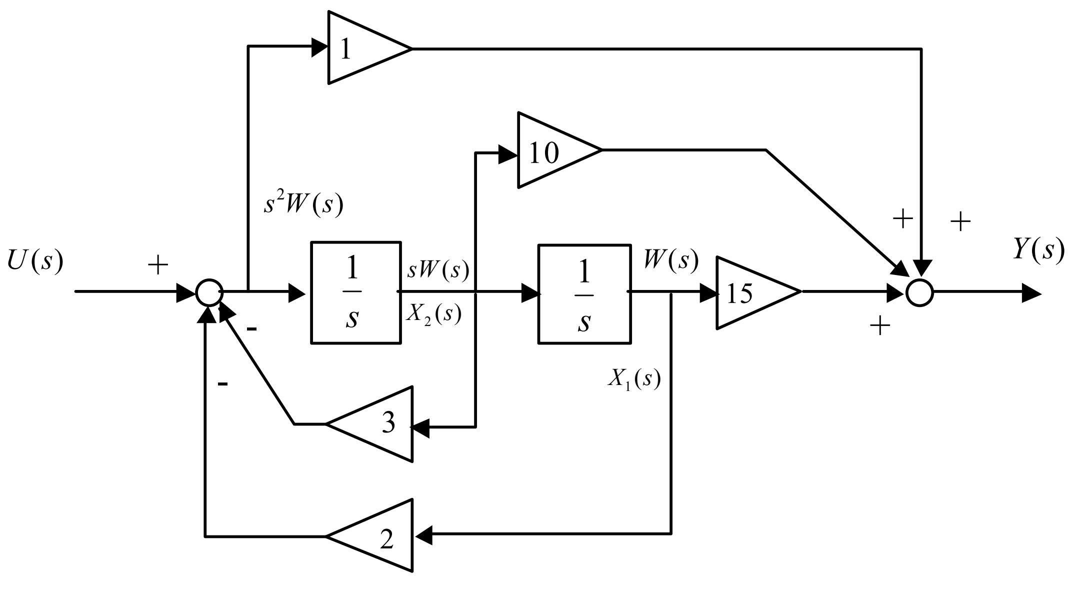
$$\mathcal{H}(s) = C(sI_n - A)^{-1}B + D$$

$$= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1 = \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{1}{s+1} & 0 \\ 0 & \frac{1}{s+2} \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix} + 1$$

$$= \begin{bmatrix} 2 & 1 \end{bmatrix} \begin{bmatrix} \frac{3}{s+1} \\ \frac{1}{s+2} \end{bmatrix} + 1 = \frac{6}{s+1} + \frac{1}{s+2} + 1 = \frac{(7s+13)+(s^2+3s+2)}{s^2+3s+2}$$

$$= \frac{s^2+10s+15}{s^2+3s+2}, \quad \mathrm{Re}\{s\} > -1$$

Controllable canonical state-space realization

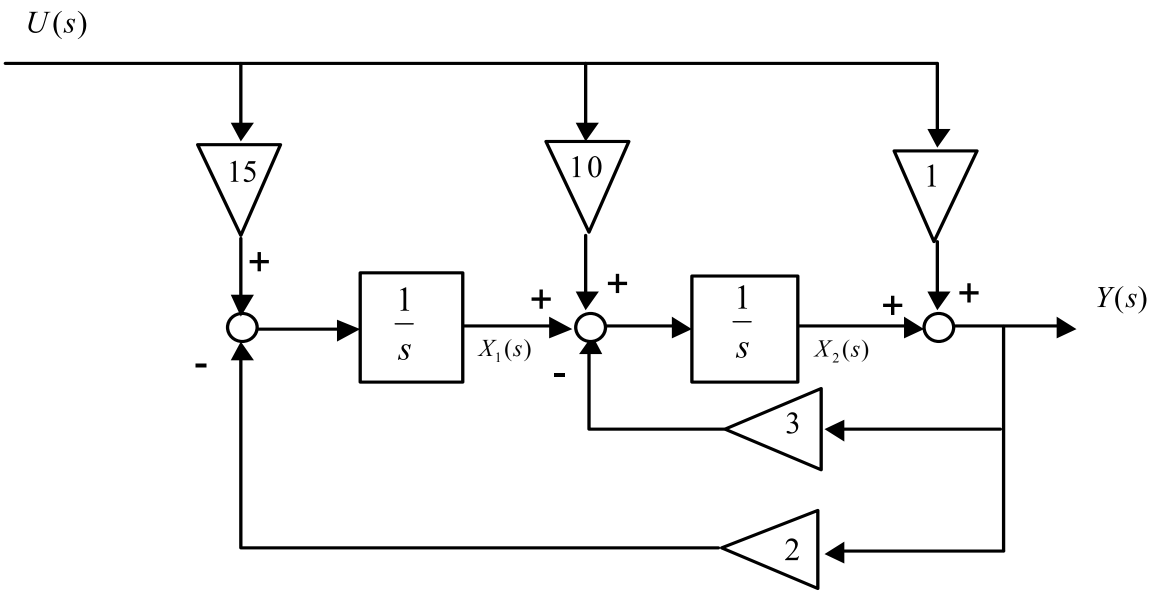


Thus,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 13 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$$

Observable canonical state-space realization.



$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 13 \\ 7 \end{bmatrix} u(t),$$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u$$

Problem 10.8

(a) Compute e^{A} for $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$.

(b) Compute the zero-input state and output responses at time $t_1 = 2$ for the causal LTI state-space system:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned},$$

where $A = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$, with initial state $x(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Problem 10.9

Consider the causal LTI state-space system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}, \quad \text{where } A = \begin{bmatrix} -11 & 1 \\ 3 & -9 \end{bmatrix}, \quad B = \begin{bmatrix} 2 \\ 3 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

(a) Is the system minimal? Is it stable? Justify.

Answer:

$$\det \begin{bmatrix} \lambda + 11 & -1 \\ -3 & \lambda + 9 \end{bmatrix} = (\lambda + 11)(\lambda + 9) - 3 = \lambda^2 + 20\lambda + 96$$
$$\Rightarrow \lambda_1 = -12, \lambda_2 = -8$$

The eigenvalues of the A matrix are negative, therefore the system is stable. To answer the question on minimality, we need to compute the transfer function first in (b).

(b) Compute the transfer function $\mathcal{H}(s)$ of the system. Specify its ROC.

$$\begin{aligned}
\mathcal{H}(s) &= C(sI_2 - A)^{-1} B \\
&= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+11 & -1 \\ -3 & s+9 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= \frac{1}{s^2 + 20s + 96} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+9 & 1 \\ 3 & s+11 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\
&= \frac{1}{s^2 + 20s + 96} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 2s+21 \\ 3s+39 \end{bmatrix} = \frac{5s+60}{s^2 + 20s + 96}, \quad \operatorname{Re}\{s\} > -8 \\
&= \frac{5(s+12)}{(s+12)(s+8)} = \frac{5}{s+8}, \quad \operatorname{Re}\{s\} > -8
\end{aligned}$$

The pole is at –8, one of the two eigenvalues: –8 and -12. This state-space system is stable but non-minimal (answer of (a))

(c) Give the state transition matrix $\Phi(t,t_0)$.

Answer:

Two eigenvectors corresponding to $\lambda_1=-12, \lambda_2=-8$ are $v_1=\begin{bmatrix}1\\-1\end{bmatrix}$, $v_2=\begin{bmatrix}1\\3\end{bmatrix}$ so that $T=[v_1 \quad v_2]=\begin{bmatrix}1 & 1\\-1 & 3\end{bmatrix}$ is diagonalizing. We have

$$\begin{aligned}\Phi(t,t_0)=e^{A(t-t_0)}&=\begin{bmatrix}1 & 1\\-1 & 3\end{bmatrix}\begin{bmatrix}e^{-12(t-t_0)} & 0\\0 & e^{-8(t-t_0)}\end{bmatrix}\begin{bmatrix}3 & -1\\1 & 1\end{bmatrix}\frac{1}{4}\\&=\frac{1}{4}\begin{bmatrix}3e^{-12(t-t_0)}+e^{-8(t-t_0)} & -e^{-12(t-t_0)}+e^{-8(t-t_0)}\\-3e^{-12(t-t_0)}+3e^{-8(t-t_0)} & e^{-12(t-t_0)}+3e^{-8(t-t_0)}\end{bmatrix}\end{aligned}$$

(d) Compute the impulse response $h(t)$ of the system using the state transition matrix (matrix exponential).

Answer:

$$\begin{aligned}h(t)&=Ce^{At}Bq(t)\\&=\frac{1}{4}[1 \quad 1]\begin{bmatrix}3e^{-12t}+e^{-8t} & -e^{-12t}+e^{-8t}\\-3e^{-12t}+3e^{-8t} & e^{-12t}+3e^{-8t}\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}q(t)\\&=\frac{1}{4}\begin{bmatrix}4e^{-8t} & 4e^{-8t}\end{bmatrix}\begin{bmatrix}2\\3\end{bmatrix}q(t)\\&=5e^{-8t}q(t)\end{aligned}$$

Problem 10.10

Consider the LTI causal state-space system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned},$$

where $x(t) \in \mathbb{R}^n$, $y(t) \in \mathbb{R}, u(t) \in \mathbb{R}$. Show that any state transformation $z = Qx$, where $Q \in \mathbb{R}^{n \times n}$ is invertible, of the above state-space system keeps the transfer function invariant. This means that there are infinitely many state-space representations of any given proper rational transfer function.